This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

NYPL RESEARCH LIBRARIES
3 3433 06910738 5

— Rothman Esq

from the Author

J. Lubbock

12/- 6838

ON

THE THEORY OF THE MOON,

AND ON

THE PERTURBATIONS OF THE PLANETS.

By J. W. LUBBOCK, Esq. F.R., R.A., and L.SS.
MEMBER OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES.

A NEW EDITION.

LONDON:
CHARLES KNIGHT, 22 LUDGATE STREET.
1834.

THE NEW YORK
PUBLIC LIBRARY
12146A
ASTOR, LENOX AND
TILDEN FOUNDATIONS
R 1921 L

PRINTED BY RICHARD TAYLOR,
RED LION COURT, FLEET STREET.

PREFACE.

THE researches of mathematicians who, since the time of the immortal Newton, have contributed to improve the science of physical astronomy, are described in the fifth volume of the *Mécanique céleste*, in the introduction to the *Théorie analytique du système du monde*, and in the *Essai historique sur le problème des trois corps*. I shall therefore briefly state in this introduction, wherein the methods of this treatise differ from those previously employed.

The analytical investigation of the theory of the moon was undertaken nearly at the same time by Clairaut, D'Alembert, and Euler, towards the middle of the last century. Laplace adopted the method of Clairaut, and carried the approximation much further, so that his coefficients agree far better with observation. The approximation has since been pushed to an almost incredible extent by M. Damoiseau, without any alteration, however, in the method employed.

Laplace insists particularly upon the necessity of having recourse to the equations in which the true longitude is the independent variable; but notwithstanding the respect which is due to so great authority, I am convinced, after much reflection, that the method which I have submitted to the Royal Society, and which forms the groundwork of this Essay, is on many accounts to be preferred. According to this method, the expressions for the parallax, and the true longitude obtained by the reversion of series in the method of Clairaut, are arrived at directly. The differential equations which I employ, and in which the mean longitude is the independent variable, are far more simple than those in which the true longitude is the independent variable; and I avoid the substitution of the true longitude of the sun in terms of the true longitude of the moon, which is necessary in the other method. The coordinate z of the moon may be obtained directly by means of the equation

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} + \frac{m_{,} z}{r_{,}^3} + \frac{3 m_{,} z r' r \cos (\lambda' - \lambda)}{r_{,}^5} = 0.$$

This equation is not complicated, but the latitude may, I think, be arrived at more easily, as hitherto, by the well known method employed by Laplace.

Many of the developments which are required may be formed with facility by means of a Table which I have given for the purpose in the Appendix. This Table may be used with equal advantage in either method.

In order to give various examples, I have shown how certain coefficients may be found; but even with the advantages which this method possesses, to determine the coefficients of all the inequalities with that degree of accuracy which the present state of astronomy demands, will require great labour, and the sacrifice of more time than I can devote to this subject; nor must it be forgotten, that if the method originally proposed by Clairaut has at last been brought to perfection, this result has been obtained through the efforts of the greatest mathematicians, continued for more than half a century. The accuracy, however, of the Lunar Tables is a matter of such extreme importance, and the calculations required in order to deduce from theory the numerical values of the coefficients of the inequalities are of such prodigious complexity, that no method of verification should be neglected by which the exactness of the coefficients now in use can be ascertained, or any new inequalities can be detected; and although, indeed, the identity of solutions of the same problem obtained by different methods can be anticipated *à priori*, yet it is always interesting and advantageous to trace their coincidence.

It seems, moreover, desirable to compare the theory of the perturbations of the moon with that of the planets, to show distinctly where the analogy ceases, and to introduce into the science of physical astronomy a uniform system, by employing, as far as is possible, methods which embrace the motions of all the heavenly bodies.

With this view I have endeavoured to explain how the perturbations of the planets and their satellites may be obtained from the equations employed in my Theory of the Moon.

The following are the chief sources of dissimilarity:

1. In the planetary theory the terms depending upon the square and the higher powers of the disturbing mass are generally insensible; in the lunar theory, on the contrary, many terms of this nature must be taken into account.

2. In the planetary theory the higher powers of the ratio of the semi-axis major of the inferior planet to that of the superior planet must be retained, whereas in the lunar theory, the ratio of the moon's distance from the earth to that of the sun is a very small quantity.

3. In the lunar theory the quantity $c_{\prime}$ may be considered equal to unity; or, in other words, the mean motion of the earth's perihelium may be neglected. If the same methods are applied to the theory of the planetary perturbations, the mean motion of the perihelium of the disturbing planet is of the same order as that of the planet whose motion is considered, and cannot, of course, be neglected.

The analytical character of these problems is not, however, essentially different owing to these considerations; and it seems to me that the method of indeterminate coefficients is complete, so that the time may be got rid of in the expressions for the radius vector and for the latitude of the planets by methods similar to those employed in the theory of the moon.

When the square of the disturbing function is neglected, the table in the Appendix may be used in the planetary theory, both in the solution of the differential equations and in the development of the disturbing function.

I obtain the inequalities of the reciprocal of the radius vector, by the method of indeterminate coefficients, from the well known equation

$$\frac{d^2 r^2}{2\,d t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int d R + r\left(\frac{d R}{d r}\right) = 0,$$

without the intervention of an auxiliary variable, as in the *Mécanique Céleste*, vol. i.; and I obtain the inequalities of longitude from the equation

$$\mathrm{d}\lambda' = \frac{h}{r'^2}\left\{1 - \frac{1}{h}\int \frac{\mathrm{d}R}{\mathrm{d}\lambda'}\,\mathrm{d}t\right\}\mathrm{d}t,$$

(in which λ' signifies the longitude reckoned upon some fixed plane, and r' the projection of r upon the same plane,) instead of the equation

$$\delta\lambda = \frac{\dfrac{2r\mathrm{d}.\delta r + \delta r\mathrm{d}r}{a^2 n\mathrm{d}t} + \dfrac{an}{\mu}\left\{3\iint \mathrm{d}t\,\mathrm{d}R + 2\int r\left(\dfrac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}t\right\}}{\sqrt{1-e^2}},$$

employed in the *Mécanique Céleste*, and in other works where the same methods are adopted.

I have shown how the development of the disturbing function may be obtained in a more simple form than that given in the *Mécanique Céleste*, through the binomial theorem. The expression which at first results is very complicated; but admits of simplification, in consequence of equations of condition which obtain between the quantities of which the general symbol is b. This method is not readily applicable to the terms which are multiplied by the higher powers of the eccentricities; but I have given another method, according to which, the terms multiplied by any given powers of the eccentricities are obtained in terms of those of the inferior orders, and which is peculiarly advantageous in the lunar theory. Numerous equations of condition arise in the latter method, between the coefficients corresponding to different arguments in the disturbing function, which I think have not before been noticed.

The stability of a system of bodies in motion subject to the law of mutual gravitation which obtains in nature, is a question of great interest. It appears to me that it may be established through considerations different from those hitherto employed, namely, by observing that the quantities which have been called c and g in the lunar theory are *rational*, so that no imaginary angles or exponentials are introduced in the expressions for the coordinates of the body in motion, which are thus functions of periodic quantities. In the theory of the moon I apprehend this theorem is certainly independent of the direction of the moon's motion. Moreover, the distance of the moon from the earth (r), and the tangent of the latitude (s), are expressed by convergent series, so that these quantities do not increase indefinitely.

Some of the terms in the lunar theory due solely to the first power of the disturbing force are the most sensible, and can be obtained very easily without tedious arithmetical calculations. These serve to illustrate the theory of the perturbations, and they might, I think, be introduced with advantage into the elements of Dynamics, where the elliptic theory has long found a place. The combinations which concur in the formation of any term, increase with appalling multiplicity, in proportion as the approximation is carried to higher powers of the disturbing mass.

Since the publication of the former edition of this treatise, M. Plana's work, entitled *Théorie du Mouvement de la Lune*, has appeared, which has changed the state of this problem, in consequence chiefly of the form which M. Plana has given to the expressions. Agreeing with M. Plana in the importance of this modification, and in the necessity of retaining all the terms of a given order when any are taken into account, I have published this new edition in order to pursue the same plan, to arrive at some few terms at least which occur in the earlier approximations, and to offer some comparisons with M. Plana's final results. In the majority of instances I have the satisfaction of obtaining by my methods (which differ entirely from M. Plana's) the same results; in those where this is not the case, the details which I have afforded will assist in deciding to what circumstance those discrepancies are to be attributed.

The change of form in the expressions alluded to, due, I believe, to M. Plana, and at all events adopted by him, consists in developing the expressions in terms of a certain quantity m, which denotes the ratio of the observed mean motion of the sun to that of the moon. The coefficients of the different powers of m in these expressions, generally fractions in their lowest terms, are definite. The expression in this form for any inequality is generally composed of various terms, depending upon powers of m, the eccentricities, &c.; but the coefficient of each of these terms being definite, each can be verified separately. This consideration is highly important as regards the accuracy of the final results (without which they can be of no practical utility), because if expressions, similar, and which ought to be identically the same, are obtained by any separate and independent method, the points of comparison are by this plan greatly multiplied, and the detection of the source of any discrepancy is greatly facilitated.

Recently, also, M. Poisson has published a very important Memoir upon this subject, in which he employs the method of the variation of the elliptic constants. This method is peculiarly applicable to those terms of which the coefficients are lowered by successive integrations, as may be seen by the very instructive examples which M. Poisson has chosen, and some others, which I have given in this treatise. The development of the disturbing function in terms of the mean longitudes, which results from the substitution of the elliptic values of the coordinates, is the foundation of that method, as of the other, which is more properly the subject of this treatise, and which seems to me to be generally preferable.

29, *Eaton Place,*
Nov. 15, 1834.

CONTENTS.

1. EQUATIONS of motion. Equations of condition between the constants h, ι, a, e and e' in the elliptic theory. Expressions for various powers of the radius vector in terms of the mean anomaly. Finite expressions for the non-periodical portions of r^2 and r^{-3} . . *page* 1 to 11
2. Equation which obtains when a quantity (v) is introduced, which in the elliptic theory is the eccentric anomaly. Expressions for the variation of the elliptic constants. Example of the application of the expressions for the variations of the elliptic constants in a very simple case. Equations which serve to determine the perturbations of a comet, Method of quadratures. If the sun or primary be a spheroid, the disturbing function R is increased by the quantity $c\left\{\frac{3\sin^2\omega\sin^2\lambda-1}{r^3}\right\}$. Expressions for the variation of the elliptic constants, when the body moves in a resisting medium . . . . *page* 11 to 22
3. Arguments which occur in the theory of the moon, with their corresponding indices. Explanation of the manner in which the Table is made which serves to perform all the developments required. Method of obtaining the development of the disturbing function R. Expression for R. Examples of the determination of the first term of various inequalities in the expression for the reciprocal of the radius vector, and of the longitude (λ), in terms of the mean longitudes. Examples of the determination of the first term of various inequalites in the expression for the tangent of the latitude (s), in terms of the true longitude. Determination of the second term in the evection. Expressions for $\frac{\mathrm{a}}{\mu}\delta R$, $\frac{\mathrm{a}}{\mu}R'$, and $\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2$. Examples of the determination of terms in the reciprocal of the radius vector, arising from the second approximation. Expression for $\int\delta\,\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t$, and

$\frac{a^2}{r^2}\int\frac{dR}{d\lambda}\,dt$. Examples of the determination of terms in the longitude arising from the second approximation. Further approximation to the value of c. Determination of the term in λ_1 multiplied by $m^2\frac{a^2}{a_{\prime}^2}$, and of the term in λ_5 multiplied by m^3, which depends on the cube of the disturbing force. Method of obtaining the inequalities due to the nonsphericity of the earth. Analytical proof of the theorems of Newton, with respect to the motion of the moon's node and the variation of the obliquity of her orbit . . . *page* 22 to 73

4. Expressions for the variation of the elliptic constants. Application of these expressions to the determination of certain inequalities and of certain terms in the expressions for c and g . . . *page* 73 to 84

5. Table of the arguments which occur in the planetary theory. Method of developing the disturbing function, depending on the binomial theorem. Expression for R. Another method of developing R, with examples of its application to various terms. Equations for determining the inequalities of the reciprocal of the radius vector; of the tangent of the latitude; and of the longitude. Comparison of some of the preceding results with those of Laplace in the *Mécanique Céleste*, vol. i. . . . . *page* 84 to 112

6. Numerical examples. Calculation of $r_{1,1}$, $\lambda_{1,1}$ in the theory of Jupiter and Saturn . . . . *page* 112 to 115

Table . . . . *page* (1 to 15)

ON

THE THEORY OF THE MOON,

AND ON

THE PERTURBATIONS OF THE PLANETS.

1. LET x, y, z denote the rectangular co-ordinates.

$$\left.\begin{array}{lll} r & \ldots\ldots\ldots & \text{distance from the sun} \\ \grave{r} & \ldots\ldots\ldots & \text{distance from the sun projected upon the plane } x\,y \\ {}^{*}\lambda & \ldots\ldots\ldots & \text{longitude reckoned upon the plane of its orbit} \\ \grave{\lambda} & \ldots\ldots\ldots & \text{longitude reckoned upon the plane } x\,y \\ s & \ldots\ldots\ldots & \text{tangent of the latitude} \\ m & \ldots\ldots & \text{the mass} \end{array}\right\}\text{of a planet.}$$

M …… the mass of the sun.

$$\sqrt{\frac{\mu}{a^3}} = n.$$

$$x = \grave{r}\cos\grave{\lambda}, \quad y = \grave{r}\sin\grave{\lambda}, \quad z = \grave{r}\,s, \quad r = \grave{r}\sqrt{(1 + s^2)}$$

$$R = m_{,}\left\{\frac{x x_{,} + y y_{,} + z z_{,}}{\{x_{,}^2 + y_{,}^2 + z_{,}^2\}^{\frac{3}{2}}} - \frac{1}{\{(x - x_{,})^2 + (y - y_{,})^2 + (z - z_{,})^2\}^{\frac{1}{2}}}\right\}$$

---

[*] Laplace uses the letter v to denote longitude, u the eccentric anomaly, and φ the inclination of the orbit to a fixed plane; but as v is very frequently used to signify velocity, and φ geographical latitude, and as the letters of the Greek alphabet are generally used for angles, I have taken the letters λ, υ, and ι for these quantities. The letter R is used throughout this treatise in the same acceptation as in the *Mécanique Céleste*. I have used the grave accent to denote projection.

B

$$R = m_{\prime}\left\{\frac{r'\{\cos(\lambda' - \lambda'_{\prime}) + ss_{\prime}\}}{r'^{2}_{\prime}(1+s_{\prime}^{2})^{\frac{3}{2}}}\right.$$

$$\left.-\frac{1}{\{r'^{2}(1+s^{2}) - 2\,r'r'_{\prime}\{\cos(\lambda' - \lambda'_{\prime}) + ss_{\prime}\} + r'^{2}_{\prime}(1+s_{\prime}^{2})\}^{\frac{1}{2}}}\right\}$$

$$= -m_{\prime}\left\{\begin{aligned}&\frac{1}{r'_{\prime}\sqrt{1+s_{\prime}^{2}}} + \frac{r'^{2}}{4\,r'^{3}_{\prime}}\{1 + 3\cos(2\lambda' - 2\lambda'_{\prime}) + 12\,ss_{\prime}\cos(\lambda' - \lambda'_{\prime}) - 2s^{2}\}\\ &\qquad + \frac{r'^{3}}{8\,r'^{4}_{\prime}}\{3(1 - 4s^{2})\cos(\lambda' - \lambda'_{\prime}) + 5\cos(3\lambda' - 3\lambda'_{\prime})\}\\ &\qquad + \frac{r'^{4}}{64\,r'^{5}_{\prime}}\{9 + 20\cos(2\lambda' - 2\lambda'_{\prime}) + 35\cos(4\lambda' - 4\lambda'_{\prime})\}\end{aligned}\right\}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) = m_{\prime}\left\{\frac{\cos(\lambda' - \lambda'_{\prime}) + ss_{\prime}}{r'^{2}_{\prime}(1+s_{\prime}^{2})^{\frac{3}{2}}}\right.$$

$$\left.+\frac{r'(1+s^{2}) - r'_{\prime}\{\cos(\lambda' - \lambda'_{\prime} + ss_{\prime}\}}{\{r'^{2}(1+s^{2}) - 2r'r'_{\prime}\{\cos(\lambda' - \lambda'_{\prime}) + ss_{\prime}\} + r'^{2}_{\prime}(1+s_{\prime}^{2})\}^{\frac{3}{2}}}\right\}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right) = -m_{\prime}\left\{\frac{r'\sin(\lambda' - \lambda'_{\prime})}{r'^{2}_{\prime}(1+s^{\prime 2})^{\frac{3}{2}}}\right.$$

$$\left.-\frac{r'r'_{\prime}\sin(\lambda' - \lambda'_{\prime})}{\{r'^{2}(1+s^{2}) - 2r'r'_{\prime}\{\cos(\lambda' - \lambda'_{\prime}) + ss_{\prime}\} + r'^{2}_{\prime}(1+s_{\prime}^{2})\}^{\frac{3}{2}}}\right\}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) = m_{\prime}\left\{\frac{r's_{\prime}}{r'^{2}_{\prime}(1+s_{\prime}^{2})^{\frac{3}{2}}}\right.$$

$$\left.+\frac{r'^{2}s - r'r'_{\prime}s_{\prime}}{\{r'^{2}(1+s^{2}) - 2r'r'_{\prime}\{\cos\lambda' - \lambda'_{\prime}) + ss_{\prime}\} + r'^{2}_{\prime}(1+s_{\prime}^{2})\}^{\frac{3}{2}}}\right\}$$

The following are the fundamental equations, which serve to determine all the circumstances of the motion of the body m.

$$\frac{\mathrm{d}^{2}x}{\mathrm{d}t^{2}} + \frac{\mu\,x}{r^{3}} + \left(\frac{\mathrm{d}R}{\mathrm{d}x}\right) = 0$$

$$\frac{\mathrm{d}^{2}y}{\mathrm{d}t^{2}} + \frac{\mu\,y}{r^{3}} + \left(\frac{\mathrm{d}R}{\mathrm{d}y}\right) = 0$$

$$\frac{\mathrm{d}^{2}z}{\mathrm{d}t^{2}} + \frac{\mu\,z}{r^{3}} + \left(\frac{\mathrm{d}R}{\mathrm{d}z}\right) = 0.$$

See *Mécanique Céleste*, vol. i. p. 254.

These equations admit of various well known transformations which it is necessary here to recapitulate, for the sake of future reference. Multiplying the first by x, the second by y, the third by z, and adding together the results,

$$\frac{x\,\mathrm{d}^2 x + y\,\mathrm{d}^2 y + z\,\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{\mu}{r} + x\left(\frac{\mathrm{d}R}{\mathrm{d}x}\right) + y\left(\frac{\mathrm{d}R}{\mathrm{d}y}\right) + z\left(\frac{\mathrm{d}R}{\mathrm{d}z}\right) = 0$$

Multiplying the same equations by $\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}$, adding together the results, and taking the integral of the sum,

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{\mathrm{d}t^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}R = 0,$$

d R denoting the differential of the disturbing function R with regard only to the coordinates of the body m.

Adding together the two last equations, the following is obtained:

$$\frac{\mathrm{d}^2 . r^2}{2\,\mathrm{d}t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}R + r\frac{\mathrm{d}R}{\mathrm{d}r} = 0^*.$$

I propose to show hereafter how, by means of this equation, the inequalities in the expression for the reciprocal of the radius vector may be obtained.

The equation

$$\frac{y\,\mathrm{d}^2 x - x\,\mathrm{d}^2 y}{\mathrm{d}t^2} + y\left(\frac{\mathrm{d}R}{\mathrm{d}x}\right) - x\left(\frac{\mathrm{d}R}{\mathrm{d}y}\right) = 0$$

gives
$$\frac{\mathrm{d} . r'^2\frac{\mathrm{d}\lambda'}{\mathrm{d}t}}{\mathrm{d}t} + \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right) = 0.$$

The integral of this equation is

$$\frac{r'^2\,\mathrm{d}\lambda'}{\mathrm{d}t} = h - \int\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}t,$$

h being a constant. I shall hereafter employ this equation in determining the inequalities of the longitude λ'.

The following equations may also be deduced from the equations of motion without difficulty.

$$\frac{r'^2\,\mathrm{d}\lambda'^2 + \mathrm{d}r'^2(1+s^2) + 2r's\,\mathrm{d}r'\,\mathrm{d}s + r'^2\,\mathrm{d}s^2}{\mathrm{d}t^2}$$

$$- \frac{2\mu}{r'(1+s^2)^{\frac{1}{2}}} + \frac{\mu}{a} + 2\int \mathrm{d}R = 0$$

$$\frac{\mathrm{d}r^2 + \frac{r^2}{1+s^2}\left(\frac{\mathrm{d}s^2}{1+s^2} + \mathrm{d}\lambda'^2\right)}{\mathrm{d}t^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}R = 0$$

---

[*] This equation is given in the *Mécanique Céleste*, vol. i. p. 255.

$$\frac{r' s\, \mathrm{d}^2 r' + 2 r' \mathrm{d} r' \mathrm{d} s + r'^2 \mathrm{d}^2 s}{\mathrm{d} t^2} + \frac{\mu s}{r'(1+s^2)^{\frac{3}{2}}} + \left(\frac{\mathrm{d} R}{\mathrm{d} s}\right) = 0$$

$$\frac{r' \mathrm{d}^2 r' - r'^2 \mathrm{d} \lambda'^2}{\mathrm{d} t^2} + \frac{\mu}{r'(1+s^2)^{\frac{3}{2}}} + r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) - s \left(\frac{\mathrm{d} R}{\mathrm{d} s}\right) = 0$$

$$\frac{r'^2 \mathrm{d}^2 s + 2 r' \mathrm{d} r' \mathrm{d} s - r'^2 s\, \mathrm{d} \lambda'^2}{\mathrm{d} t^2} + (1+s^2)\left(\frac{\mathrm{d} R}{\mathrm{d} s}\right) - r' s \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) = 0$$

$$\frac{\mathrm{d}^2 . r'^2 (1+s^2)}{2\, \mathrm{d} t} - \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}} + \frac{\mu}{a} + 2 \int \mathrm{d} R + r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) = 0.$$

Making λ' the independent variable instead of *t*,

$$r' \frac{\mathrm{d}^2 r'}{\mathrm{d} t^2} - r' \frac{\mathrm{d} r' \mathrm{d}^2 t}{\mathrm{d} t^3} - r'^2 \frac{\mathrm{d} \lambda'^2}{\mathrm{d} t^2} + \frac{\mu}{r'(1+s^2)^{\frac{3}{2}}} + r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) - s \left(\frac{\mathrm{d} R}{\mathrm{d} s}\right) = 0$$

$r'^4 \dfrac{\mathrm{d} \lambda'^2}{\mathrm{d} t^2} = h^2 - 2 \displaystyle\int r'^2 \left(\frac{\mathrm{d} R}{\mathrm{d} \lambda'}\right) \mathrm{d} \lambda'$, *h* being the same constant as before,

$$4 r'^3 \mathrm{d} r' \frac{\mathrm{d} \lambda'^2}{\mathrm{d} t^2} - 2 r'^4 \frac{\mathrm{d} \lambda'^2 \mathrm{d}^2 t}{\mathrm{d} t^3} = -2 r'^2 \left(\frac{\mathrm{d} R}{\mathrm{d} \lambda'}\right) \mathrm{d} \lambda'$$

$$\left\{\frac{\mathrm{d}^2 . \frac{1}{r'}}{\mathrm{d} \lambda'^2} + \frac{1}{r'}\right\} \left\{1 - \frac{2}{h^2} \int r'^2 \left(\frac{\mathrm{d} R}{\mathrm{d} \lambda'}\right) \mathrm{d} \lambda'\right\} - \frac{\mu}{h^2 (1+s^2)^{\frac{3}{2}}}$$

$$- \frac{r'}{h^2} \left\{ r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) - s \left(\frac{\mathrm{d} R}{\mathrm{d} s}\right) - \frac{1}{r'} \left(\frac{\mathrm{d} R}{\mathrm{d} \lambda'}\right) \frac{\mathrm{d} r'}{\mathrm{d} \lambda'} \right\} = 0$$

$$\left\{\frac{\mathrm{d}^2 s}{\mathrm{d} \lambda'^2} + s\right\} \left\{1 - \frac{2}{h^2} \int r'^2 \left(\frac{\mathrm{d} R}{\mathrm{d} \lambda'}\right) \mathrm{d} \lambda'\right\}$$

$$+ \frac{r'^2}{h^2} \left\{ (1+s^2) \left(\frac{\mathrm{d} R}{\mathrm{d} s}\right) - r' s \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) - \left(\frac{\mathrm{d} R}{\mathrm{d} \lambda'}\right) \frac{\mathrm{d} s}{\mathrm{d} \lambda'} \right\} = 0.$$

These equations are identical with the equations (L) of the *Mécanique Céleste*, vol. i. p. 151, and those of the *Mechanism of the Heavens*, p. 425, observing that

$$\frac{\mathrm{d} R}{\mathrm{d} r} = \frac{\mathrm{d} R}{\mathrm{d} u} \frac{\mathrm{d} u}{\mathrm{d} r'} = - \frac{\mathrm{d} R}{r'^2 \mathrm{d} u}, \qquad \frac{\mathrm{d} r}{\mathrm{d} \lambda'} = \frac{\mathrm{d} u}{\mathrm{d} \lambda'} \frac{\mathrm{d} r}{\mathrm{d} u} = - \frac{r^2 \mathrm{d} u}{\mathrm{d} \lambda'}.$$

In the latter work, the letter ***R*** represents the same quantity as in the *Mécanique Céleste*, but with a contrary sign.

If $\mathrm{d} t = \sqrt{\frac{a}{\mu}}\, r\, \mathrm{d} v$, and *v* be taken for the independent variable,

$$\frac{r\, \mathrm{d}^2 r}{\mathrm{d} t^2} = \frac{r\, \mathrm{d}^2 r}{\mathrm{d} t^2} - \frac{r\, \mathrm{d} r\, \mathrm{d}^2 t}{\mathrm{d} t^3}.$$

The equation

$$\frac{\mathrm{d}^2 r^2}{2\,\mathrm{d}t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}R + r\frac{\mathrm{d}R}{\mathrm{d}r} = 0$$

gives

$$\frac{\mathrm{d}^2 r}{\mathrm{d}v^2} - a + r + \frac{a\,r}{\mu}\left\{2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right\} = 0.$$

In the elliptic theory, that is when the disturbing force is neglected, the following equations obtain:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}v^2} - a + r = 0, \qquad v \text{ being the eccentric anomaly.}$$

$$\mathrm{d}.\frac{r'^2\,\mathrm{d}\lambda'}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}^2.\frac{1}{r'}}{\mathrm{d}\lambda'^2} + \frac{1}{r'} - \frac{\mu}{h^2(1+s^2)^{\frac{3}{2}}} = 0 \qquad\qquad \frac{\mathrm{d}^2 s}{\mathrm{d}\lambda'} + s = 0$$

of which the integrals are

$$r = a\,\{1 - e\cos(v - \alpha)\}$$

$$r'^2\,\mathrm{d}\lambda' = h\,\mathrm{d}t \qquad \frac{1}{r'} = \frac{\mu\cos\iota^2}{h^2}\{(1+s^2)^{\frac{1}{2}} + e'\cos(\lambda' - \varpi')\}$$

$$s = \tan\iota\sin(\lambda' - \nu)$$

$$\frac{1}{r} = \frac{1 + e\cos(\lambda - \varpi)}{a(1 - e^2)}.$$

In order to complete the elliptic theory, it is necessary to obtain the equations of condition which obtain between these constants, h, a, e, e', ϖ, ν, ϖ', &c.

Making $s = 0$; $\lambda' = \nu$, $r' = r$, $\frac{\mathrm{d}r'}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}$, and the equation of p. 3, line 20, gives

$$\frac{\mathrm{d}r^2}{\mathrm{d}t^2} + \frac{h^2}{r^2\cos^2\iota} - \frac{2\mu}{r} + \frac{\mu}{a} = 0$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}v}\,\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\frac{h^2}{\cos^2\iota} = a\,\mu\,\{(1 - e\cos v)\,(1 + e\cos v) - e^2\sin^2 v\}$$

$$= a\,\mu\,\{1 - e^2\}$$

$$a^3 r^{-3} = 1 + \frac{3}{2} e^2 \left(1 + \frac{5}{4} e^2\right) + 3 e \left(1 + \frac{9}{8} e^2\right) \cos \xi$$

$$+ \frac{9}{2} e^2 \left(1 + \frac{7}{9} e^2\right) \cos 2\xi + \frac{53}{8} e^3 \cos 3\xi + \frac{77}{8} e^4 \cos 4\xi$$

$$a^2 r^{-2} = 1 + \frac{e^2}{2} \left(1 + \frac{3}{4} e^2\right) + 2 e \left(1 + \frac{3}{8} e^2\right) \cos \xi$$

$$+ \frac{5}{2} e^2 \left(1 + \frac{2}{15} e^2\right) \cos 2\xi + \frac{13}{4} e^3 \cos 3\xi + \frac{103}{24} e^4 \cos 4\xi$$

$$a r^{-1} = 1 + e \left(1 - \frac{e^2}{8}\right) \cos \xi + e^2 \left(1 - \frac{e^2}{3}\right) \cos 2\xi$$

$$+ \frac{9}{8} e^3 \cos 3\xi + \frac{4}{3} e^4 \cos 4\xi$$

$$\frac{r^2}{a^2} = 1 + \frac{3 e^2}{2} - 2 e \left(1 - \frac{e^2}{8}\right) \cos \xi - \frac{e^2}{2} \left(1 - \frac{e^2}{3}\right) \cos 2\xi$$

$$- \frac{e^3}{4} \cos 3\xi - \frac{e^4}{6} \cos 4\xi$$

$$\frac{r^3}{a^3} = 1 + 3 e^2 \left(1 + \frac{e^2}{8}\right) - 3 e \left(1 + \frac{3}{8} e^2\right) \cos \xi$$

$$+ \frac{5}{8} e^4 \cos 2\xi + \frac{e^3}{8} \cos 3\xi + \frac{e^4}{8} \cos 4\xi.$$

$$\frac{r^4}{a^4} = 1 + 5 e^2 - 4 e \cos \xi + e^2 \cos 2\xi.$$

The constant or non-periodical portion of powers of r in terms of the mean longitude can easily be found in finite terms, as I shall now show in the case of r^2 and r^{-3}, which expressions will be useful hereafter in the Lunar theory in determining the constant or non-periodical portion of the disturbing function.

$$n\, d t = \frac{r}{a} d v \qquad\qquad n r^2 d t = \frac{r^3}{a} d v$$

$$r^3 = a^3 \{1 - 3 e \cos v + 3 e^2 \cos^2 v + e^3 \cos^3 v\}$$

$$= a^3 \left\{1 + \frac{3}{2} e^2 + \text{a series of cosines of multiples of } v\right\}$$

Let $r^2 = a^2 \{A + B \cos \xi + C \cos 2\xi + \&c.\}$

$$A + B \cos \xi + C \cos 2\xi + \&c.$$

$$= \left\{1 + \frac{3}{2}e^2 + \text{certain cosines of multiples of } \upsilon\right\} d\,\upsilon$$

$A\,\xi$ + a series of sines of multiples of ξ

$$= \left(1 = \frac{3\,e^2}{2}\right)\upsilon + \text{a certain series of sines of multiples of } \upsilon.$$

Making $\xi = 180°$ and $\upsilon = 180°$ which is allowable, I find

$$A = 1 + \frac{3}{2}e^2.$$

I now proceed to find the constant or non-periodical portion of r^{-3}.

$$\text{Since } r^2\,d\,\lambda = a^2\,n\,\sqrt{1 - e^2}\,d\,t$$

$$\frac{d\lambda}{r} = \frac{a^2 n \sqrt{1 - e^2}}{r^3}\,d\,t \qquad\qquad r = \frac{a\,(1 - e^2)}{1 + e\cos(\lambda - \varpi)}$$

$$\frac{\{1 + e\cos(\lambda - \varpi)\}\,d\,\lambda}{a\,(1 - e^2)} = \frac{a^2 n \sqrt{1 - e^2}}{r^3}\,d\,t$$

Integrating this equation after supposing

$$r^{-3} = a^{-3}\{A + B\cos\xi + C\cos 2\,\xi + \&c.\}$$

$$\lambda - \varpi + e\sin(\lambda - \varpi) = A\,\{1 - e^2\}^{\frac{3}{2}} + B\sin\xi + \&c.$$

Making $\xi = 180°$ and $\lambda - \varpi = 180°$ at the same time, which is allowable, I find

$$A = \frac{1}{(1 - e^2)^{\frac{3}{2}}} = 1 + \frac{3}{2}e^2\left(1 + \frac{5}{4}e^2\right) + \&c.$$

$$nt + \varepsilon = \lambda - 2e\sin(c\lambda - \varpi) + \frac{3}{4}e^2\sin(2c\lambda - 2\varpi) + \&c.$$

$$\lambda = nt + \varepsilon + 2e\left\{1 - \frac{e^2}{8}\right\}\sin\xi + \frac{5}{4}e^2\left\{1 - \frac{11}{30}e^2\right\}\sin 2\,\xi$$

$$+ \frac{13}{12}e^3\sin 3\,\xi + \frac{103}{96}e^4\sin 4\xi + \&c. \qquad \text{See } \textit{Méc. Cél.} \text{ tom. i. p. 181.}$$

$$\tan(\lambda' - \nu) = \cos\iota\,\tan(\lambda - \beta)$$

$$\lambda' = \lambda - \beta + \nu - \frac{\gamma^2}{4}\sin(2\,\lambda - 2\,\beta) + \&c.$$

Hence, by Lagrange's theorem, if $\eta = g\,n\,t + \varepsilon - \beta$, g being equal to unity in the elliptic theory,

$$\sin(2\,\lambda - 2\,\beta) = (1 - 4\,e^2)\sin 2\,\eta + 2\,e\sin(\xi - 2\,\eta) + 2\,e\sin(\xi + 2\,\eta)$$

$$-\frac{3}{4}e^2\sin(2\xi-2\eta)+\frac{13}{4}e^2\sin(2\xi+2$$

$$\lambda' = nt+\varepsilon-\beta+\nu+2e\left\{1-\frac{e^2}{8}\right\}\sin\xi$$

$$+\frac{5}{4}e^2\left\{1-\frac{11}{30}e^2\right\}\sin 2\xi+\frac{13}{12}e^3\sin 3\xi+\frac{103}{96}e^4\sin$$

$$-\frac{\gamma^2}{2}(1-4e^2)\sin 2\eta-\frac{e\gamma^2}{2}\sin(\xi-2\eta)-\frac{e\gamma^2}{2}\sin(\xi+$$

$$+\frac{3}{16}e^2\gamma^2\sin(2\xi-2\eta)-\frac{13}{16}e^2\gamma^2\sin(2\xi+2\eta)$$

$$s=\gamma(1-e^2)\sin\eta+\gamma e\sin(\xi-\eta)+\gamma e\sin(\xi+\eta)$$

$$+\frac{\gamma e^2}{8}\sin(2\xi-\eta)+\frac{9}{8}\gamma e^2\sin(2\xi+\eta)$$

$$\cos v=\left\{1-\frac{3}{8}e^2\right\}\cos(nt-\varpi)-\frac{e}{2}$$

$$+\frac{e}{2}\cos(2nt-2\varpi)+\frac{3}{8}e^2\cos(3nt-3\varpi)$$

$$\sin v=\left\{1-\frac{e^2}{8}\right\}\sin(nt-\varpi)$$

$$+\frac{e}{2}\sin(2nt-2\varpi)+\frac{3}{8}e^2\sin(3nt-3\varpi)$$

$$\cos(v+\varpi)=\cos v\cos\varpi-\sin v\sin\varpi$$

$$=\left\{1-\frac{e^2}{4}\right\}\cos nt-\frac{e}{2}\cos\varpi$$

$$+\frac{e}{2}\cos(2nt-\varpi)+\frac{3}{8}e^2\cos(3nt-2\varpi)$$

$$\sin(v+\varpi)=\sin v\cos\varpi+\cos v\sin\varpi$$

$$=\left\{1-\frac{e^2}{4}\right\}\sin nt-\frac{e}{2}\sin\varpi$$

$$+\frac{e}{2}\sin(2nt-\varpi)+\frac{3}{8}e^2\sin(3nt-2\varpi)$$

$$r\cos\lambda=r\cos(\lambda-\varpi)\cos\varpi-r\sin(\lambda-\varpi)\sin\varpi$$

$$=a\{(\cos v-e)\cos\varpi-(1-e^2)^{\frac{1}{2}}\sin v\sin\varpi\}$$

$$r \sin \lambda = r \sin (\lambda - \varpi) \cos \varpi + r \cos (\lambda - \varpi) \sin \varpi$$

$$= a \{(1 - e^2)^{\frac{1}{2}} \sin \upsilon \cos \varpi + (\cos \upsilon - e) \sin \varpi\}$$

$$\genfrac{}{}{0pt}{}{r \cos \lambda}{r \sin \lambda} = a \left\{ \left(1 - \frac{e^2}{4}\right) \genfrac{}{}{0pt}{}{\cos}{\sin} (\upsilon + \varpi) - e \genfrac{}{}{0pt}{}{\cos}{\sin} \varpi \pm \frac{e^2}{4} \genfrac{}{}{0pt}{}{\cos}{\sin} (\upsilon - \varpi) \right\}$$

Let $n t - n_{\prime} t$ be called τ[*]

$c n t - \varpi$ ………… ξ

$c_{\prime} n_{\prime} t - \varpi_{\prime}$ ………… $\xi_{\prime}$

The quantities c and g are equal to unity in the elliptic theory: the difference is of the order of the disturbing force, as will be shown hereafter.

$$r r_{\prime} \genfrac{}{}{0pt}{}{\cos}{\sin} (\lambda - \lambda_{\prime}) = a a_{\prime} \left\{ \left(1 - \frac{e^2}{2} - \frac{e_{\prime}^{2}}{2}\right) \genfrac{}{}{0pt}{}{\cos}{\sin} \tau \right.$$

$$- \frac{3}{2} e \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau - \xi) + \frac{e}{2} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau + \xi)$$

$$+ \frac{e_{\prime}}{2} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau - \xi_{\prime}) - \frac{3}{2} e_{\prime} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau + \xi_{\prime}) - \frac{e^2}{8} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau - 2 \xi)$$

$$+ \frac{3}{8} e^2 \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau + 2 \xi) - \frac{3}{4} e e_{\prime} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau - \xi - \xi_{\prime}) - \frac{3}{4} e e_{\prime} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau + \xi + \xi_{\prime})$$

$$+ \frac{9}{4} e e_{\prime} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau - \xi + \xi_{\prime}) + \frac{e e_{\prime}}{4} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau + \xi - \xi_{\prime}) + \frac{3}{8} e_{\prime}^{2} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau - 2 \xi_{\prime})$$

$$\left. + \frac{e_{\prime}^{2}}{8} \genfrac{}{}{0pt}{}{\cos}{\sin} (\tau + 2 \xi_{\prime}) \right\}$$

This expression will be useful hereafter in obtaining the development of R in the planetary theory.

2. I now revert to the equation

$$\frac{d^2 r}{d \upsilon^2} - a + r + \frac{a r}{\mu} \left\{ 2 \int d R + r \left(\frac{d R}{d r}\right) \right\} = 0.$$

When the disturbing force is neglected, the integral of this equation is $r = a \{1 - e \cos (\upsilon - \alpha)\}$, υ being the eccentric anomaly.

---

[*] In the Philosophical Transactions I have called these quantities t, x and z, in order to agree with the notation of M. Damoiseau. So much inconvenience, however, arises from using the letter t with two different significations, that I have reluctantly here adopted other letters.

$$nt + \epsilon - \varpi = v - \alpha - e\sin(v - \alpha)$$

$$\tan\frac{\lambda - \varpi}{2} = \left\{\frac{1+e}{1-e}\right\}^{\frac{1}{2}}\tan\frac{v-\alpha}{2}.$$

If Q be put for the quantity $\frac{ar}{\mu}\left\{2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right\}$ and the constant α, which may afterwards be replaced, be omitted for the present,

$$r = a\{1 - e\cos v\} - \sin v\int Q\cos v\,\mathrm{d}v + \cos v\int Q\sin v\,\mathrm{d}v$$

$$\mathrm{d}t = \sqrt{\frac{a}{\mu}}\left\{a\{1 - e\cos v\} - \sin v\int Q\cos v\,\mathrm{d}v\right.$$

$$\left. + \cos v\int Q\sin v\,\mathrm{d}v\right\}\mathrm{d}v$$

$$nt + \epsilon - \varpi = v - e\sin v - \frac{1}{a}\left\{\int Q\,\mathrm{d}v - \cos v\int Q\cos v\,\mathrm{d}v\right.$$

$$\left. - \sin v\int Q\sin v\,\mathrm{d}v\right\}$$

If $v = \mathrm{f}(nt + \epsilon - \varpi)$ in the elliptic theory, then neglecting the square of the disturbing force,

$$v = \mathrm{f}(nt + \epsilon - \varpi) + \frac{\mathrm{d.f}(nt + \epsilon - \varpi)}{\mathrm{d}t}\,\frac{1}{a}\left\{\int Q\,\mathrm{d}v\right..$$

$$\left. - \cos v\int Q\cos v\,\mathrm{d}v - \sin v\int Q\sin v\,\mathrm{d}v\right\}$$

If δv, δr denote the values of these parts of r and v which are due to the first power of the disturbing force,

$$\delta v = \frac{\mathrm{d}v}{an\,\mathrm{d}t}\left\{\int Q\,\mathrm{d}v - \cos v\int Q\cos v\,\mathrm{d}v - \sin v\int Q\sin v\,\mathrm{d}v\right\}$$

$$\delta r = ae\sin v\,\delta v - \sin v\int Q\cos v\,\mathrm{d}v + \cos v\int Q\sin v\,\mathrm{d}v$$

$$= \frac{\sin v}{1 - e\cos v}\int Q\{e - \cos v\}\,\mathrm{d}v - \frac{e - \cos v}{1 - e\cos v}\int Q\sin v\,\mathrm{d}v.$$

In the elliptic theory,

$$\mathrm{d}v = \frac{n\,\mathrm{d}t}{1 - e\cos v}, \quad \frac{\cos v - e}{1 - e\cos v} = \cos\lambda, \quad \frac{\sin v\,(1 - e^2)^{\frac{1}{2}}}{1 - e\cos v} = \sin\lambda$$

Therefore,

$$\delta r = \frac{\begin{array}{l} a\,n \cos\lambda \int r \sin\lambda \left\{ 2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right\}\mathrm{d}t \\ - a\,n\sin\lambda \int r\cos\lambda\left\{2\int\mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right\}\mathrm{d}t \end{array}}{\mu\,(1-e^2)^{\frac{1}{2}}}$$

which is the equation X. of the *Mécanique Céleste*, tom. i. p. 258.

Multiplying the equation of p. 3, l. 14, by $\frac{2r'^2\,\mathrm{d}\lambda'}{\mathrm{d}t}$, and integrating, $\frac{r'^4\,\mathrm{d}\lambda'^2}{\mathrm{d}t^2} + 2\int r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda' = h_0^2$.

If $h^2 = h_0^2 - 2\int r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'$, h being variable, and h_0 the value of h at a given epoch, $h\,\mathrm{d}h = -r'^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'$, $\quad r^2\,\mathrm{d}\lambda' = h\,\mathrm{d}t$, and making λ' the independent variable instead of t,

$$2r'\,\mathrm{d}r'\,\mathrm{d}\lambda' = \mathrm{d}h\,\mathrm{d}t + h\,\mathrm{d}^2t$$

$$r'\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - \frac{r'\,\mathrm{d}r'\,\mathrm{d}^2t}{\mathrm{d}t^3} - \frac{r'^2\,\mathrm{d}\lambda'^2}{\mathrm{d}t^2} + \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}} + r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) = 0$$

$$r'\left(\frac{\mathrm{d}^2r'}{\mathrm{d}t^2}\right) + \frac{r'\,\mathrm{d}r'\,\mathrm{d}h}{h\,\mathrm{d}t^2} - \frac{2r'^2\,\mathrm{d}r'^2\,\mathrm{d}\lambda}{h\,\mathrm{d}t^3} - \frac{r'^2\,\mathrm{d}\lambda'^2}{\mathrm{d}t^2} + \frac{\mu}{r'(1+s^2)^{\frac{1}{2}}}$$
$$+ r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) = 0$$

$$\frac{\mathrm{d}^2.\frac{1}{r'}}{\mathrm{d}\lambda'^2} + \frac{1}{r'} - \frac{\mu}{h^2(1+s^2)^{\frac{3}{2}}} - \frac{r'}{h^2}\left\{r'\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)\right.$$
$$\left. - \frac{1}{r'}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\frac{\mathrm{d}r'}{\mathrm{d}\lambda'}\right\} = 0$$

$$\frac{\mathrm{d}^2s}{\mathrm{d}\lambda'^2} + s + \frac{r'^2}{h^2}\left\{(1+s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) - r's\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right\} = 0$$

In order to obtain the *variations* of the elliptic constants, these constants may be supposed to change, subject to the condition that they still satisfy these differential equations, and that the form of the first differential coefficients $\frac{\mathrm{d}r}{\mathrm{d}\lambda'}$, $\frac{\mathrm{d}s}{\mathrm{d}\lambda'}$, &c. remains unaltered.

So, for example, since $s = \tan\iota\,\sin(\lambda' - \nu)$

$$\sin(\lambda' - \nu)\frac{\mathrm{d}\,\iota}{\cos\iota^2} - \tan\iota\cos(\lambda' - \nu)\,\mathrm{d}\,\nu = 0$$

$$\cos(\lambda' - \nu)\frac{\mathrm{d}\,\iota}{\cos\iota^2} + \tan\iota\sin(\lambda' - \nu)\,\mathrm{d}\,\nu + \frac{r'^2}{h^2}\left\{(1 + s^2)\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right) - r's\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r'}\right) - \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\right)\left(\frac{\mathrm{d}\,s}{\mathrm{d}\,\lambda'}\right)\right\}\mathrm{d}\,\lambda' = 0$$

whence by elimination,

$$\mathrm{d}\,\nu + \frac{r'^2\sin(\lambda' - \nu)}{h^2\tan\iota}\left\{(1 + s^2)\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right) - r's\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r'}\right) - \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\right)\left(\frac{\mathrm{d}\,s}{\mathrm{d}\,\lambda'}\right)\right\}\mathrm{d}\,\lambda' = 0$$

$$\mathrm{d}\,\iota + \frac{r^2\cos\iota^2\cos(\lambda' - \nu)}{h^2}\left\{(1 + s^2)\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right) - r's\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r'}\right) - \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\right)\left(\frac{\mathrm{d}\,s}{\mathrm{d}\,\lambda'}\right)\right\}\mathrm{d}\,\lambda' = 0$$

The theorems of Newton with respect to the motion of the moon's node and the change of the inclination of her orbit, may be deduced without difficulty from these expressions, as will be shown hereafter. The general expressions for $\mathrm{d}\,e'$ and $\mathrm{d}\,\varpi'$ may be obtained in a similar manner, but are more complicated.

In order to obtain the expression for $\mathrm{d}\,e$, I revert to the equation

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\,v^2} - a + r + \frac{a\,r}{\mu}\left\{2\int\mathrm{d}\,R + r\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r}\right)\right\} = 0$$

The integral of this equation in the elliptic theory is

$$r = a\,\{1 - e\cos(v - \alpha)\}$$

If the constants in this elliptic integral are supposed to vary, subject to the condition that they still satisfy this differential equation, and that the form of the first differential coefficient $\frac{\mathrm{d}\,r}{\mathrm{d}\,v}$ remains unaltered,

$$\mathrm{d}^2 t = \frac{r\,\mathrm{d}\,v\,\mathrm{d}\,a}{2\sqrt{a\,\mu}} + \sqrt{\frac{a}{\mu}}\,\mathrm{d}\,r\,\mathrm{d}\,v, \qquad \frac{r\,\mathrm{d}^2 r}{\mathrm{d}\,t^2} = \frac{r\,\mathrm{d}\,r^2}{\mathrm{d}\,t^2} - \frac{r\,\mathrm{d}\,r\,\mathrm{d}^2 t}{\mathrm{d}\,t^3}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\,v^2} - a + r + \frac{a\,r^2}{\mu}\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r}\right) - \frac{\mathrm{d}\,r\,\mathrm{d}\,a}{2\,a\,\mathrm{d}\,v^2} = 0$$

$$\big(1 - e\cos(v - \alpha)\big)\,\mathrm{d}\,a - a\cos(v - \alpha)\,\mathrm{d}\,e - a\,e\sin(v - \alpha)\,\mathrm{d}\,\alpha = 0$$

$$e\sin(v-\alpha)\,\mathrm{d}\,a+a\sin(v-\alpha)\,\mathrm{d}\,e-a\,e\cos(v-\alpha)\,\mathrm{d}\,\alpha$$

$$+\frac{a\,r^2}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}\,v+\frac{a^2\,e}{\mu}\sin(v-\alpha)\,\mathrm{d}\,R=0$$

$$\mathrm{d}a=-\frac{2\,a^2}{\mu}\mathrm{d}\,R$$

$$\mathrm{d}\,e=\frac{a}{\mu}\{2e-2\cos(v-\alpha)-e\sin(v-\alpha)^2\}\,\mathrm{d}\,R$$

$$-\frac{r^2}{\mu}\sin(v-\alpha)\cdot\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}\,v$$

$$e\,d\,\alpha=\frac{a}{\mu}\sin(v-\alpha)\{e\cos(v-\alpha)-2\}\,\mathrm{d}\,R$$

$$+\frac{r^2}{\mu}\cos(v-\alpha)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}\,v$$

$$\int n\,\mathrm{d}\,t+\epsilon-\varpi=v-\alpha-e\sin(v-\alpha)$$

$$\mathrm{d}\epsilon-\mathrm{d}\,\varpi=-\mathrm{d}\,\alpha-\sin(v-\alpha)\,\mathrm{d}\,e+e\cos(v-\alpha)\,\mathrm{d}\,\alpha.$$

The equations of p. 13. are susceptible of simplification when the square and higher powers of the disturbing function are neglected. In this case, if the orbit be supposed to coincide with the plane $x\,y$, $\tan\iota=0$,

$$\frac{1}{r}=\frac{\mu}{h^2}\{1+e\cos(\lambda-\varpi)\}$$

$$\frac{\mathrm{d}^2.\frac{1}{r}}{\mathrm{d}\lambda^2}+\frac{1}{r}-\frac{\mu}{h^2}-\frac{r}{h^2}\left\{r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\frac{1}{r}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right\}=0$$

$$-2\{1+e\cos(\lambda-\varpi)\}\,h\,\mathrm{d}\,h+h^2\cos(\lambda-\varpi)\,\mathrm{d}\,e$$

$$+h^2\,e\sin(\lambda-\varpi)\,\mathrm{d}\,\varpi=0$$

$$2\,e\sin(\lambda-\varpi)\,h\,\mathrm{d}\,h-h^2\sin(\lambda-\varpi)\,\mathrm{d}\,e+h^2\,e\cos(\lambda-\varpi)\,\mathrm{d}\,\varpi$$

$$-h^2\,r\left\{r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\frac{1}{r}\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right\}\mathrm{d}\,\lambda=0$$

$$\mathrm{d}\,e+\{2\cos(\lambda-\varpi)+e+e\cos^2(\lambda-\varpi)\}\,r^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\mathrm{d}\,\lambda$$

$$+h^2\,r^2\,\frac{\sin(\lambda-\varpi)}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}\,\lambda=0$$

$$e\,\mathrm{d}\varpi + \{2 + e\cos(\lambda - \varpi)\}\sin(\lambda - \varpi)\,r^2\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda$$

$$- h^2 r^2 \cos(\lambda - \varpi)\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}\lambda = 0$$

In order to give a very simple application of these formulæ, sup

$$R = \frac{m_{\prime}}{r}, \quad \text{then} \frac{\mathrm{d}R}{\mathrm{d}r} = -\frac{m_{\prime}}{r^2}, \quad \frac{\mathrm{d}R}{\mathrm{d}\lambda} = 0, \quad \mathrm{d}h = 0$$

$$\mathrm{d}e - \frac{m_{\prime}}{\mu}\sin(\lambda - \varpi)\,\mathrm{d}\lambda = 0$$

$$e\,\mathrm{d}\varpi + \frac{m_{\prime}}{\mu}\cos(\lambda - \varpi)\,\mathrm{d}\lambda = 0$$

$$\sin\varpi\,\mathrm{d}e + e\cos\varpi\,\mathrm{d}\varpi + \frac{m_{\prime}}{\mu}\cos\lambda\,\mathrm{d}\lambda = 0$$

$$\cos\varpi\,\mathrm{d}e - e\sin\varpi\,\mathrm{d}\varpi - \frac{m_{\prime}}{\mu}\sin\lambda\,\mathrm{d}\lambda = 0$$

Integrating

$$e\sin\varpi = -\frac{m_{\prime}}{\mu}\sin\lambda + \text{constant}$$

$$e\cos\varpi = -\frac{m_{\prime}}{\mu}\cos\lambda + \text{constant}$$

$$\frac{1}{r} = \frac{\mu}{h^2}\{1 + e\cos(\lambda - \varpi)\}$$

$$= \frac{\mu}{h^2}\{1 + e(\cos\lambda\cos\varpi + \sin\lambda\sin\varpi)\}$$

if $\varpi = 0$ when $\lambda = 0$,

$$e\sin\varpi = -\frac{m_{\prime}}{\mu}\sin\lambda, \qquad e\cos\varpi = -\frac{m}{\mu}\cos\lambda + (e) + \frac{m_{\prime}}{\mu}$$

(e) being a constant.

$$\frac{1}{r} = \frac{\mu}{h^2}\left\{1 - \frac{m_{\prime}}{\mu} + \left\{(e) + \frac{m_{\prime}}{\mu}\right\}\cos\lambda\right\}$$

which agrees with the known form of the result.

When $\iota = 0$, as before, by the equation p. 5, at foot, $\mu\,a\,(1 -$
$= h^2$; also $e = e'$.

$$\mathrm{d}e = \frac{a}{\mu}\{2e - 2\cos(\upsilon - \alpha) - e\sin(\upsilon - \alpha)^2\}\,\mathrm{d}R$$

$$- \frac{r^2}{\mu} \sin v - \alpha) \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, r}\right) \mathrm{d}\, v$$

$$e\, \mathrm{d}\, a = \frac{a}{\mu} \left\{ e \cos (v - \alpha) - 2 \right\} \sin (v - \alpha)\, \mathrm{d}\, R$$

$$+ \frac{r^2}{\mu} \cos v - \alpha) \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, r}\right) \mathrm{d}\, v$$

$$\int n\, \mathrm{d}\, t + \varepsilon - \varpi = v - \alpha - e \sin (v - \alpha)$$

$$\mathrm{d}\, \varepsilon - \mathrm{d}\, \varpi = - \{ 1 - e \cos (v - \alpha \}\, \mathrm{d}\, \alpha - \sin (v - \alpha) \mathrm{d}\, e$$

$$\mathrm{d}\, \varepsilon - \mathrm{d}\, \varpi = - \frac{r^2}{a^2 \sqrt{1 - e^2}} \mathrm{d}\, \varpi - \left\{ \frac{r}{a\, (1 - c^2)} + 1 \right\} \sin (v - \alpha)\, \mathrm{d}\, e$$

$$\frac{\frac{h^2}{\mu}}{1 + e \cos (\lambda - \varpi)} = a \{ 1 - e \cos (v - \alpha) \}$$

$$\mathrm{d}\, e = - \frac{a\, n\, \mathrm{d}\, t}{\mu \sqrt{1 - e^2}} \{ 2 \cos (\lambda - \varpi) + e + e \cos (\lambda - \varpi)^2 \} \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, \lambda}\right)$$

$$- \frac{a^2 n\, \mathrm{d}\, t \sqrt{1 - e^2}}{\mu} \sin (\lambda - \varpi) \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, r}\right)$$

$$e\, \mathrm{d}\, \varpi = - \frac{a\, n\, \mathrm{d}\, t}{\mu \sqrt{1 - e^2}} \{ 2 + e \cos (\lambda - \varpi) \} \sin (\lambda - \varpi) \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, \lambda}\right)$$

$$+ \frac{a^2 n\, \mathrm{d}\, t \sqrt{1 - e^2}}{\mu} \cos (\lambda - \varpi) \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, r}\right)$$

The two last equations, obtained indirectly, are given in the *Mécanique Céleste*, vol. i. p. 347, and in the *Mechanism of the Heavens*, p. 226.

$$\mathrm{d}\, \varepsilon - \mathrm{d}\, \varpi = - \sqrt{1 - e^2}\, \mathrm{d}\, \varpi + \frac{2\, a^2 n\, (1 - e^2)\, \mathrm{d}\, t}{\mu \{ 1 + e \cos (\lambda - \varpi) \}} \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, r}\right)$$

The last equations serve to determine by the method of quadratures the peturbations of a comet.

Let $(\Delta\, e)_n$ be the variation of any element e during the variation $\Delta\, v$ of v at any given epoch n, then neglecting the square and higher powers of $\Delta\, v$,

$$(\Delta\, e)_n = \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_n \Delta\, v$$

If the values of $(\Delta\, e)_n$ be calculated for the epochs 0, 1, 2 . . . m corresponding to the values v, $v + i\, \Delta\, v$, $v + 2\, i\, \Delta\, v$, &c., differing from each other by $i \, \Delta \, v$, then the whole variation ($\delta \, e$) of e corresponding to the variation $i \, \Delta \, v$ of v

$$= i \, \{(\Delta \, e)_0 + (\Delta \, e)_1 \, \ldots\ldots\ldots + (\Delta \, e)_{m-1}\}$$

$$+ \frac{i-1}{2} \{(\Delta \, e)_m - (\Delta \, e)_0\}$$

$$- \frac{(i-1)\,(i+1)}{12\, i} \{(\Delta^2 e)_m - (\Delta^2 e)_0\} + \&c.$$

$$= i \left\{ \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_0 + \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_1 \ldots\ldots\ldots + \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_{m-1} \right\} \Delta \, v$$

$$+ \frac{i-1}{2} \left\{ \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_m - \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_0 \right\} \Delta \, v$$

$$- \frac{(i-1)\,(i+1)}{12\, i} \left\{ \Delta \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_m - \Delta \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_0 \right\} \Delta \, v.$$

When the interval $\Delta \, v$ is indefinitely diminished, $i \, \Delta \, v$ is still equal to the variation of v between the epochs for which the quantities $\left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_0$, $\left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_1$ &c. are calculated, and

$$\delta \, e = i \, \Delta \, v \left\{ \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_0 + \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_1 \ldots\ldots + \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_{m-1} \right.$$

$$\left. + \frac{1}{2} \left\{ \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_m - \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_0 \right\} - \frac{1}{12} \left\{ \Delta \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_m - \Delta \left(\frac{\mathrm{d}\, e}{\mathrm{d}\, v}\right)_0 \right\} + \&c. \right\}$$

If the radius be taken for unity, and $i \, \Delta \, v$ is the mth part of the circumference; $i \, \Delta \, v = \frac{2 \times 3{\cdot}14159}{m}$, or, in other words, the resulting values of $\delta \, e$ and $\delta \, a$ in the equations given above must be multiplied by $2 \times 3{\cdot}14159$, and divided by 360° expressed in the same unit as $i \, \Delta \, v$.

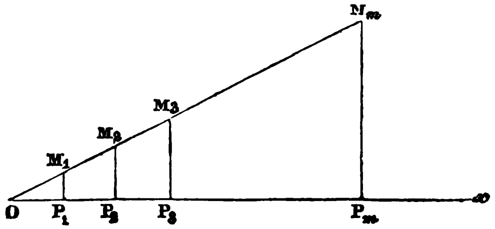
n is equal to the angular circumference divided by the periodic time expressed in the same unit as t; so that if a degree be taken as the unity of angular circumference, $n = 360°$ divided by the periodic time expressed in the same unit as t.

The preceding series, which is to be found in the *Mécanique Céleste*, vol. iv. p. 206, is substantially identical with that given in the *Traité elémentaire de calcul différentiel et de calcul intégral* of Lacroix, p. 578,

$$\int y \, \mathrm{d}\, x = h \, \Sigma \, y + \frac{h}{2} \, y - \frac{h^2}{12} \frac{\mathrm{d}\, y}{\mathrm{d}\, x} + \&c.$$

In order to take a very simple application of this series, let $y = a \, x$,

$P_1 P_2 = h = 1$, and suppose it were required to determine *by quadratures* the area of the triangle $O\,P_m\,M_m = \frac{a\,x^2}{2}$.



According to the preceding expression

$$\int y\,\mathrm{d}\,x = P_1\,M_1 + P_2\,M_2 \,\ldots\ldots\ldots\, P_{m-1}\,M_{m-1}$$

$$+ \frac{P_m\,M_m}{2} - \frac{a}{12} + \text{constant}$$

constant $= \frac{a}{12}$ and

$$\int y\,\mathrm{d}\,x = P_1\,M_1 + P_2\,M_2 \,\ldots\ldots\ldots + P_{m-1}\,M_{m-1}$$

$$+ \frac{P_m\,M_m}{2}$$

$$= a\,\{x_1 + x_2 \,\ldots\ldots\ldots + x_{m-1}$$

$$+ \frac{x_m}{2}$$

Suppose $O\,P_m = x_m = 4$, then

$$\int y\,\mathrm{d}\,x = a\left\{1 + 2 + 3 + \frac{4}{2}\right\}$$

$$= 8\,a = \frac{a\,x^2}{2}$$

In the same manner if $y = a\,x^2$, after determining the constant,

$$\int y\,\mathrm{d}\,x = \Sigma y + \frac{y}{2} - \frac{a\,x}{6}$$

and between the same limits as before

$$\int y\,\mathrm{d}\,x = a\left\{1 + 2^2 + 3^3 + \frac{4^2}{2} - \frac{4}{6}\right\}$$

$$= \frac{64\,a}{3} = \frac{a\,x^3}{3}$$

The expression

$$h\,\Sigma y = \int y\,\mathrm{d}x + h\,A y + h^2\,B\frac{\mathrm{d}y}{\mathrm{d}x} + h^3\,C\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \&\text{c.}$$

is of very extensive application. The quantities A, B, C, &c., may be determined directly from the equation itself, by taking $y = x^n$ and making $x = 1 = h$ after integration, because then the term Σy vanishes.

By making $y = x$, $y = x^2$, $y = x^3$, &c. successively, we get

$$\frac{1}{2} + A = 0$$

$$\frac{1}{3} + A + 2B = 0$$

$$\frac{1}{4} + A + 3B + 3.2C = 0$$

$$\frac{1}{5} + A + 4B + 4.3C + 4.3.2D = 0$$

whence $A = -\frac{1}{2}$, $B = \frac{1}{12}$, $C = 0$, $D = -\frac{1}{720}$

The preceding methods are used to determine by quadratures the perturbations of a comet. When, however, the perturbations are required through only a short interval, as during the time of its *apparition*, then the equations which I have given (*On the determination of the distance of a comet from the earth*, p. 40,) present advantages. By means of these equations the peturbations of the rectangular coordinates, δx, δy, δz, and $\delta\frac{\mathrm{d}x}{\mathrm{d}t}$, $\delta\frac{\mathrm{d}y}{\mathrm{d}t}$, and $\delta\frac{\mathrm{d}z}{\mathrm{d}t}$ are obtained; and from these the changes in the elements of the osculating ellipse may be easily found, if required.

In the elliptic movement or first approximation

$\lambda = nt + \varepsilon$ + a series of sines of arcs multiples of nt, &c.

$\lambda_{\prime} = n_{\prime}t + \varepsilon_{\prime}$ + a series of sines of arcs multiples of $n_{\prime}t$, &c.

$\frac{a}{r}$ = constant + a series of cosines.

s = a series of sines.

$s_{\prime}$ = a series of sines.

These values being substituted in the proper equations give $\frac{\mathrm{d}e}{\mathrm{d}\lambda}$, $\frac{\mathrm{d}h}{\mathrm{d}\lambda}$ and $\frac{\mathrm{d}\iota}{\mathrm{d}\lambda}$ each equal to a series of sines, and $\frac{\mathrm{d}\varpi}{\mathrm{d}\lambda}$, $\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}$ and $\frac{\mathrm{d}\varepsilon}{\mathrm{d}\lambda}$ each equal to a series of cosines + a constant quantity.

If the sun or primary be a spheroid, ω the angle which the plane of the sun's equator makes with the plane of the orbit of the planet; and if the longitude be reckoned from the line of intersection of the sun's equator with the orbit of the planet; R is increased by the quantity $c\left\{\frac{3\sin^2\omega\sin^2\lambda - 1}{r^3}\right\}$, c being a constant dependent upon the figure of the sun; but the partial differential coefficients of this quantity, which are introduced into the values of $\mathrm{d}\,e$, $\mathrm{d}\,\varpi$, &c., do not change the form of the expressions for those quantities.

If the planet move in a medium which resists according to any power q of the velocity, if ρ be a constant and v the velocity, the term

$2\rho\int v^{q+1}\,\mathrm{d}\,t$ must be added to $2\int \mathrm{d}\,R$,

$$\rho v^{q-1}\left\{(1+s^2)\frac{\mathrm{d}\,r'}{\mathrm{d}\,t}+r's\frac{\mathrm{d}\,s}{\mathrm{d}\,t}\right\} \text{ to } \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r'}\right),$$

$$\rho v^{q+1}r'^2\frac{\mathrm{d}\,\lambda'}{\mathrm{d}\,t} \text{ to } \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\right), \text{ and } \rho v^{q-1}r'\frac{\mathrm{d}\,.\,r's}{\mathrm{d}\,t} \text{ to } \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right)$$

in the equations of p. 4.

If the orbit of the planet be supposed to coincide with the plane xy so that $s = 0$,

$$v^2 = \frac{\mu\,a\,(1+e\cos v)}{1-e\cos v}$$

and the following equations result after many reductions which may be compared with those given by M. Poisson, *Traité de Mécanique*, vol. i. p. 443, by making $q = 2$.

$$= -2\rho\,a\left(\frac{\mu}{a}\right)^{\frac{q-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{q+1}{2}}\frac{(1-e\cos v)}{n}\,\mathrm{d}\,v$$

$$= -2\rho\left(\frac{\mu}{a}\right)^{\frac{q-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{q-1}{2}}\left(\frac{1-e^2}{n}\right)\cos v\,\mathrm{d}\,v$$

$$= -2\rho\left(\frac{\mu}{a}\right)^{\frac{q-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{q-1}{2}}\frac{\sqrt{1-e^2}}{n}\sin v\,\mathrm{d}\,v$$

$$= 2\rho\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{q-1}{2}}\frac{\{1-e^3\cos v-\sqrt{1-e^2}\}}{n\,e}\sin v\,\mathrm{d}\,v$$

The form of these equations differs from that which obtained before, now the variations of ϖ and ε are periodical, while those of a and e have terms which vary with the time, and as the periods of the periodic inequalities of all the elliptic constants due to the action of the resisting medium are fractional parts of the periodic time of the planet, the longitude of the perihelion is the same after every revolution.

Making $q = 2$, retaining only the first term and integrating, I get

$$\delta a = -2\rho a^2 v \qquad \delta e = -2\rho a \sin v$$
$$e\delta\varpi = 2\rho a \cos v \qquad \delta\varepsilon = -\rho a e \cos v$$

which results agree with those of M. Poisson, p. 444, except the last.

3. The preceding equations, which obtain in the planetary theory, are also applicable to the theory of the moon, provided

The mass of the sun . . . . $= m_{,}$
The mass of the earth + the mass of the moon . . $= \mu$
The distance of the moon from the earth . . . . . $= r$
The distance of the sun from the earth . . . . $= r_{,}$
The longitude of the moon . . . . $= \lambda'$
The longitude of the sun . . . . $= \lambda_{,}$
The tangent of the moon's latitude . . . . $= s$

I shall now proceed to consider particularly the theory of the moon's motion round the earth.

I show first how the development of R may be obtained in terms of the mean longitudes of the sun and moon, and I give this development as far as quantities of the order of the fourth powers of the eccentricities. The developments of $\mathrm{d}R$, $\frac{r\,\mathrm{d}R}{\mathrm{d}r}$ and $\frac{\mathrm{d}R}{\mathrm{d}\lambda}$ are very easily deducible from that of R, in the first approximation, and do not require separate consideration. The method of developing the disturbing function here given has the advantage of detecting numerous equations of condition which exist between the coefficients of the different terms, all of which may of course be employed in verifying the result. Whether the differential equations be treated by the methods proposed in this work, or whether the method of the variation of the elliptic constants be adopted as suggested by M. Poisson, this development of the disturbing function is equally useful, being the foundation of either method.

I next give examples of the determination of terms in the reciprocal of the radius vector due to the first approximation, by means of the equation

$$\frac{\mathrm{d}^2 . r^2}{2\,\mathrm{d}t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}R + \frac{r\,\mathrm{d}R}{\mathrm{d}r} = 0$$

and of terms in the longitude due to the first approximation by means of the equation

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t} = \frac{h}{r'^2} - \frac{1}{r'^2}\int \frac{\mathrm{d}R}{\mathrm{d}\lambda'}\,\mathrm{d}t$$

The inequalities of the latitude may be obtained in a similar manner by means of the equation

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} + \left(\frac{d R}{d z}\right) = 0$$

Here, however, it seems to me better to adhere to the method of Clairaut, and to employ the equation

$$\left\{\frac{d^2 s}{d \lambda'^2} + s\right\} \left\{1 - \frac{2}{h^2}\int r'^3 \left(\frac{d R}{d \lambda'}\right) d \lambda'\right\}$$
$$+ \frac{r^2}{h^2}\left\{(1 + s^2)\left(\frac{d R}{d s}\right) - r' s\left(\frac{d R}{d r'}\right) - \left(\frac{d R}{d \lambda'}\right)\frac{d s}{d \lambda'}\right\} = 0$$

Having obtained the terms in the reciprocal of the radius vector, in the longitude, and in the latitude of the moon which are given by the first approximation, the developments of $d R$, $\frac{r\,d R}{d r}$, &c. may be carried further, and terms depending upon higher powers of m may be successively obtained, m being the ratio of the sun's mean motion to that of the moon.

In future I shall distinguish the arguments by the numerical indices which are contained in the first column of the following Table. The numbers in the second column are the indices of the arguments in M. Damoiseau's *Mémoire sur la théorie de la lune.*

0	...	0	23	21	$2\xi+\xi_{,}$	46	...	$2\tau+3\xi-\xi_{,}$
1	30	2τ *	24	53	$2\tau-2\xi-\xi_{,}$	47	...	$2\xi+2\xi_{,}$
2	1	ξ	25	54	$2\tau+2\xi+\xi_{,}$	48	75	$2\tau-2\xi-2\xi_{,}$
3	31	$2\tau-\xi$ †	26	20	$2\xi-\xi_{,}$	49	...	$2\tau+2\xi+2\xi_{,}$
4	32	$2\tau+\xi$	27	51	$2\tau-2\xi+\xi_{,}$	50	...	$2\xi-2\xi_{,}$
5	16	$\xi_{,}$ ‡	28	52	$2\tau+2\xi-\xi_{,}$	51	...	$2\tau-2\xi+2\xi_{,}$
6	33	$2\tau-\xi_{,}$	29	23	$\xi+2\xi_{,}$	52	...	$2\tau+2\xi-2\xi_{,}$
7	34	$2\tau+\xi_{,}$	30	59	$2\tau-\xi-2\xi_{,}$	53	...	$\xi+3\xi_{,}$
8	2	2ξ	31	...	$2\tau+\xi+2\xi_{,}$	54	...	$2\tau-\xi-3\xi_{,}$
9	35	$2\tau-2\xi$	32	22	$\xi-2\xi_{,}$	55	...	$2\tau+\xi+3\xi_{,}$
10	36	$2\tau+2\xi$	33	61	$2\tau-\xi+2\xi_{,}$	56	...	$\xi-3\xi_{,}$
11	19	$\xi+\xi_{,}$	34	60	$2\tau+\xi-2\xi_{,}$	57	...	$2\tau-\xi+3\xi_{,}$
12	41	$2\tau-\xi-\xi_{,}$	35	...	$3\xi_{,}$	58	...	$2\tau+\xi-3\xi_{,}$
13	42	$2\tau+\xi+\xi_{,}$	36	...	$2\tau-3\xi_{,}$	59	...	$4\xi_{,}$
14	18	$\xi-\xi_{,}$	37	...	$2\tau+3\xi_{,}$	60	...	$2\tau-4\xi_{,}$
15	39	$2\tau-\xi+\xi_{,}$	38	9	4ξ	61	...	$2\tau+4\xi_{,}$
16	40	$2\tau+\xi-\xi_{,}$	39	67	$2\tau-4\xi$	62	3	2η
17	17	$2\xi_{,}$	40	...	$2\tau+4\xi$	63	37	$2\tau-2\eta$
18	43	$2\tau-2\xi_{,}$	41	27	$3\xi+\xi_{,}$	64	38	$2\tau+2\eta$
19	44	$2\tau+2\xi_{,}$	42	73	$2\tau-3\xi-\xi_{,}$	65	5	$\xi-2\eta$
20	4	3ξ	43	...	$2\tau+3\xi+\xi_{,}$	66	6	$\xi+2\eta$
21	45	$2\tau-3\xi$	44	26	$3\xi-\xi_{,}$	67	49	$2\tau-\xi-2\eta$
22	46	$2\tau+3\xi$	45	...	$2\tau-3\xi+\xi_{,}$	68	47	$2\tau-\xi+2\eta$

* Variation. † Evection. ‡ Annual Equation.

69	48	$2\tau+\xi-2\eta$	109	92	$\tau+\xi+\xi_{,}$	148	...	$2\tau+\eta$
70	50	$2\tau+\xi+2\eta$	110	89	$\tau-\xi+\xi_{,}$	149	...	$\xi-\eta$
71	24	$\xi_{,}-2\eta$	111	...	$\tau+\xi-\xi_{,}$	150	...	$\xi+\eta$
72	25	$\xi_{,}+2\eta$	112	...	$\tau-2\xi_{,}$	151	...	$2\tau-\xi-\eta$
73	57	$2\tau-\xi_{,}-2\eta$	113	...	$\tau+2\xi_{,}$	152	...	$2\tau-\xi+\eta$
74	56	$2\tau-\xi_{,}+2\eta$	114	...	$\tau-2\eta$	153	...	$2\tau+\xi-\eta$
75	55	$2\tau+\xi_{,}-2\eta$	115	...	$\tau+2\eta$	154	...	$2\tau+\xi+\eta$
76	58	$2\tau+\xi_{,}+2\eta$	116	100	3τ	155	...	$\xi_{,}-\eta$
77	7	$2\xi-2\eta$	117	101	$3\tau-\xi$	156	...	$\xi_{,}+\eta$
78	8	$2\xi+2\eta$	118	102	$3\tau+\xi$	157	...	$2\tau-\xi_{,}-\eta$
79	65	$2\tau-2\xi-2\eta$	119	103	$3\tau-\xi_{,}$	158	...	$2\tau-\xi_{,}+\eta$
80	63	$2\tau-2\xi+2\eta$	120	104	$3\tau+\xi_{,}$	159	...	$2\tau+\xi_{,}-\eta$
81	64	$2\tau+2\xi-2\eta$	121	...	$3\tau-2\xi$	160	...	$2\tau+\xi_{,}+\eta$
82	...	$2\tau+2\xi+2\eta$	122	...	$3\tau+2\xi$	161	...	$2\xi-\eta$
83	...	$\xi+\xi_{,}-2\eta$	123	...	$3\tau-\xi-\xi_{,}$	162	...	$2\xi+\eta$
84	...	$\xi+\xi_{,}+2\eta$	124	...	$3\tau+\xi+\xi_{,}$	163	...	$2\tau-2\xi-\eta$
85	...	$2\tau-\xi-\xi_{,}-2\eta$	125	...	$3\tau-\xi+\xi_{,}$	164	...	$2\tau-2\xi+\eta$
86	...	$2\tau-\xi-\xi_{,}+2\eta$	126	...	$3\tau+\xi-\xi_{,}$	165	...	$2\tau+2\xi-\eta$
87	...	$2\tau+\xi+\xi_{,}-2\eta$	127	...	$3\tau-2\xi_{,}$	166	...	$2\tau+2\xi+\eta$
88	...	$2\tau+\xi+\xi_{,}+2\eta$	128	...	$3\tau+2\xi_{,}$	167	...	$\xi+\xi_{,}-\eta$
89	...	$\xi-\xi_{,}-2\eta$	129	...	$3\tau-2\eta$	168	...	$\xi+\xi_{,}+\eta$
90	...	$\xi-\xi_{,}+2\eta$	130	...	$3\tau+2\eta$	169	...	$2\tau-\xi-\xi_{,}-\eta$
91	...	$2\tau-\xi+\xi_{,}-2\eta$	131	120	4τ	170	...	$2\tau-\xi-\xi_{,}+\eta$
92	...	$2\tau-\xi+\xi_{,}+2\eta$	132	121	$4\tau-\xi$	171	...	$2\tau+\xi+\xi_{,}-\eta$
93	...	$2\tau+\xi-\xi_{,}-2\eta$	133	122	$4\tau+\xi$	172	...	$2\tau+\xi+\xi_{,}+\eta$
94	...	$2\tau+\xi-\xi_{,}+2\eta$	134	123	$4\tau-\xi_{,}$	173	...	$\xi-\xi_{,}-\eta$
95	...	$2\xi_{,}-2\eta$	135	124	$4\tau+\xi_{,}$	174	...	$\xi-\xi_{,}+\eta$
96	...	$2\xi_{,}+2\eta$	136	125	$4\tau-2\xi$	175	...	$2\tau-\xi+\xi_{,}-\eta$
97	...	$2\tau-2\xi_{,}-2\eta$	137	126	$4\tau+2\xi$	176	...	$2\tau-\xi+\xi_{,}+\eta$
98	...	$2\tau-2\xi_{,}+2\eta$	138	131	$4\tau-\xi-\xi_{,}$	177	...	$2\tau+\xi-\xi_{,}-\eta$
99	...	$2\tau+2\xi_{,}-2\eta$	139	...	$4\tau+\xi+\xi_{,}$	178	...	$2\tau+\xi-\xi_{,}+\eta$
100	...	$2\tau+2\xi_{,}+2\eta$	140	129	$4\tau-\xi+\xi_{,}$	179	...	$2\xi_{,}-\eta$
101	80	τ *	141	...	$4\tau+\xi-\xi_{,}$	180	...	$2\xi_{,}+\eta$
102	81	$\tau-\xi$	142	...	$4\tau-2\xi_{,}$	181	...	$2\tau-2\xi_{,}-\eta$
103	82	$\tau+\xi$	143	...	$4\tau+2\xi_{,}$	182	...	$2\tau-2\xi_{,}+\eta$
104	83	$\tau-\xi_{,}$	144	127	$4\tau-2\eta$	183	...	$2\tau+2\xi_{,}-\eta$
105	84	$\tau+\xi_{,}$	145	...	$4\tau+2\eta$	184	...	$2\tau+2\xi_{,}+\eta$
106	85	$\tau-2\xi$	146	...	η	185	...	$\tau-\eta$
107	86	$\tau+2\xi$	147	...	$2\tau-\eta$	186	...	$\tau+\eta$
108	91	$\tau-\xi-\xi_{,}$						

$$\cos 2\tau \cos 2\tau = \underset{[131]}{\frac{1}{2}\cos 4\tau} + \underset{[0]}{\frac{1}{2}}$$

$$\cos 2\tau \cos \xi = \underset{[4]}{\frac{1}{2}\cos(2\tau+\xi)} + \underset{[-3]}{\frac{1}{2}\cos(-2\tau+\xi)}$$

Hence the argument 131 is produced *by addition* through
square of the term corresponding to argument 1. Similarly
argument 4 is produced *by addition* through the multiplication
argument 1 by argument 2, and the argument — 3 *by subtractio*
These results, extended through all the arguments, are exhibi

---

* Parallactic inequality.

in the Table at the end of this work, by means of which any multiplication of one series by another may be effected, and any coefficient obtained at pleasure, when the approximation is limited to those terms which the Table reaches, as will be shown by repeated examples. By means of this Table all the developments which are required in the theory of the moon may be obtained.

This Table may also be used in forming the developments required in the method employed by MM. Laplace and Damoiseau. For this purpose it is merely necessary to make

$$\begin{array}{lll} \tau = \lambda' - \lambda_{\prime} & \text{instead of} & nt - n_{\prime}t \\ \xi = c\lambda' - \varpi & \ldots\ldots & cnt - \varpi \\ \xi_{\prime} = c\lambda_{\prime} - \varpi_{\prime} & \ldots\ldots & c_{\prime}n_{\prime}t - \varpi_{\prime} \\ \text{and } \eta = g\lambda' - \nu & \ldots\ldots & gnt - \nu \end{array}$$

the indices remaining the same.

$$R = m_{\prime}\left\{\frac{r' r_{\prime} \cos(\lambda - \lambda_{\prime})}{r_{\prime}^3} - \frac{1}{\{r^2 - 2r' r_{\prime} \cos(\lambda' - \lambda_{\prime}) + r_{\prime}^2\}^{\frac{1}{2}}}\right\}$$

$$= m_{\prime}\left\{-\frac{1}{r_{\prime}} + \frac{r^2}{2r_{\prime}^3} - \frac{3}{8}\frac{\{2r' r_{\prime} \cos(\lambda' - \lambda_{\prime}) - r^2\}^2}{r_{\prime}^5}\right.$$

$$\left. - \frac{15}{18}\frac{\{2r' r_{\prime} \cos(\lambda' - \lambda_{\prime}) - r^2\}^3}{r_{\prime}^7}\right\}$$

$$= m_{\prime}\left\{-\frac{1}{r_{\prime}} + \frac{r^2}{2r_{\prime}^3} - \frac{3}{2}\frac{r'^2 r_{\prime}^2}{r_{\prime}^5} \cos(\lambda' - \lambda_{\prime})^2\right.$$

$$\left. + \frac{3}{2}\frac{r^2 r' r_{\prime}}{r_{\prime}^5} \cos(\lambda - \lambda_{\prime}) - \frac{5}{2}\frac{r'^3 r_{\prime}^3}{r_{\prime}^7} \cos(\lambda' - \lambda_{\prime})^3\right\}$$

$$= m_{\prime}\left\{-\frac{1}{r_{\prime}} - \frac{r'^2}{4r_{\prime}^3}\{1 + 3\cos(2\lambda' - 2\lambda_{\prime}) - 2s^2\}\right.$$

$$\left. - \frac{r'^3}{8r_{\prime}^4}\{3(1 - 4s^2)\cos(\lambda' - \lambda_{\prime}) + 5\cos(3\lambda' - 3\lambda_{\prime})\}\right\}$$

$$\text{Let } R = m_{\prime}\{R_0 + \underset{[1]}{R_1 \cos\tau} + \underset{[2]}{e R_2 \cos\xi} + \underset{[3]}{e R_3 \cos(2\tau - \xi)} + \text{\&c.}\}$$

$$r^{-1} = 1 + \underset{[2]}{e\left(1 - \frac{e^2}{8}\right)\cos\xi} + \underset{[8]}{e^2\left(1 - \frac{e^2}{3}\right)\cos 2\xi}$$

$$+\underset{[20]}{\frac{9}{8}e^3\cos 3\xi}+\underset{[35]}{\frac{4}{3}e^4\cos 4\xi}$$

$$\frac{\mathrm{d}r}{\mathrm{d}e}=e-\underset{[2]}{\left(1-\frac{9}{8}e^2\right)\cos\xi}-\underset{[8]}{e\left(1-\frac{4e^2}{3}\right)\cos 2\xi}$$

$$-\underset{[20]}{\frac{9}{8}e^2\cos 3\xi}-\underset{[35]}{\frac{4e^3}{3}\cos 4\xi}$$

$$\frac{\mathrm{d}r}{r\,\mathrm{d}e}=\underset{[0]}{\frac{e}{2}\left(1+\frac{e^2}{4}\right)}-\underset{[2]}{\left(1-\frac{9}{8}e^2\right)\cos\xi}$$

$$-\underset{[8]}{\frac{3}{2}e\left(1-\frac{11}{9}e^2\right)\cos 2\xi}$$

$$-\underset{[20]}{\frac{17}{8}e^2\cos 3\xi}-\underset{[35]}{\frac{71}{24}e^3\cos 4\xi}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}e}=\underset{[2]}{2\left(1-\frac{3e^2}{8}\right)\sin\xi}+\underset{[8]}{\frac{5}{2}e\left(1-\frac{28}{15}e\right)\sin 2\xi}$$

$$+\underset{[20]}{\frac{13}{4}e^2\sin 3\xi}+\underset{[35]}{\frac{103}{24}e^3\sin 4\xi}$$

$$\frac{\mathrm{d}R}{\mathrm{d}e}=\frac{\mathrm{d}R}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}e}+\frac{\mathrm{d}R}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}e}$$

$$=\frac{r\,\mathrm{d}R}{\mathrm{d}r}\frac{\mathrm{d}r}{r\,\mathrm{d}e}+\frac{\mathrm{d}R}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}e}$$

$\dfrac{r\,\mathrm{d}R}{\mathrm{d}r}=\dfrac{a\,\mathrm{d}R}{\mathrm{d}a}$ $\qquad$ $\dfrac{\mathrm{d}R}{\mathrm{d}\lambda}=\dfrac{\mathrm{d}R}{\mathrm{d}\tau}$. See *Méc. Cél.* vol. i. p. 267.

Considering the terms in R multiplied by $\dfrac{a^2}{a_{,}^3}$, $\dfrac{r\,\mathrm{d}R}{\mathrm{d}r}=2R$

$$R_0=-\frac{a^2}{4a_{,}^3}+\&c.\qquad R_1=-\frac{3a^2}{4a^3}+\&c.$$

$$R_{101}=-\frac{3a^3}{8a_{,}^4}+\&c.\qquad R_{116}=-\frac{5a^3}{8a_{,}^4}+\&c.$$

Multiplying by means of the Table, neglecting the squares of the eccentricities, I find

$$R_2 = -\frac{a\,\mathrm{d}R_0}{\mathrm{d}a} = \frac{a^2}{2a_,^3}$$

$$R_3 = -\frac{a\,\mathrm{d}R_1}{2\,\mathrm{d}a} - 2\,R_1 = \frac{9\,a^2}{4\,a_,^3}$$

$$R_4 = -\frac{a\,\mathrm{d}R_1}{2\,\mathrm{d}a} + 2\,R_1 = -\frac{3\,a^2}{4\,a_,^3}$$

Neglecting the cubes of the eccentricities, we have

$$2\,R_8 = -\frac{a\,\mathrm{d}R_2}{2\,\mathrm{d}a} - \frac{3\,a\,\mathrm{d}R_0}{2\,\mathrm{d}a} = \frac{a^2}{4\,a_,^3}$$

$$2\,R_9 = -\frac{a\,\mathrm{d}R_3}{2\,\mathrm{d}a} - 2\,R_3 - \frac{3\,a\,\mathrm{d}R_1}{4\,\mathrm{d}a} - \frac{5\,.\,2\,R_1}{4} = -\frac{15\,a^2}{4\,a_,^3}$$

$$2\,R_{10} = -\frac{a\,\mathrm{d}R_4}{2\,\mathrm{d}a} + 2\,R_4 - \frac{3\,a\,\mathrm{d}R_1}{4\,\mathrm{d}a} + \frac{5\,.\,2\,R_1}{4} = -\frac{3\,a^2}{2\,a_,^2}$$

These equations may be formed at once from the Table by inspection, taking care to write R with the sign + in the term multiplied by $\frac{\mathrm{d}R}{\mathrm{d}\tau}$ when the index is found in the upper line in the Table, as in the case of the argument (10); and with the sign − when in the lower, as in the case of the argument (9). The term multiplied by $\frac{a\,\mathrm{d}R}{\mathrm{d}a}$ always takes its sign from the factor arising from $\frac{\mathrm{d}r}{r\,\mathrm{d}e}$.

By means of the Table, any term in R depending on the eccentricities may be found at pleasure, and the development, *Phil. Trans.* 1831, p. 263, may be verified with great facility; thus,

$$4\,R_{38} = -\frac{a\,\mathrm{d}R_{20}}{2\,\mathrm{d}a} - \frac{3\,a\,\mathrm{d}R_8}{4\,\mathrm{d}a} - \frac{17\,a\,\mathrm{d}R_2}{16\,\mathrm{d}a} - \frac{71\,a\,\mathrm{d}R_0}{24\,\mathrm{d}a}.$$

I find, on reference to the development in question,

$$R_{20} = \frac{a^2}{16\,a_,^3} \qquad R_8 = \frac{a^2}{8\,a_,^3} \qquad R_2 = \frac{a^2}{2\,a_,^3} \qquad R_0 = -\frac{a^2}{4\,a_,^3};$$

whence

$$a\frac{\mathrm{d}R_{20}}{\mathrm{d}a} = \frac{a^2}{8\,a_,^3} \qquad \frac{a\,\mathrm{d}R_8}{\mathrm{d}a} = \frac{a^2}{4\,a_,^3} \qquad \frac{a\,\mathrm{d}R_2}{\mathrm{d}a} = \frac{a^2}{a_,^3} \qquad \frac{a\,\mathrm{d}R_0}{\mathrm{d}a} = -\frac{a^2}{2\,a_,^3}$$

$$4\,R_{38} = \left\{\frac{1}{2\,.\,8} - \frac{3}{4\,.\,4} - \frac{17}{16} + \frac{71}{24\,.\,2}\right\}\frac{a^2}{a_,^3} = \frac{4\,a^2}{24\,a_,^3} \qquad R_{38} = \frac{a^2}{24\,a_,^3}.$$

The terms depending upon the inclination of the moon's orbit may be obtained in a similar manner.

If R be considered as a function of r, λ' and s,

$$\frac{\mathrm{d}R}{\mathrm{d}\gamma}=\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\frac{\mathrm{d}\lambda'}{\mathrm{d}\gamma}+\frac{\mathrm{d}R}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\gamma}.$$

In this case

$$\frac{\mathrm{d}R}{\mathrm{d}s}=m_{,}\left\{\frac{r^2}{r_{,}^3}+\frac{r^2}{2r_{,}^3}\{1+\cos(2\lambda-2\lambda_{,})\}\right\}s \text{ nearly}$$

$$\frac{r^2}{2r_{,}^3}+\frac{r^2}{4r^{\prime 3}}\{1+3\cos(2\lambda-2\lambda_{,})\}$$

$$=\frac{a^2}{a_{,}^3}\left\{\frac{3}{4}\left\{1+\frac{3}{2}e^2+\frac{3}{2}e_{,}^2\right\}\right.$$

$$+\frac{3}{4}\left\{1-\frac{5}{2}e^2-\frac{5}{2}e_{,}^2\right\}\cos 2\tau-\frac{3}{2}e\cos\xi$$

$$-\frac{9}{4}e\cos(2\tau-\xi)+\frac{3}{4}e\cos(2\tau+\xi)+\frac{9}{4}e_{,}\cos\xi_{,}$$

$$+\frac{21}{8}e_{,}\cos(2\tau-\xi_{,})-\frac{3}{8}e_{,}\cos(2\tau+\xi_{,})-\frac{3}{8}e^2\cos 2\xi$$

$$+\frac{15}{8}e^2\cos(2\tau-2\xi)+\frac{3}{4}e^2\cos(2\tau+2\xi)-\frac{9}{4}ee_{,}\cos(\xi+\xi_{,})$$

$$-\frac{63}{8}ee_{,}\cos(2\tau-\quad-\xi_{,})-\frac{3}{8}ee_{,}\cos(2\tau+\xi+\xi_{,})$$

$$-\frac{9}{4}ee_{,}\cos(\xi-\xi_{,})+\frac{9}{8}ee_{,}\cos(2\tau-\xi+\xi_{,})$$

$$\left.+\frac{21}{8}ee_{,}\cos(2\tau+\xi-\xi_{,})+\frac{27}{8}e_{,}^2\cos 2\xi_{,}+\frac{51}{8}e_{,}^2\cos(2\tau-2\xi_{,})\right\}$$

The following expression for $\frac{\mathrm{d}\lambda'}{\mathrm{d}\gamma}$ is easily deduced from that for λ', given in p. 10.

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}\gamma}=-\underset{[62]}{\frac{\gamma}{2}(1-4e^2)\sin 2\eta}-\underset{[65]}{\gamma e\sin(\xi-2\eta)}-\underset{[66]}{\gamma e\sin(\xi+2\eta)}$$

$$+\underset{[77]}{\frac{3}{8}\gamma e^2\sin(2\xi-2\eta)}-\underset{[78]}{\frac{13}{8}\gamma e^2\sin(2\xi+2\eta)}$$

$$\frac{s\,d\,s}{d\,\gamma} = \frac{\gamma}{2} - \underset{[62]}{\frac{\gamma}{2}(1 - 4\,e^2)\cos 2\,\eta} + \underset{[65]}{\gamma\,e\cos(\xi - 2\,\eta)} - \underset{[66]}{\gamma\,e\cos(\xi + 2\,\eta)}$$

$$- \underset{[77]}{\frac{3}{8}\gamma\,e^2\cos(2\,\xi - 2\,\eta)} - \underset{[78]}{\frac{13}{8}\gamma\,e^2\cos(2\,\xi + 2\,\eta)}$$

$$2\,R_{63} = \left\{-\frac{3\,.\,1}{4\,.\,2} - \frac{3\,.\,1}{4\,.\,2}\right\}\frac{a^2}{a_{,}^3} = -\frac{3\,a^2}{4\,a_{,}^3} \qquad R_{63} = -\frac{3\,a^2}{8\,a_{,}^3}$$

$$2\,R_{64} = \left\{\frac{3\,.\,1}{4\,.\,2} - \frac{3\,.\,1}{4\,.\,2}\right\}\frac{a^2}{a_{,}^3} = 0 \qquad R_{64} = 0$$

$$2\,R_{69} = \left\{-\frac{3\,.\,1}{4\,.\,2} + \frac{3}{4} - \frac{3\,.\,1}{4\,.\,2} + \frac{3}{4}\right\}\frac{a^2}{a_{,}3} = \frac{3\,a^2}{4\,a_{,}^3} \quad R_{69} = \frac{3\,a^2}{8\,a_{,}^3}$$

It follows immediately, from the preceding method of development, that when a term in the disturbing function vanishes, such as R_{19}, corresponding to the argument $2\,\tau + 2\,\xi_{,}$, all the terms corresponding to arguments which result from any combination of this quantity with any multiples of ξ and η must also vanish, which seems to me to constitute a remarkable theorem on account of its extreme generality. Thus, R_{19} being equal to zero, R_{31}, R_{33}, R_{49}, R_{51}, R_{99} and R_{100} are all of necessity equal to zero likewise. So again, since R_{64} is equal to zero, it follows of necessity that R_{74} and R_{76} are also equal to zero. I have shown in the *Phil. Trans.* that the term vanishes in the disturbing function corresponding to the argument $3\,\tau + 3\,\xi_{,}$, and it follows therefore of necessity that all the terms corresponding to any combinations of this quantity with any multiples of ξ and η are also equal to zero. Amongst these occurs the argument $3\,\tau - \xi + 3\,\xi_{,} - 2\,\eta$[*], important on account of the small diviser which would be introduced by integration in determining the corresponding inequality. M. Poisson first ascertained by a different process that the coefficient of the corresponding term in the disturbing function vanishes, and moreover he has shown, by means of the theory of the variation of the elliptic constants, that the first term in the corresponding inequality in the moon's longitude depends upon the coefficient in question, and therefore also vanishes.

By proceeding in the way explained above, all the coefficients in the development of R are made to depend ultimately upon R_0, R_1, R_{101}, &c. This method is far more simple than that which I formerly employed in the *Philosophical Transactions*, 1832, for the development of R in the lunar theory.

---

[*] This is the argument noticed by Laplace, *Mécanique Céleste*, vol. iii. p. 178: "Cette inégalité dont la période est de 184 ans, dépend du produit de ces quatre quantités, le carré de l'inclinaison de l'orbe lunaire à l'écliptique, l'excentricité de cet orbe, le cube de l'excentricité de l'orbe solaire, et le rapport de la parallaxe du soleil à celle de la lune: elle paroît ainsi devoir être insensible; mais les grands diviseurs qu'elle acquiert par les intégrations peuvent le rendre sensible."

Finally, the following development results, arising from the substitution in the disturbing function of the elliptic expressions for the coordinates of the sun and moon.

$$R = m_{,}\left\{ -\frac{1}{r_{,}} - \frac{1}{4}\left\{ 1 + \frac{3}{2}e^2 + \frac{3}{2}e_{,}^2 + \frac{9}{4}e^2 e_{,}^2 + \frac{15}{8}e_{,}^4 \right.\right.$$

$$\left. - \frac{3}{2}\gamma^2 - \frac{9}{4}\gamma^2 e^2 - \frac{9}{4}\gamma^2 e_{,}^2 + \frac{3}{2}\gamma^4 + \frac{9\,a^2}{16\,a_{,}^2}\right\}\frac{a^2}{a_{,}^3}$$

$$[0]$$

$$-\frac{3}{4}\left\{1 - \frac{5}{2}e^2 - \frac{5}{2}e_{,}^2 + \frac{23}{16}e^4 + \frac{25}{4}e^2 e_{,}^2\right.$$

$$\left. + \frac{13}{16}e_{,}^4 + \frac{5\,a^2}{12\,a_{,}^2}\right\}\cos^4\frac{\iota}{2}{}^{*}\,\frac{a^2}{a_{,}^3}\cos 2\tau$$

$$[1]$$

$$+\frac{1}{2}\left\{1 - \frac{e^2}{8} + \frac{3}{2}e_{,}^2 - \frac{3}{2}\gamma^2 + \frac{9\,a^2}{8\,a_{,}^2}\right\}\frac{a^2}{a_{,}^3}e\cos\xi$$

$$[2]$$

$$+\frac{9}{4}\left\{1 - \frac{13}{24}e^2 - \frac{5}{2}e_{,}^2 - \frac{\gamma^2}{2} + \frac{5\,a^2}{9\,a_{,}^2}\right\}\frac{a^2}{a_{,}^3}e\cos(2\tau - \xi)$$

$$[3]$$

$$-\frac{3}{4}\left\{1 - \frac{19}{8}e^2 - \frac{5}{2}e_{,}^2 - \frac{\gamma^2}{2}\right\}\frac{a^2}{a_{,}^3}e\cos(2\tau + \xi)$$

$$[4]$$

$$-\frac{3}{4}\left\{1 + \frac{3}{2}e^2 + \frac{9}{8}e_{,}^2 - \frac{3}{2}\gamma^2 + \frac{35\,a^2}{48\,a_{,}^2}\right\}\frac{a^2}{a_{,}^3}e_{,}\cos\xi_{,}$$

$$[5]$$

$$-\frac{21}{8}\left\{1 - \frac{5}{2}e^2 - \frac{123}{56}e_{,}^2 - \frac{\gamma^2}{2} + \frac{15\,a^2}{28\,a_{,}^2}\right\}\frac{a^2}{a_{,}^3}e_{,}\cos(2\tau - \xi_{,})$$

$$[6]$$

$$+\frac{3}{8}\left\{1 - \frac{5}{2}e^2 - \frac{e_{,}^2}{8} - \frac{\gamma^2}{2} - \frac{5\,a^2}{4\,a_{,}^2}\right\}\frac{a^2}{a_{,}^3}e_{,}\cos(2\tau + \xi_{,})$$

$$[7]$$

---

* $\cos\frac{\iota}{2} = 1 - \frac{\gamma^2}{2} + \frac{7}{16}\gamma^4 -$, &c.

Develop of *R.*

$$-\frac{1}{8}\left\{1-\frac{e^2}{3}+\frac{3}{2}e_{\prime}^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_{\prime}^3}e^2\cos 2\,\xi$$

[8]

$$-\frac{15}{8}\left\{1-\frac{5}{2}e_{\prime}^2-\frac{\gamma^2}{2}\right\}\frac{a^2}{a_{\prime}^3}e^2\cos(2\,\tau-2\,\xi)$$

[9]

$$-\frac{3}{4}\left\{1-\frac{5}{2}e^2-\frac{5}{2}e_{\prime}^2-\frac{\gamma^2}{2}\right\}\frac{a^2}{a_{\prime}^3}e^2\cos(2\,\tau+2\,\xi)$$

[10]

$$+\frac{3}{4}\left\{1-\frac{e^2}{8}+\frac{9}{8}e_{\prime}^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_{\prime}^3}e\,e_{\prime}\cos(\xi+\xi)$$

[11]

$$+\frac{63}{8}\left\{1-\frac{91}{168}e^2-\frac{123}{56}e_{\prime}^2-\frac{\gamma^2}{2}\right\}\frac{a^2}{a_{\prime}^3}e\,e_{\prime}\cos(2\,\tau-\xi-\xi_{\prime})$$

[12]

$$+\frac{3}{8}\left\{1-\frac{19}{8}e^2-\frac{e_{\prime}^2}{8}-\frac{\gamma^2}{2}\right\}\frac{a^2}{a_{\prime}^3}e\,e_{\prime}\cos(2\,\tau+\xi+\xi_{\prime})$$

[13]

$$+\frac{3}{4}\left\{1-\frac{e^2}{8}+\frac{9}{8}e_{\prime}^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_{\prime}^3}e\,e_{\prime}\cos(\xi-\xi_{\prime})$$

[14]

$$-\frac{9}{8}\left\{1-\frac{13}{24}e^2-\frac{e_{\prime}^2}{8}-\frac{\gamma^2}{2}\right\}\frac{a^2}{a_{\prime}^3}e\,e_{\prime}\cos(2\,\tau-\xi+\xi_{\prime})$$

[15]

$$-\frac{21}{8}\left\{1-\frac{19}{8}e^2-\frac{123}{56}e_{\prime}^2-\frac{\gamma^2}{2}\right\}\frac{a^2}{a_{\prime}^3}e\,e_{\prime}\cos(2\,\tau+\xi-\xi_{\prime})$$

[16]

$$-\frac{9}{8}\left\{1+\frac{3}{2}e^2+\frac{7}{9}e_{\prime}^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_{\prime}^3}e_{\prime}^2\cos 2\,\xi_{\prime}$$

[17]

$$-\frac{51}{8}\left\{1-\frac{5}{2}e^2-\frac{115}{51}e_{\prime}^2-\frac{\gamma^2}{2}\right\}\cos^4\frac{\iota}{2}\,\frac{a^2}{a_{\prime}^3}e_{\prime}^2\cos(2\,\tau-2\,\xi_{\prime})$$

[18]

$$+\frac{1}{16}\frac{a^2}{a_{\prime}^3}e^3\cos 3\,\xi+\frac{7}{32}\frac{a^2}{a_{\prime}^3}e^3\cos(2\,\tau-3\,\xi)-\frac{25}{32}\frac{a^2}{a_{\prime}^3}e^3\cos(2\,\tau+3\,\xi)$$

[20] [21] [22]

$$+\frac{3\,a^2}{16\,a_{\prime}^3}e^2e_{\prime}\cos(2\,\xi+\xi_{\prime})-\frac{105}{16}\frac{a^2}{a_{\prime}^3}e^2e_{\prime}\cos(2\,\tau-2\,\xi-\xi_{\prime})$$

[23] [24]

velopment R.

$$+\frac{3}{8}\frac{a^2}{a_{,}^3}e^2 e_{,}\cos(2\tau+2\xi+\xi_{,})+\frac{3}{16}\frac{a^2}{a_{,}^3}e^2 e_{,}\cos(2\xi-\xi_{,})$$

[25] [26]

$$+\frac{15}{16}\frac{a^2}{a_{,}^3}e^2 e_{,}\cos(2\tau-2\xi+\xi_{,})-\frac{21}{8}\frac{a^2}{a_{,}^3}e^2 e_{,}\cos(2\tau+2\xi-\xi_{,})$$

[27] [28]

$$+\frac{9}{8}\frac{a^2}{a_{,}^3}e\,e_{,}^2\cos(\xi+2\xi_{,})+\frac{153}{8}\frac{a^2}{a_{,}^3}e\,e_{,}^2\cos(2\tau-\xi-2\xi_{,})$$

[29] [30]

$$+\frac{9}{8}\frac{a^2}{a_{,}^3}e\,e_{,}^2\cos(\xi-2\xi_{,})-\frac{51}{8}\frac{a^2}{a_{,}^3}e\,e_{,}^2\cos(2\tau+\xi-2\xi_{,})$$

[32] [34]

$$-\frac{53}{32}\frac{a^2}{a_{,}^3}e_{,}^3\cos 3\xi_{,}-\frac{845\,a^2}{64\,a_{,}^3}e_{,}^3\cos(2\tau-3\xi_{,})$$

[35] [36]

$$-\frac{1}{64}\frac{a^2}{a_{,}^3}e_{,}^3\cos(2\tau+3\xi_{,})+\frac{1}{24}\frac{a^2}{a_{,}^3}e^4\cos 4\xi$$

[37] [38]

$$+\frac{3}{64}\frac{a^2}{a_{,}^3}e^4\cos(2\tau-4\xi)-\frac{27}{32}\frac{a^2}{a_{,}^3}e^4\cos(2\tau+4\xi)$$

[39] [40]

$$+\frac{3}{32}\frac{a^2}{a_{,}^3}e^3 e_{,}\cos(3\xi+\xi_{,})$$

[41]

$$+\frac{49}{64}\frac{a^2}{a_{,}^3}e^3 e_{,}\cos(2\tau-3\xi-\xi_{,})+\frac{25}{64}\frac{a^2}{a_{,}^3}e^3 e_{,}\cos(2\tau-3\xi-\xi_{,}$$

[42] [43]

$$+\frac{3}{32}\frac{a^2}{a_{,}^3}e^3 e_{,}\cos(3\xi-\xi_{,})-\frac{7}{64}\frac{a^2}{a_{,}^3}e^3 e_{,}\cos(2\tau-3\xi+\xi_{,})$$

[44] [45]

$$-\frac{175}{64}\frac{a^2}{a'^3}e^3 e_{,}\cos(2\tau+3\xi-\xi_{,})+\frac{9}{32}\frac{a^2}{a^3}e^2 e_{,}^2\cos(2\xi+2\xi_{,})$$

[46] [47]

$$-\frac{255}{16}e^2 e_{,}^2\cos(2\tau-2\xi-2\xi_{,})+\frac{9}{32}\frac{a^2}{a_{,}^3}e^2 e_{,}^2\cos(2\xi-2\xi_{,})$$

[48] [50]

$$-\frac{51}{8}\frac{a^2}{a_{,}^3}e^2 e_{,}^2\cos(2\tau+2\xi-2\xi)+\frac{53}{32}\frac{a^2}{a_{,}^3}e\,e_{,}^3\cos(\xi+3\xi_{,})$$

[52] [53]

$$+\frac{2535}{64}\frac{a^2}{a_{\prime}^{3}}e\,e^{\prime 3}\cos(2\tau-\xi-3\xi_{\prime})-\frac{1}{64}\frac{a^2}{a_{\prime}^{3}}e\,e_{\prime}^{3}\cos(2\tau+\xi+3\xi_{\prime})$$ Developme of *R.*

[54] [55]

$$+\frac{53}{32}\frac{a^2}{a_{\prime}^{3}}e\,e_{\prime}^{3}\cos(\xi-3\xi_{\prime})+\frac{3}{64}\frac{a^2}{a_{\prime}^{3}}e\,e_{\prime}^{3}\cos(2\tau-\xi+3\xi_{\prime})$$

[56] [57]

$$+\frac{45}{64}\frac{a^2}{a_{\prime}^{3}}e\,e_{\prime}^{3}\cos(2\tau+\xi-3\xi_{\prime})+\frac{591}{64}\frac{a^2}{a_{\prime}^{3}}e_{\prime}^{4}\cos 4\xi_{\prime}$$

[58] [59]

$$-\frac{2453}{128}\frac{a^2}{a_{\prime}^{3}}e_{\prime}^{4}\cos(2\tau-4\xi_{\prime})+\frac{741}{128}\frac{a^2}{a_{\prime}^{3}}e_{\prime}^{4}\cos(2\tau+4\xi_{\prime})$$

[60] [61]

$$-\frac{3}{8}\left\{1-\frac{5}{2}e^2+\frac{3}{2}e_{\prime}^{2}\right\}\frac{a^2}{a_{\prime}^{3}}\gamma^2\cos 2\eta$$

[62]

$$-\frac{3}{8}\left\{1+\frac{3}{2}e^2-\frac{5}{2}e_{\prime}^{2}+\frac{\gamma^2}{8}\right\}\frac{a^2}{a_{\prime}^{3}}\gamma^2\cos(2\tau-2\eta)$$

[63]

$$+\frac{9}{8}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e\cos(\xi-2\eta)-\frac{3}{8}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e\cos(\xi_{\prime}+2\eta)$$

[65] [66]

$$+\frac{3}{8}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e\cos(2\tau-\xi-2\eta)+\frac{3}{8}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e\cos(2\tau+\xi-2\eta)$$

[67] [69]

$$-\frac{9}{16}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e_{\prime}\cos(\xi_{\prime}-2\eta)-\frac{9}{16}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e_{\prime}\cos(\xi_{\prime}+2\eta)$$

[71] [72]

$$-\frac{21}{16}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e_{\prime}\cos(2\tau-\xi_{\prime}-2\eta)+\frac{3}{16}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e_{\prime}\cos(2\tau+\xi_{\prime}+2\eta)$$

[73] [75]

$$-\frac{15}{16}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e^2\cos(2\xi_{\prime}-2\eta)-\frac{3}{8}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e^2\cos(2\xi+2\eta)$$

[77] [78]

$$+\frac{3}{32}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e^2\cos(2\tau-2\xi-2\eta)+\frac{3}{32}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e^2\cos(2\tau+2\xi-2\eta)$$

[79] [81]

$$+\frac{27}{16}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e\,e_{\prime}\cos(\xi+\xi_{\prime}-2\eta)-\frac{9}{16}\frac{a^2}{a_{\prime}^{3}}\gamma^2 e\,e_{\prime}\cos(\xi+\xi_{\prime}+2\eta)$$

[83] [84]

evelopment
f *R.*

$$+\frac{21}{16}\frac{a^2}{a_{,}^3}\gamma^2 e\,e_{,}\cos(2\tau-\xi-\xi_{,}-2\eta)$$

[85]

$$-\frac{3}{16}\frac{a^2}{a_{,}^3}\gamma^2 e\,e_{,}\cos(2\tau+\xi+\xi_{,}-2\eta)+\frac{27}{16}\frac{a^2}{a_{,}^3}\gamma^2 e\,e_{,}\cos(\xi-\xi_{,}-$$

[87] [89]

$$-\frac{9}{16}\frac{a^2}{a_{,}^3}\gamma^2 e\,e_{,}\cos(\xi-\xi_{,}+2\eta)-\frac{3}{16}\frac{a^2}{a_{,}^3}\gamma^2 e\,e_{,}\cos(2\tau-\xi+\xi_{,}-$$

[90] [91]

$$+\frac{21}{16}\frac{a^2}{a_{,}^3}\gamma^2 e\,e_{,}\cos(2\tau+\xi-\xi_{,}-2\eta)-\frac{27}{16}\frac{a^2}{a_{,}^3}\gamma^2 e_{,}^2\cos(2\xi_{,}-2$$

[93] [95]

$$-\frac{27}{16}\frac{a^2}{a'^3}\gamma^2 e_{,}^2\cos(2\xi_{,}+2\eta)-\frac{51}{16}\frac{a^2}{a_{,}^3}\gamma^2 e_{,}^2\cos(2\tau-2\xi_{,}-2\eta)$$

[96] [97]

$$-\frac{3}{8}\left\{1+3e^2+3e_{,}^2-\frac{11}{4}\gamma^2\right\}\frac{a^3}{a_{,}^4}\cos\tau+\frac{15}{16}\frac{a^3}{a_{,}^4}e\cos(\tau-\xi$$

[101] [102]

$$+\frac{3}{16}\frac{a^3}{a_{,}^4}e\cos(\tau+\xi)-\frac{9}{8}\frac{a^3}{a_{,}^4}e_{,}\cos(\tau-\xi_{,})-\frac{3}{8}\frac{a^3}{a_{,}^4}e_{,}\cos(\tau+$$

[103] [104]. [105

$$-\frac{33}{64}\frac{a^3}{a_{,}^4}e^2\cos(\tau-2\xi)+\frac{9}{64}\frac{a^3}{a_{,}^4}e^2\cos(\tau+2\xi)$$

[106] [107]

$$+\frac{45}{16}\frac{a^3}{a_{,}^4}e\,e_{,}\cos(\tau-\xi-\xi_{,})+\frac{3}{16}\frac{a^3}{a_{,}^4}e\,e_{,}\cos(\tau+\xi+\xi_{,})$$

[108] [109]

$$+\frac{15}{16}\frac{a^3}{a_{,}^4}e\,e_{,}\cos(\tau-\xi+\xi_{,})+\frac{9}{16}\frac{a^3}{a_{,}^4}e\,e_{,}\cos(\tau+\xi-\xi_{,})$$

110] [111]

$$-\frac{159}{64}\frac{a^3}{a_{,}^4}e_{,}^2\cos(\tau-2\xi_{,})-\frac{33}{64}\frac{a^3}{a_{,}^4}e_{,}^2\cos(\tau+2\xi_{,})$$

[112] [113]

$$-\frac{9}{16}\frac{a^3}{a_{,}^4}\gamma^2\cos(\tau-2\eta)-\frac{15}{128}\frac{a^3}{a_{,}^4}\gamma^2\cos(\tau+2\eta)$$

[114] [115]

$$-\frac{5}{8}\left\{1-6e^2-6e_{,}^2-\frac{3}{4}\gamma^2\right\}\frac{a^3}{a_{,}^4}\cos 3\tau+\frac{45}{16}\frac{a^3}{a_{,}^4}e\cos(3\tau-$$

[116] [117

$$-\frac{15}{16}\frac{a^3}{a_{,}^4}e\cos(3\tau+\xi)-\frac{25}{8}\frac{a^3}{a_{,}^4}e_{,}\cos(3\tau-\xi_{,})$$

[118] [119]

$$+\frac{5}{8}\frac{a^3}{a_{,}^4}e_{,}\cos(3\tau+\xi_{,})-\frac{285}{64}\frac{a^3}{a_{,}^4}e^2\cos(3\tau-2\xi)$$

[120] [121]

$$-\frac{75}{64}\frac{a^3}{a_{,}^4}e^2\cos(3\tau+2\xi)-\frac{225}{16}\frac{a^3}{a_{,}^4}e\,e_{,}\cos(3\tau-\xi-\xi_{,})$$

[122] [123]

$$+\frac{15}{16}\frac{a^3}{a_{,}^4}e\,e_{,}\cos(3\tau+\xi+\xi_{,})-\frac{45}{16}\frac{a^3}{a_{,}^4}e\,e_{,}\cos(3\tau-\xi+\xi_{,})$$

[124] [125]

$$-\frac{75}{16}\frac{a^3}{a_{,}^4}e\,e_{,}\cos(3\tau+\xi-\xi_{,})-\frac{635}{64}\frac{a^3}{a_{,}^4}e_{,}^2\cos(3\tau-2\xi_{,})$$

[126] [127]

$$-\frac{5}{64}\frac{a^3}{a_{,}^4}e_{,}^2\cos(3\tau+2\xi)-\frac{15}{32}\frac{a^3}{a_{,}^4}\gamma^2\cos(3\tau-2\eta)$$

[128] [129]

The equation for determining the coefficients of the terms in the expression for the radius vector is

$$\frac{\mathrm{d}^2.\,\mathrm{r}^2}{2\,\mathrm{d}t^2}-\frac{\mu}{r}+\frac{\mu}{a}+2\int\mathrm{d}R+r\frac{\mathrm{d}R}{\mathrm{d}r}=0.$$

From this equation any power of r might in fact be determined by the method of indeterminate coefficients. I propose to obtain r^{-1}, as this appears to me the most convenient proceeding. The horizontal parallax results immediately from this power of the radius vector. Comparatively few terms would be sufficient if this expression were only wanted to obtain the parallax with sufficient precision, but r^{-2} enters into the differential equation from which the longitude must be determined, and r^{-2} is obtained immediately by squaring r^{-1}. In integrating the equation above, r^{-1} is to be considered equal to a series of cosines, of which the arguments are detailed p. 23, ξ and η being equal to $c\,n\,t+\epsilon-\varpi$ and $g\,n\,t+\epsilon-\varpi$.

If the elliptic value of r be called r, and the part of $\frac{1}{r}$ due to the perturbations be called $\delta\,\frac{1}{r}$,

$$\frac{d^2 . r^2}{2\,d\,t^2} - \frac{d^2 . r^3\,\delta\,\frac{1}{r}}{d\,t^2} + \frac{3\,d^2 . r^4\left(\delta\,\frac{1}{r}\right)^2}{2\,d\,t^2} - \frac{2\,d^2 . r^5\left(\delta\,\frac{1}{r}\right)^3}{d\,t^2}$$

$$-\frac{\mu}{r} + \frac{\mu}{a} + 2\int d\,R + r\left(\frac{d\,R}{d\,r}\right) = 0.$$

I denote the coefficient corresponding to the n^{th} argument in the expression for $a\,\delta\,\frac{1}{r}$ by r_n; and by $\mathfrak{r}_n$, that part of the coefficient of the n^{th} argument in the development of the quantity $r^3\,\delta\,\frac{1}{r} - \frac{3}{2}\,r^4\left(\delta\,\frac{1}{r}\right)^2$ + &c., which is independent of r_n with a contrary sign. Considering the terms in R multiplied by $\frac{a^2}{a_{,}^3}$ only,

$$\frac{r\,d\,R}{d\,r} = 2\,R.$$

Considering again those separately which are multiplied by $\frac{a^3}{a_{,}^4}$,

$$\frac{r\,d\,R}{d\,r} = 3\,R.$$

I denote as usual by m the ratio of the earth's mean motion to that of the moon, so that

$$m^2 = \frac{n_{,}^2}{n^2} = \frac{m_{,}{}^{*}\,a^3}{\mu\,a_{,}^3}.$$

I now proceed to obtain successively the inequalities of the reciprocal of the radius vector from the equation given above by the method of indeterminate coefficients.

$$\frac{r\,d\,R}{d\,r} = 2\,R \qquad\qquad r_0 = 2\,R_0 = -\frac{m^2}{2},$$

$$r_1\left\{\left\{1 + 3\,e^2\left(1 + \frac{e^2}{8}\right)\right\}\left\{2 - 2\,m\right\}^2 - 1\right\} = (2 - 2\,m)^2\,\mathfrak{r}_1$$

$$-2\left\{\left\{\frac{2}{2 - 2\,m} + 1\right\}\frac{m_{,}}{\mu}\,a\,R_1 + \frac{m}{2 - 2\,m)}\,\frac{m_{,}}{\mu}\,a\,R'_1\right\}$$

---

* More properly $\frac{m_{,}\,a^3}{\mu\,a_{,}^3}$, as will be hereafter explained.

The term multiplied by R' is throughout insensible in the first approximation. Expanding according to ascending powers of m, when all the terms but the first are neglected; and to the same degree of approximation $r_1 = m^2$;

$$\mathfrak{r}_2 = 3\, r_0 \qquad \mathfrak{r}_3 = \frac{3}{2}\, m^2 \qquad \mathfrak{r}_4 = \frac{3}{2}\, m^2 \qquad \mathfrak{r}_5 = 0 \qquad \mathfrak{r}_6 = 0 \qquad \mathfrak{r}_7 = 0$$

$$r_3 \left\{ \{1 + 3\, e^2\} \{2 - 2\, m - \mathrm{c}\}^2 - 1 \right\} = (2 - 2\, m - \mathrm{c})^2\, \mathfrak{r}_3$$

$$-2 \left\{ \left\{ \frac{2 - \mathrm{c}}{2 - 2\, m - \mathrm{c}} + 1 \right\} \frac{m_,}{\mu}\, a\, R_3 + \frac{m}{(2 - 2\, m - \mathrm{c})} \frac{m_,}{\mu}\, a\, R'_3 \right\}$$

It will be shown hereafter that $\mathrm{c} = 1 - \frac{3}{4}\, m^2$

$$r_3 = -\frac{1}{4\, m}\, \mathfrak{r}_3 + \frac{2\,.\,2\,.\,9}{4\,.\,4}\, m = \frac{15}{8}\, m$$

$$r_4 \left\{ \{1 + 3\, e^2\} \{2 - 2 m + \mathrm{c}\}^2 - 1 \right\} = (2 - 2\, m + \mathrm{c})^2\, \mathfrak{r}_4$$

$$-2 \left[\left\{ \frac{2 + \mathrm{c}}{2 - 2\, m + \mathrm{c}} + 1 \right\} \frac{m_,}{\mu}\, a\, R_4 + \frac{2}{(2 - 2\, m + \mathrm{c})} \frac{m_,}{\mu} \right] a\, R'_4$$

$$r_4 = \frac{9}{8}\, \mathfrak{r}_4 + \frac{2\,.\,3}{4\,.\,4}\, m^2 = \frac{33}{16}\, m^2$$

$$r_5 \left\{ \{1 + 3\, e^2\}\, m^2 - 1 \right\} = m^2\, \mathfrak{r}_5 - 2 \left\{ \frac{m_,}{\mu}\, a\, R_5 + \frac{m_,}{\mu}\, a\, R'_5 \right\}$$

$$r_5 = -\frac{2\,.\,3}{4}\, m^2 = -\frac{3}{2}\, m^2$$

$$r_6 \left\{ \{1 + 3\, e^2\} \{2 - 3\, m\}^2 - 1 \right\} = (2 - 3\, m)^2\, \mathfrak{r}_6$$

$$-2 \left[\left\{ \frac{2}{2 - 3\, m} + 1 \right\} \frac{m_,}{\mu}\, a\, R_6 + \frac{m}{(2 - 3\, m)} \frac{m_,}{\mu}\, a\, R'_6 \right]$$

$$r_6 = \frac{4}{3}\, \mathfrak{r}_6 + \frac{2\,.\,2\,.\,2}{3\,.\,8}\, m^2 = \frac{7}{2}\, m^2$$

$$r_7 \left\{ \{1 + 3\, e^2\} \{2 - m\}^2 - 1 \right\} = (2 - m)^2\, \mathfrak{r}_7$$

$$-2\left\{\left\{\frac{2}{2-m}+1\right\}\frac{m_{,}}{\mu}\,a\,R_7+\frac{m}{(2-m)}\,\frac{m_{,}}{\mu}\,a\,R'_7\right\}$$

$$r_7=\frac{4}{3}\,\mathfrak{r}_7-\frac{2.2.3}{3.8}\,m^2=-\frac{m^2}{2}$$

$$\mathfrak{r}_9=\frac{3}{2}\,r_3=\frac{75}{16}\,m \qquad \mathfrak{r}_{10}=\frac{3}{2}\,r_4=\frac{99}{32}\,m^2$$

$$\mathfrak{r}_{11}=\frac{3}{2}\,r_5=-\frac{9}{4}\,m^2 \qquad \mathfrak{r}_{12}=\frac{3}{2}\,r_6=\frac{21}{4}\,m^2$$

$$\mathfrak{r}_{13}=\frac{3}{2}\,r_7=-\frac{3}{4}\,m^2 \qquad \mathfrak{r}_{14}=\frac{3}{2}\,r_5=-\frac{9}{4}\,m^2$$

$$\mathfrak{r}_{15}=\frac{3}{2}\,r_7=-\frac{3}{4}\,m^2 \qquad \mathfrak{r}_{16}=\frac{3}{2}\,r_6=\frac{21}{4}\,m^2$$

$$\mathfrak{r}_{17}=0 \qquad \mathfrak{r}_{18}=0 \qquad \mathfrak{r}_{19}=0$$

$$r_9\left\{\{1+3e^2\}\{2-2m-2\mathrm{c}\}^2-1\right\}=(2-2m-2\mathrm{c})^2\,\mathfrak{r}_9$$

$$-2\left\{\left\{\frac{2-2\mathrm{c}}{2-2m-2\mathrm{c}}+1\right\}\frac{m_{,}a}{\mu}\,R_9+\frac{m}{(2-2m-2\mathrm{c})}\,\frac{m_{,}}{\mu}\,a\,R'_9\right\}$$

$$r_9=-\frac{15}{4}\,m^2$$

$$r_{10}\left\{\{1+3e^2\}\{2-2m+2\mathrm{c}\}^2-1\right\}=(2-2m+2\mathrm{c})^2\,\mathfrak{r}_{10}$$

$$-2\left\{\left\{\frac{2+2\mathrm{c}}{2-2m+2\mathrm{c}}+1\right\}\frac{m_{,}a}{\mu}\,R_{10}+\frac{m}{(2-2m+2\mathrm{c})}\,\frac{m_{,}}{\mu}\,a\,R'_{10}\right\}$$

$$r_{10}=\frac{16.99}{15.32}\,m^2+\frac{2.2.3}{16.4}\,m^2=\frac{7}{2}\,m^2$$

$$r_{11}\left\{\{1+3e^2\}\{\mathrm{c}+m\}^2+1\right\}=(\mathrm{c}+m)^2\,\mathfrak{r}_{11}$$

$$-2\left\{\left\{\frac{\mathrm{c}}{\mathrm{c}+m}+1\right\}\frac{m_{,}}{\mu}\,a\,R_{11}+\frac{m}{(\mathrm{c}+m)}\,\frac{m_{,}}{\mu}\,a\,R'_{11}\right\}$$

$$r_{11}=-\frac{9}{2.4}\,m-\frac{2.2.3}{2.4}\,m=-\frac{21}{8}\,m$$

$$r_{12}\left\{\{1+3e^2\}\{2-3m-\mathrm{c}\}^2-1\right\}=(2-3m-\mathrm{c})^2\,\mathfrak{r}_{12}$$

$$-2\left\{\left\{\frac{2-\mathrm{c}}{2-3m-\mathrm{c}}+1\right\}\frac{m_{,}a}{\mu}R_{12}+\frac{m}{(2-3m-\mathrm{c})}\frac{m_{,}}{\mu}a\,R'_{12}\right\}$$

$$\mathfrak{r}_{12}=-\frac{21}{6.24}m+\frac{2.2.63}{6.8}m=\frac{35}{8}m$$

$$r_{13}\left\{\{1+3e^2\}\{2-m+\mathrm{c}\}^2-1\right\}=(2-m+\mathrm{c})^2\,\mathfrak{r}_{13}$$

$$-2\left\{\left\{\frac{2+\mathrm{c}}{2-m+\mathrm{c}}+1\right\}\frac{m_{,}a}{\mu}R_{13}+\frac{m}{(2-m+\mathrm{c})}\frac{m_{,}}{\mu}a\,R'_{13}\right\}$$

$$r_{13}=-\frac{9.3}{8.4}m^2-\frac{2.2.3}{8.8}m^2=-\frac{33}{32}m^2$$

$$r_{14}\left\{\{1+3e^2\}\{\mathrm{c}-m^2\}-1\right\}=(\mathrm{c}-m)^2\,\mathfrak{r}_{14}$$

$$-2\left\{\left\{\frac{\mathrm{c}}{\mathrm{c}-m}+1\right\}\frac{m_{,}a}{\mu}R_{14}+\frac{m}{(\mathrm{c}-m)}\frac{m_{,}}{\mu}a\,R'_{14}\right\}$$

$$r_{14}=\frac{9}{2.4}m+\frac{2.2.3}{2.4}m=\frac{21}{8}m$$

$$r_{15}\left\{\{1+3e^2\}\{2-m-\mathrm{c}\}^2-1\right\}=(2-m-\mathrm{c})^2\,\mathfrak{r}_{15}$$

$$-2\left\{\left\{\frac{2-\mathrm{c}}{2-m-\mathrm{c}}+1\right\}\frac{m_{,}}{\mu}a\,R_{15}+\frac{m}{(2-m-\mathrm{c})}\frac{m_{,}}{\mu}a\,R'_{15}\right\}$$

$$r_{15}=\frac{3}{2.4}m-\frac{2.2.9}{2.8}m=-\frac{15}{8}m$$

$$r_{16}\left\{\{1+3e^2\}\{2-3m+\mathrm{c}\}^2-1\right\}=(2-3m+\mathrm{c})^2\,\mathfrak{r}_{16}$$

$$-2\left\{\left\{\frac{2+\mathrm{c}}{2-3m+\mathrm{c}}+1\right\}\frac{m_{,}a}{\mu}R_{16}+\frac{m}{(2-3m+\mathrm{c})}\frac{m_{,}a}{\mu}R'_{16}\right\}$$

$$r_{16}=\frac{9.21}{8.4}m^2+\frac{2.2.21}{8.8}m^2=\frac{231}{32}m^2$$

$$r_{17}\left\{\{1+3e^2\}\{4m^2-1\}\right\}=m^2\mathfrak{r}_{17}-2\left\{\frac{m_{,}a}{\mu}R_{16}+\frac{m_{,}}{2\mu}aR'_{17}\right\}$$

$$r_{17} = -\frac{9}{4}m^2$$

$$r_{18}\left\{\{1 + 3e^2\}\{2 - 4m\}^2 - 1\right\} = (2 - 4m)^2\,\mathfrak{r}_{18}$$

$$-2\left\{\left\{\frac{2}{2-4m} + 1\right\}\frac{m_{,}a}{\mu}R_{18} + \frac{m}{(2-4m)}\frac{m_{,}a}{\mu}R'_{18}\right\}$$

$$r_{18} = \frac{2.2.51}{3.8} = \frac{17}{2}m^2$$

$$r_{19}\{1 + 3e^2\}3 = 4\,\mathfrak{r}_{19} - 2\left\{2\frac{m_{,}a}{\mu}R_{19} + \frac{m}{2}\frac{m_{,}a}{\mu}R'_{19}\right\}$$

The preceding examples will serve to show how the inequalities of the reciprocal of the radius vector may generally be determined; in some cases, however, even to obtain the first term it is necessary to carry further the development of the disturbing function.

Thus,

$$r_{65}\{(\mathrm{c} - 2\mathrm{g})^2(1 - 3r_0) - 1\} = (\mathrm{c} - 2\mathrm{g})^2\,\mathfrak{r}_{65} - 2.2.m^2R_{65}$$

$$\mathfrak{r}_{65} = \frac{3}{2}r_{62} \qquad r_{62} = \frac{m^2}{2} \qquad \mathrm{c} = 1 - \frac{3}{4}m^2 \qquad \mathrm{g} = 1 + \frac{3}{4}m^2$$

$$r_0 = \frac{m^2}{6} \qquad R_{65} = \frac{9}{8} + \frac{1}{2}r_{65} \qquad (\mathrm{c} - 2\mathrm{g})^2(1 - 3r_0) = 1 + 4m^2$$

$$r_{65} = \frac{3}{4.2.2} - \frac{2.2}{4}\left(\frac{9}{8} + \frac{1}{2}r_{65}\right)$$

$$r_{65} = -\frac{5}{8}.$$

So that, although the elliptic value of $\frac{\mathrm{a}}{r}$ contains no term multiplied by γ^2, there arises in $\frac{\mathrm{a}}{r}$ the term

$$-\frac{5}{8}e\gamma^2\cos(\xi - 2\eta)$$

independent of m.

Again, in determining r_{105}, which multiplies $\cos(\tau + \xi_{,})$, it is necessary to take into account the term

$$\frac{m_{,}a^2}{2a_{,}^3}r_{105}\cos(\tau + \xi_{,})$$

in the development of R, which gives in

$$2\int \mathrm{d}R + \frac{r\,\mathrm{d}R}{\mathrm{d}r}, \text{ the term } \qquad \frac{2\,m_{,}a^2}{a_{,}^3}\,r_{105}\cos(\tau + \xi_{,})$$

and the equation for determining r_{105} is

$$r_{105}\{1 - 3\,r_0 + 2\,m^2\} = \frac{15}{8}\,m^2\,\frac{a}{a_{,}} \qquad r_0 = \frac{m^2}{6}$$

$$r_{105} = \frac{5\,a}{4\,a_{,}}$$

r_{105} corresponds to the quantity f' of the *Méc. Cél.*, vol. i. p. 279, $\tau + \xi_{,}$ being equal to $n\,t + \varepsilon - \varpi_{,}$, when $c_{,}$ is made equal to unity.

The expression for the longitude (λ') may be found from the equation

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t} = \frac{h}{r'^2} - \frac{1}{r'^2}\int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\,\mathrm{d}t$$

See p. 3.

$$\frac{\mathrm{d}R}{\mathrm{d}\lambda'} = \frac{\mathrm{d}R}{\mathrm{d}\tau}.$$

See *Méc. Cél.*, vol. i. p. 267.

Neglecting γ^2 and the higher terms, and also certain constant factors, which may afterwards be replaced so as to preserve homogeneity,

$$\begin{aligned}
-\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t = {} & \frac{3}{4}\left(1 - \frac{5}{2}\,e^2 - \frac{5}{2}\,e_{,}^2\right) m^2\cos 2\,\tau \\
& - \frac{9}{2}\,m^2\,e\cos(2\,\tau - \xi) + \frac{m^2}{2}\,e\cos(2\,\tau + \xi) \\
& + \frac{21}{8}\,m^2\,e_{,}\cos(2\,\tau - \xi_{,}) - \frac{3}{8}\,m^2\,e_{,}\cos(2\,\tau + \xi_{,}) \\
& - \frac{15}{8}\,m\,e^2\cos(2\,\tau - 2\,\xi) + \frac{3}{8}\,m^2\,e^2\cos(2\,\tau + 2\,\xi) \\
& - \frac{63}{4}\,m^2\,e\,e_{,}\cos(2\,\tau - \xi - \xi_{,}) - \frac{m^2}{4}\,e\,e_{,}\cos(2\,\tau + \xi + \xi_{,}) \\
& + \frac{9}{4}\,m^2\,e\,e_{,}\cos(2\,\tau - \xi - \xi_{,}) + \frac{7}{4}\,m^2\,e\,e_{,}\cos(2\,\tau + \xi - \xi_{,}) \\
& + \frac{51}{8}\,m^2\,e_{,}^2\cos(2\,\tau - 2\,\xi_{,})
\end{aligned}$$

$$\frac{a^2}{r^2} = 1 + \frac{e^2}{2} + 2\,e\cos\xi + \frac{5}{2}\,e^2\cos 2\,\xi + \&\text{c.}$$

See p. 8.

Expanding according to ascending powers of m, retaining only the first term, the coefficients corresponding to the n^{th} argument in the expression for λ being called λ_n, and in $-\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t$, $\mathfrak{R}_n$;

$$\lambda_1=\{2r_1+\mathfrak{R}_1\}\frac{1}{(2-2m)}=\left\{2m^2+\frac{3}{4}m^2\right\}\frac{1}{2}=\frac{11}{8}m^2$$

$$\lambda_3=\{2r_3+r_1+\mathfrak{R}_3+\mathfrak{R}_1\}\frac{1}{(2-2m-\mathrm{c})}=\frac{15}{4}m$$

$$\lambda_4=\{2r_4+r_1+\mathfrak{R}_4+\mathfrak{R}_1\}\frac{1}{(2-2m+\mathrm{c})}$$

$$=\left\{\frac{33}{8}m^2+m^2+\frac{m^2}{2}+\frac{3}{4}m^2\right\}\frac{1}{3}=\frac{17}{8}m^2$$

$$\lambda_5=\frac{2r_5}{m}=-3m$$

$$\lambda_6=\{2r_6+\mathfrak{R}_6\}\frac{1}{(2-3m)}=\left\{7m^2+\frac{21}{8}m^2\right\}\frac{1}{2}=\frac{77}{16}m^2$$

$$\lambda_7=\{2r_7+\mathfrak{R}_7\}\frac{1}{(2-m)}=\left\{-m^2-\frac{3}{8}m^2\right\}\frac{1}{2}=-\frac{11}{16}m^2$$

$$\lambda_9=\{2r_9+r_3+\mathfrak{R}_9+\mathfrak{R}_3\}\frac{1}{(2-2m-2\mathrm{c})}=0.$$

The first term in λ_9 depends upon a further approximation. It can be obtained very easily by means of the expressions for the variations of the elliptic constants:

$$\lambda_{10}=\{2r_{10}+r_4+r_1+\mathfrak{R}_{10}+\mathfrak{R}_4+\frac{5}{4}\mathfrak{R}_1\}\frac{1}{(2-2m+2\mathrm{c})}$$

$$=\left\{7m^2+\frac{33}{16}m^2+m^2+\frac{3}{8}m^2+\frac{m^2}{2}+\frac{15}{16}m^2\right\}\frac{1}{4}=\frac{95}{32}m^2$$

$$\lambda_{11}=\{2r_{11}+r_5\}\frac{1}{(\mathrm{c}+m)}=-\frac{21}{4}m$$

$$\lambda_{12}=\{2r_{12}+r_6+\mathfrak{R}_{12}+\mathfrak{R}_6\}\frac{1}{(2-3m-\mathrm{c})}=\frac{35}{4}m$$

$$\lambda_{13}=\{2r_{13}+r_7+\mathfrak{R}_{13}+\mathfrak{R}_7\}\frac{1}{(2-m+\mathrm{c})}$$

$$=\left\{-\frac{33}{16}m^2-\frac{m^2}{2}-\frac{m^2}{4}-\frac{3}{8}m^2\right\}\frac{1}{3}=-\frac{17}{16}m^2$$

$$\lambda_{14} = \{2\,r_{14} + r_5\}\frac{1}{(\mathrm{c} - m)} = \frac{21}{4}\,m$$

$$\lambda_{15} = \{2\,r_{15} + r_7 + \mathfrak{R}_{15} + \mathfrak{R}_7\}\frac{1}{(2 - m - \mathrm{c})} = -\frac{15}{4}\,m$$

$$\lambda_{16} = \{2\,r_{16} + r_6 + \mathfrak{R}_{16} + \mathfrak{R}_6\}\frac{1}{(2 - 3\,m + \mathrm{c})}$$

$$= \left\{\frac{231}{16}m^2 + \frac{7}{2}m^2 + \frac{7}{4}m^2 + \frac{21}{8}m^2\right\}\frac{1}{3} = \frac{119}{16}\,m^2$$

$$\lambda_{17} = \frac{r_{17}}{m} = -\frac{9}{4}\,m$$

$$\lambda_{18} = \{2\,r_{18} + \mathfrak{R}_{18}\}\frac{1}{(2 - 4\,m)} = \left\{17\,m^2 + \frac{51}{8}\,m^2\right\}\frac{1}{2} = \frac{187}{16}\,m^2$$

$$\lambda_{19} = \{2\,r_{19} + \mathfrak{R}_{19}\}\frac{1}{2} = 0.$$

These examples will serve to show how the terms given by the first approximation in the expressions of M. Plana may be successively verified.

The inequalities of latitude may be determined by means of the equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\,t^2} + \frac{\mu\,z}{r^3} + \frac{m_{,}\,z}{\{r^2 - 2\,r\,r'\cos(\lambda' - \lambda_{,}) + r_{,}^{\,2}\}^{\frac{3}{2}}} = 0$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\,t^2} + \frac{\mu\,z}{r^3} + \frac{m_{,}\,z}{r_{,}^{\,3}} + \frac{3\,m_{,}\,z\,r'\,r\cos(\lambda' - \lambda)}{r_{,}^{\,5}} = 0.$$

This equation is apparently not complicated; but it will be found that it is far simpler to determine the inequalities of latitude according to the method of Clairaut, to which I shall therefore adhere.

The following is the differential equation employed:

$$\left\{\frac{\mathrm{d}^2 s}{\mathrm{d}\,\lambda'^2} + s\right\}\left\{1 - \frac{2}{h^2}\int r^2\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\mathrm{d}\,\lambda\right\}$$

$$+ \frac{r'^2}{h^2}\left\{(1 + s^2)\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right) - r'\,s\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r}\right) - \left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\right)\frac{\mathrm{d}\,s}{\mathrm{d}\,\lambda'}\right\} = 0.$$

Substituting in this equation in the terms multiplied by $m_{,}$, for s, $\gamma\sin(\mathrm{g}\,\lambda' - \nu)$, and for $\frac{\mathrm{d}\,s}{\mathrm{d}\,\lambda'}$, $\gamma\cos(\mathrm{g}\,\lambda' - \nu)$, neglecting the square of the disturbing force and the cube of s, I obtain

$$\frac{\mathrm{d}^2 s}{\mathrm{d}\,\lambda^2} + s - \frac{3\,r^4\,\gamma}{2\,h^2\,r_{,}^{\,3}}\left\{\sin(2\,\lambda - 2\,\lambda_{,} - \mathrm{g}\,\lambda + \nu) - \sin(\mathrm{g}\,\lambda - \nu)\right\}$$

$$-\frac{r^5}{8h^2}\frac{\gamma}{r_,^4}\left\{\frac{21}{2}\cos(\lambda-\lambda_,-g\lambda+\nu)+\frac{15}{2}\cos(\lambda-\lambda_,+g\lambda-\nu)\right.$$

$$\left.+9\cos(3\lambda-3\lambda_,-g\lambda+\nu)+6\cos(3\lambda-3\lambda_,+g\lambda-\nu)\right\}=0.$$

The simplest method of substituting for $\lambda_,$ in terms of λ seems to me to be by first putting for $\lambda_,$ its value in terms of $n_, t$, and since in the elliptic theory

$$\lambda = nt + 2e\left(1-\frac{e^2}{8}\right)\sin\xi + \frac{5}{4}e^2\sin 2\xi + \&c.$$

$$m\lambda = n_,t + 2me\left(1-\frac{e^2}{8}\right)\sin\xi + \frac{5}{4}\,me^2\left(1-\frac{11}{30}\,e^2\right)\sin 2\xi$$

$$+\frac{13}{12}me^3\sin\xi + \frac{103}{96}me^4\sin 4\xi + \&c.$$

It is evident by Lagrange's theorem, that when the higher powers of m are neglected, that is, in the second approximation, it is sufficient to write $m\lambda$ for $n_, t$ and $c_, m\lambda$ for $\xi_,$; also $c_,$ may be considered equal to unity.

$$n_,t + \epsilon_, = \lambda_, - 2e_,\sin(c_,\lambda_, - \varpi_,) + \frac{3}{4}e_,^2\sin(2c_,\lambda_, - 2\varpi_,) + \&c.;$$

therefore, by Lagrange's theorem,

$$\sin(2\lambda - 2\lambda_, - g\lambda + \nu) = \sin(2\lambda - 2n_,t - g\lambda + \nu)$$

$$-2\left\{-2e_,\sin\xi_, + \frac{3}{4}e_,^2\sin 2\xi_,\right\}\cos(2\lambda - 2n_,t - g\lambda + \nu)$$

$$+\frac{d\{8e_,^2\sin^2\xi_,\cos(2\lambda - 2n_,t - g\lambda + \nu)\}}{2n_,dt}$$

Finally, writing $m\lambda$ for $n_, t$, and omitting ν, which accompanies $g\lambda$, $\sin(2\lambda - 2\lambda_, - g\lambda)$

$$= (1 - 4e_,^2)\sin(2\lambda - 2m\lambda - g\lambda)$$

$$- 2e_,\sin(2\lambda - 2m\lambda + c_,m\lambda - g\lambda)$$

$$+ 2e_,\sin(2\lambda - 2m\lambda - c_,m\lambda - g\lambda)$$

$$+ \frac{3}{4}e_,^2\sin(2\lambda - 2m\lambda + 2c_,m\lambda - g\lambda)$$

$$+ \frac{13}{4}e_,^2\sin(2\lambda - 2m\lambda - 2c_,m\lambda - g\lambda)$$

$$\frac{3\,r^4}{2\,h^2\,r_{,}^{3}} = m^2 \left\{ \frac{3}{2} + \frac{9}{4}\,e_{,}^{2} + 3\,e^2 - 6\,e\cos \mathrm{c}\,\lambda \right.$$

$$+ \frac{9}{2} e_{,} \cos \mathrm{c}_{,}\, m\,\lambda + \frac{15}{2}\, e^2 \cos 2\, \mathrm{c}\,\lambda$$

$$- 9\, e\, e_{,} \cos (\mathrm{c}\,\lambda - \mathrm{c}_{,}\, m\,\lambda) - 9\, e\, e_{,} \cos (\mathrm{c}\,\lambda + \mathrm{c}_{,}\, m\,\lambda)$$

$$\left. + \frac{27}{4} e_{,}^{2} \cos 2\, \mathrm{c}_{,}\, m\,\lambda \right\}$$

omitting the constants ϖ and ϖ, which accompany c λ and $\mathrm{c}_{,}\, m\,\lambda$, and ν which accompanies g λ.

Writing $E\,\lambda$ instead of $\lambda - m\,\lambda$,

$$\frac{d^2 s}{d\lambda^2} + s + \frac{3}{2}\left\{1 + 2\,e^2 + \frac{3}{2}\,e_{,}^{2}\right\} m^2 \gamma \sin \mathrm{g}\,\lambda$$

$$- \frac{3}{2}\left\{1 + 2\,e^2 - \frac{5}{2}\,e_{,}^{2}\right\} m^2 \gamma \sin (2\,E\,\lambda - \mathrm{g}\,\lambda)$$

$$+ 3\, m^2 e\,\gamma \sin (\mathrm{c}\,\lambda - \mathrm{g}\,\lambda) - 3\, m^2 e\,\gamma \sin (\mathrm{c}\,\lambda + \mathrm{g}\,\lambda)$$

$$+ 3\, m^2 e\,\gamma \sin (2\,E\,\lambda - \mathrm{c}\,\lambda - \mathrm{g}\,\lambda)$$

$$+ 3\, m^2 e\gamma \sin (2 E\,\lambda + \mathrm{c}\,\lambda - \mathrm{g}\,\lambda) - \frac{9}{4}\, m^2 e\,\gamma \sin (\mathrm{c}_{,}\, m\,\lambda - \mathrm{g}\,\lambda)$$

$$+ \frac{9}{4}\, m^2 e_{,} \gamma \sin (\mathrm{c}_{,}\, m\,\lambda + \mathrm{g}\,\lambda) - \frac{21}{4}\, m^2 e_{,} \gamma \sin (2\,E\,\lambda - \mathrm{c}_{,}\, m\,\lambda - \mathrm{g}\,\lambda)$$

$$+ \frac{3}{4}\, m^2 e_{,} \gamma \sin (2\,E\,\lambda + \mathrm{c}_{,} m\,\lambda - \mathrm{g}\,\lambda) - \frac{15}{4}\, m^2 e^2 \gamma \sin (2\,\mathrm{c}\,\lambda - \mathrm{g}\,\lambda)$$

$$+ \frac{15}{4} m^2 e^2 \gamma \sin (2\,\mathrm{c}\,\lambda + \mathrm{g}\,\lambda) - \frac{15}{4} m^2 e^2 \gamma \sin (2 E\,\lambda - 2 \mathrm{c}\lambda - \mathrm{g}\lambda)$$

$$- \frac{15}{4} m^2 e^2 \gamma \sin (2\,E\,\lambda + 2\,\mathrm{c}\,\lambda - \mathrm{g}\,\lambda) + \frac{9}{2} m^2 e e_{,} \gamma \sin (\mathrm{c}\lambda + \mathrm{c}_{,} m\lambda - \mathrm{g}\lambda)$$

$$- \frac{9}{2}\, m^2 e\, e_{,}\, \gamma \sin (\mathrm{c}\,\lambda + \mathrm{c}_{,}\, m\,\lambda + \mathrm{g}\,\lambda)$$

$$+ \frac{21}{2}\, m^2 e\, e_{,}\, \gamma \sin (2\,E\,\lambda - \mathrm{c}\,\lambda - \mathrm{c}_{,}\, m\,\lambda - \mathrm{g}\,\lambda)$$

$$- \frac{3}{2}\, m^2 e\, e_{,}\, \gamma \sin (2\,E\,\lambda + \mathrm{c}\,\lambda + \mathrm{c}_{,}\, m\,\lambda - \mathrm{g}\,\lambda)$$

$$+ \frac{9}{2} m^2 e e_{,} \gamma \sin (\mathrm{c}\lambda - \mathrm{c}_{,} m \lambda - \mathrm{g}\lambda)$$

$$- \frac{9}{2} m^2 e e_{,} \gamma \sin (\mathrm{c}\lambda - \mathrm{c}_{,} m \lambda + \mathrm{g}\lambda)$$

$$- \frac{3}{2} m^2 e e_{,} \gamma \sin (2 E\lambda - \mathrm{c}\lambda + \mathrm{c}_{,} m \lambda - \mathrm{g}\lambda)$$

$$+ \frac{21}{2} m^2 e e_{,} \gamma \sin (2 E\lambda + \mathrm{c}\lambda - \mathrm{c}_{,} m \lambda \quad \mathrm{g}\lambda)$$

$$- \frac{27}{8} m^2 e_{,}^2 \gamma \sin (2 \mathrm{c}_{,} m \lambda - \mathrm{g}\lambda) + \frac{27}{8} m^2 e_{,}^2 \gamma \sin (2 \mathrm{c}_{,} m \lambda + \mathrm{g}\lambda)$$

$$- \frac{51}{4} m^2 e_{,}^2 \gamma \sin (2 E\lambda - 2 \mathrm{c}_{,} m \lambda - \mathrm{g}\lambda)$$

Integrating [*]

$$- \mathrm{g}^2 + 1 + \frac{3}{2} \left\{ 1 + 2 e^2 + \frac{3}{2} e_{,}^2 \right\} m^2 = 0$$

g[†] $= 1 + \frac{3}{4} \left\{ 1 + 2 e^2 + \frac{3}{2} e_{,}^2 \right\} m^2$

$$s = \left\{ \frac{3}{8} m + \frac{3}{4} m e^2 - \frac{15}{16} m e_{,}^2 \right\} \gamma \sin (2 E \lambda - \mathrm{g}\lambda)$$

---

[*] For the method of integration see *Méc. Cél.*, vol. i. p. 240.

[†] According to M. Plana,

$$\begin{aligned}
\mathrm{g} = 1 &+ \frac{3}{4} m^2 - \frac{9}{32} m^3 - \frac{273}{128} m^4 - \frac{9797}{2048} m^5 - \frac{199273}{24576} m^6 - \frac{6657733}{589824} m^7 \\
&+ \frac{3}{2} m^2 e^2 + \frac{189}{32} m^3 e^2 + \frac{3027}{128} m^4 e^2 + \frac{179955}{2048} m^5 e^2 \\
&- \frac{3}{8} m^2 \gamma^2 + \frac{27}{64} m^3 \gamma^2 + \frac{843}{521} m^4 \gamma^2 + \frac{6393}{4096} m^5 \gamma^2 \\
&+ \frac{9}{8} m^2 e_{,}^2 - \frac{33}{32} m^3 e_{,}^2 - \frac{3261}{256} m^4 e_{,}^2 - \frac{72873}{1024} m^5 e_{,}^2 \\
&+ \frac{27}{32} m^2 e^2 \gamma^2 - \frac{189}{16} m^3 e^2 \gamma^2 + \frac{9}{4} e^2 e_{,}^2 m^2 + \frac{693}{32} m^3 e^2 e_{,}^2 \\
&- \frac{9}{16} e_{,}^2 \gamma^2 m^2 + \frac{99}{64} m^3 e_{,}^2 \gamma^2 - \frac{21}{64} m^2 e^4 + \frac{675}{256} m^3 e^4 \\
&+ \frac{9}{32} m^2 \gamma^4 - \frac{27}{64} m^3 \gamma^4 + \frac{45}{32} m^2 e_{,}^4 - \frac{9}{4} m^3 e_{,}^4 + \frac{45}{32} m^2 \frac{a^2}{a_{,}^2} + \frac{1935}{512} m^3 \frac{a^2}{a_{,}^2}.
\end{aligned}$$

See *Théorie de la Lune*, vol. i. p. 486, and vol. iii. p. 195.

$$-3m^2 e\gamma \sin(c\lambda - g\lambda) - m^2 e\gamma \sin(c\lambda + g\lambda)$$

$$-3m^2 e\gamma \sin(2E\lambda - c\lambda - g\lambda)$$

$$+m^2 e\gamma \sin(2E\lambda + c\lambda - g\lambda)$$

$$+\frac{9}{8} m e_{,}\gamma \sin(c_{,}m\lambda - g\lambda) + \frac{9}{8} m e_{,}\gamma \sin(c_{,}m\lambda + g\lambda)$$

$$+\frac{7}{8} m e_{,}\gamma \sin(2E\lambda - c_{,}m\lambda - g\lambda)$$

$$-\frac{3}{8} m e_{,}\gamma \sin(2E\lambda + c_{,}m\lambda - g\lambda) + \frac{5}{8} e^2\gamma \sin(2c\lambda - g\lambda)$$

$$+\frac{15}{32} m^2 e^2\gamma \sin(2c\lambda + g\lambda) - \frac{15}{16} m e^2\gamma \sin(2E\lambda - 2c\lambda - g\lambda)$$

$$-\frac{15}{32} m^2 e^2\gamma \sin(2E\lambda + 2c\lambda - g\lambda)$$

$$-\frac{9}{2} m^2 e e_{,}\gamma \sin(c\lambda + c_{,}m\lambda - g\lambda)$$

$$-\frac{3}{2} m^2 e e_{,}\gamma \sin(c\lambda + c_{,}m\lambda + g\lambda)$$

$$-\frac{21}{2} m^2 e e_{,}\gamma \sin(2E\lambda - c\lambda - c_{,}m\lambda - g\lambda)$$

$$-\frac{m^2}{2} e e_{,}\gamma \sin(2E\lambda + c\lambda + c_{,}m\lambda - g\lambda)$$

$$-\frac{9}{2} m^2 e e_{,}\gamma \sin(c\lambda - c_{,}m\lambda - g\lambda)$$

$$-\frac{3m^2}{2} e e_{,}\gamma \sin(c\lambda - c_{,}m\lambda + g\lambda)$$

$$+\frac{3}{2} m^2 e e_{,}\gamma \sin(2E\lambda - c\lambda + c_{,}m\lambda - g\lambda)$$

$$+\frac{7}{2} m^2 e e_{,}\gamma \sin(2E\lambda + c\lambda - c_{,}m\lambda - g\lambda)$$

$$+\frac{27}{32} m e_{,}^2\gamma \sin(2c_{,}m\lambda - g\lambda)$$

$$+\frac{27}{32} m e_{,}^2\gamma \sin(2c_{,}m\lambda + g\lambda)$$

$$+\frac{51}{32} m e_{,}^2 \sin(2E\lambda - 2c_{,}m\lambda - g\lambda)$$

This expression for s agrees, so far, with that given by M. Plana, vol. i. p. 498.

From this expression for s in terms of the true longitude (λ'), the expression may be found for the same quantity in terms of the mean longitude ($n\,t$). By Taylor's theorem, if $\lambda = n\,t + \alpha$,

$$s = (s) + \left(\frac{\mathrm{d}\,s}{\mathrm{d}\,\lambda}\right)\alpha + \left(\frac{\mathrm{d}^2\,s}{\mathrm{d}\,\lambda^2}\right)\frac{\alpha^2}{2} + \left(\frac{\mathrm{d}^3\,s}{\mathrm{d}\,\lambda^3}\right)\frac{\alpha^3}{2.3} + \&\text{c.}$$

(s) being the quantity arising from the substitution of $n\,t$ for λ', in the expression for s in terms of λ'.

Having found s,

$$\tan^{-1} s = s - \frac{s^3}{3} + \frac{s^5}{5} - \frac{s^7}{7} + \&\text{c.}$$

The relation between the constants h and a may be obtained by means of the equations

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,t} = \frac{h}{r^2} - \frac{1}{r^2}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t$$

$$\frac{r^2\,\mathrm{d}\,\lambda^2 + \mathrm{d}\,r^2}{\mathrm{d}\,t^2} - \frac{2\,\mu}{r} + \frac{\mu}{a} + 2\int\mathrm{d}\,R = 0$$

where, for simplicity, I have neglected quantities depending upon γ^2.

Neglecting, $m^2\,e^2$, $m^2\,e_{,}^2$, &c. $\left(\frac{\mathrm{d}\,r}{\mathrm{d}\,t}\right)^2 = \frac{e^2\,n^2}{2}$

$2\int\mathrm{d}\,R$ can give no constant term.

Considering only the constant part of the last equation, since the coefficients corresponding to all the arguments must be separately and identically equal to zero, to the degree of approximation intended,

$$h^2\left\{1 + \frac{e^2}{2} + 2\,r_0\right\} = \mu\,a\left\{1 - \frac{e^2}{2} + 2\,r_0\right\}$$

$$h^2 = \mu\,a\,(1 - e^2). \qquad n = \sqrt{\frac{\mu}{a^3}}$$

If $\frac{h}{a^2}\left\{1 + \frac{e^2}{2} + 2\,r_0\right\} = \mathrm{n}$ or $n\left\{1 + 2\,r_0\right\} = \mathrm{n}$

$\mathrm{d}\,\lambda = \mathrm{n}\,\mathrm{d}\,t + \&\text{c.}$

$r_0 = -\frac{m^2}{2}$. If $\sqrt{\frac{\mu}{a^3}}\left\{1 - m^2\right\} = \sqrt{\frac{\mu}{\mathrm{a}^3}}$

$$a = \mathrm{a}\left\{1 - \frac{2}{3}m^2\right\}$$

$$\frac{1 + r_0}{a} = \frac{1 + \frac{2}{3}m^2 - \frac{m^2}{2}}{\mathrm{a}} = \frac{1 + \frac{m^2}{6}}{\mathrm{a}}$$

If $\frac{a}{r} = 1 + r_0 + e\left(1 - \frac{e^2}{8}\right)\cos\xi + \&c.$

$$\xi = c\,n\,t + \varepsilon - \varpi \qquad n = \sqrt{\frac{\mu}{a^3}} \qquad r_0 = -\frac{m^2}{2}$$

$$c^2\left\{1 - \frac{e^2}{8} - \mathrm{r}_2\right\} = 1 - \frac{e^2}{8} - 2\left\{2\,\frac{m_{\prime}}{\mu}\,a\,R_2 + \frac{m_{\prime}}{c\,\mu}\,a\,R'_2\right\}$$

$$c^2\{1 - 3\,r_0\} = 1 - 2\,m^2.$$

But if r is considered as a function of a and $\mathrm{n}\,t$, so that

$$\xi = \mathrm{c}\,\mathrm{n}\,t + \varepsilon - \varpi$$

$$\frac{\mathrm{a}}{r} = 1 + \frac{m^2}{6} + \left(1 + \frac{2}{3}m^2\right)e\cos\xi + \&c.$$

If $\frac{m^2}{6}$ be denoted by r_0, and $\frac{2}{3}m^2$ by r_2,

$$\frac{\mathrm{a}}{r} = 1 + \mathrm{r}_0 + e\,(1 + \mathrm{r}_2)\cos\xi + \&c.$$

$$\frac{r^2}{\mathrm{a}^2} = 1 - 2\,\mathrm{r}_0 - 2\,e\,(1 + \mathrm{r}_2 - 3\,\mathrm{r}_0)\cos\xi + \&c.$$

$$\mathrm{c}^2\{1 + \mathrm{r}_2 - 3\,\mathrm{r}_0\} = 1 + \mathrm{r}_2 - 2\,m^2$$

$$\mathrm{c}^2 = 1 - \frac{3\,m^2}{2} \qquad \mathrm{c} = 1 - \frac{3}{4}m^2.$$

By retaining other terms a further approximation to the value of c may be obtained, including the terms multiplied by e^2, $e_{\prime}^2$, and γ^2; but having already found the development of R in terms of the mean longitudes, it is more simple to derive at once these terms in the values of c and g from the expressions for the variations of the elliptic constants, as I shall show hereafter.

In the present approximation, in order to have the coefficient of $\sin\xi$ in the expression for the longitude, it is sufficient to take

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{h}{r^2} = \mathrm{n} + 2\frac{\sqrt{\mathrm{a}}}{\mathrm{a}^2}\left\{1 + \frac{2}{3}r_0\right\}\left\{1 - \frac{4\,r_0}{3}\right\}\left\{1 - \frac{r_0}{3}\right\}e\cos\xi$$

$$\lambda = \mathrm{n}\,t + \frac{2(1-r_0)}{\mathrm{c}}\,e\sin\xi + \&\text{c.}$$

$$= \mathrm{n}\,t + 2\left\{1 + \frac{m^2}{2} + \frac{3\,m^2}{4}\right\} e\sin\xi$$

$$= \mathrm{n}\,t + 2\left\{1 + \frac{5}{4}\,m^2\right\} e\sin\xi$$

$$\frac{\mathrm{a}}{r} = 1 + \frac{m^2}{6} + \left\{1 + \frac{2}{3}\,m^2\right\} e\cos\xi + \&\text{c.}$$

The quantities a and n being connected by the equation

$$\mathrm{n} = \sqrt{\frac{\mu}{\mathrm{a}^3}}\,.$$

M. Plana has instead of these equations the following:

$$\lambda = \mathrm{n}\,t + 2\left\{1 + \frac{3}{4}\,m^2\right\} e\sin\xi + \&\text{c.}$$

$$\frac{\mathrm{a}}{r} = 1 + \frac{m^2}{6} + \left(1 + \frac{m^2}{6}\right) e\cos\xi + \&\text{c.}$$

The latter expressions may be derived from mine by the substitution of $e\left(1 - \frac{m^2}{2}\right)$ instead of e, which substitution is not inadmissible, on account of the arbitrary nature of the quantity e. I shall, however, retain my expressions, although regretting these discrepancies, as they influence many terms in the subsequent approximations.

I shall now proceed to consider some of the terms which depend upon a further approximation, first determining the second term in the *evection.*

The term $\frac{15}{8}m$ in r_3, already found, see p. 37, gives the term $\frac{15}{16}m^3 e\cos(2\tau - \xi)$ in δR.

$$\frac{\{2-2m-\mathrm{c}\}^2}{\{2-2m-\mathrm{c}\}^2 - 1} = \frac{1-4m}{-4m + \frac{11}{2}m^2} = -\frac{1}{4m}\left\{1 - 4m + \frac{11}{8}\,m\right\}$$

$$= -\frac{1}{4m} + \frac{21}{32}$$

Again,

$$-\frac{2}{\{2-2m-\mathrm{c}\}^2 - 1}\left\{\frac{2-\mathrm{c}}{2-2m-\mathrm{c}} - 1\right\}$$

$$= \frac{2}{4m}\left\{1 + \frac{11}{8}m\right\}\left\{\frac{1}{1-2m} + 1\right\}$$

$$= \frac{1}{m}\left\{1 + \frac{11}{8}m\right\}\left\{1 + m\right\}$$

$$= \frac{1}{m} + \frac{19}{8}$$

Hence

$$r_3 = -\left\{\frac{1}{4m} - \frac{21}{32}\right\}\left\{\frac{3}{2}m^2 + \frac{19\,.\,3}{6\,.\,2}m^3 + \frac{3\,.\,15}{6\,.\,8}m^3\right\}$$

$$+\left\{\frac{1}{m} + \frac{19}{8}\right\}\left\{\frac{9}{4}m^2 + \frac{15}{16}m^3\right\}$$

$$= \frac{15}{8}m + \frac{187}{32}m^2$$

$$\lambda_3 = \left\{2r_3 + r_1 - \frac{\{1 + 2r_3\}}{\{1-2m\}}\frac{9}{2}m^2\right\}\frac{1}{(1-2m)}$$

$$= \left\{\frac{15}{4}m + \frac{187}{16}m^2 + m^2\right.$$

$$\left. - \left\{1 + 2m + \frac{15}{4}m\right\}\frac{9}{2}m^2\right\}\frac{1}{(1-2m)}$$

$$= \frac{15}{4}m + \frac{203}{16}m^2 - \frac{9}{2}m^2 + \frac{15}{2}m^2 + \frac{3}{4}m^2$$

$$= \frac{15}{4}m + \frac{263}{16}m^2$$

In order to obtain the terms in $\delta\frac{1}{r}$ depending upon the next approximation, it is necessary to obtain the following developments: δR, $\frac{3}{2}r^4\left(\delta\frac{1}{r}\right)^2$ and $\delta\,\mathrm{d}R$. The expression for $\frac{r\,\mathrm{d}R}{\mathrm{d}r}$ results immediately from that for R, and does not require separate consideration: this is the case however far the approximation be carried. $\delta\,\mathrm{d}R$ is easily deduced from δ R by the addition of a few terms, which result from differentiating $\delta\frac{1}{r}$ and $\delta\lambda$ with respect to $n_{,}t$ only, and in the first approximation, $\frac{\mathrm{d}R}{\mathrm{d}\lambda}$ results by a very simple process from R. In order to obtain the terms in $\delta\lambda$ depending upon the next approximation it is necessary to obtain the developments of $\delta \frac{\mathrm{d}R}{\mathrm{d}\lambda}$, $\left(\delta \frac{1}{r}\right)^2$ and of $\frac{\mathrm{a}^2}{r^2} \int \frac{\mathrm{d}R}{\mathrm{d}\lambda} \mathrm{d}t$.

Neglecting γ^2

$$\delta R = \left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) \delta r + \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) \delta \lambda = -a \left(\frac{\mathrm{d}R}{\mathrm{d}a}\right) r \delta \frac{1}{r} + \left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right) \delta \lambda.$$

Considering r as a function of n t, a must be written throughout instead of a, $r_0 = \frac{m^2}{6}$, and I find

$$\begin{aligned} -\frac{\mathrm{a}^2}{\mu}\left(\frac{\mathrm{d}R}{\mathrm{d}\,\mathrm{a}}\right) = {} & \frac{m^2}{2} + \frac{3}{2} m^2 \cos 2\tau - m^2 e \cos \xi - \frac{9}{2} m^2 e \cos (2\tau - \xi) \\ & + \frac{3}{2} m^2 e \cos (2\tau + \xi) + \frac{3}{2} m^2 e_{,} \cos \xi_{,} \\ & + \frac{21}{4} m^2 e_{,} \cos (2\tau - \xi_{,}) - \frac{3}{4} m^2 e_{,} \cos (2\tau + \xi_{,}) \end{aligned}$$

Taking $\mathrm{r} = \mathrm{a}(1 - e \cos \xi)$ and

$$\begin{aligned} \mathrm{a}\,\delta \frac{1}{r} = {} & \frac{m^2}{6} + m^2 \cos 2\tau + \frac{2}{3} m^2 e \cos \xi \\ & + \left\{\frac{15}{8} m + \frac{187}{32} m^2\right\} e \cos (2\tau - \xi) \\ & + \frac{33}{16} m^2 e \cos (2\tau + \xi) - \frac{3}{2} m^2 e_{,} \cos \xi_{,} + \frac{7}{2} m^2 e_{,} \cos (2\tau - \xi_{,}) \\ & - \frac{m^2}{2} e_{,} \cos (2\tau + \xi_{,}) \end{aligned}$$

$$\begin{aligned} \mathrm{r}\,\delta \frac{1}{r} = {} & \frac{m^2}{6} + m^2 \cos 2\tau + \frac{m^2}{2} e \cos \xi + \left\{\frac{15}{8} m + \frac{171}{32} m^2\right\} e \cos (2\tau - \xi) \\ & + \frac{25}{16} m^2 e \cos (2\tau + \xi) - \frac{3}{2} m^2 e_{,} \cos \xi_{,} + \frac{7}{2} m^2 e_{,} \cos (2\tau - \xi_{,}) \\ & - \frac{m^2}{2} e_{,} \cos (2\tau + \xi_{,}) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{a}}{\mu} \frac{\mathrm{d}R}{\mathrm{d}\tau} = {} & \frac{3}{2} m^2 \sin 2\tau - \frac{9}{2} m^2 e \sin (2\tau - \xi) + \frac{3}{2} m^2 e \sin (2\tau + \xi) \\ & + \frac{21}{4} m^2 e_{,} \sin (2\tau - \xi_{,}) - \frac{3}{4} m^2 e_{,} \sin (2\tau + \xi_{,}) \end{aligned}$$

$$\delta\lambda = \frac{11}{8} m^2 \sin 2\tau + \frac{5}{2} m^2 e \sin \xi + \left\{ \frac{15}{4} m + \frac{263}{16} m^2 \right\} e \sin (2\tau - \xi)$$

$$+ \frac{17}{8} m^2 e \sin (2\tau + \xi) - 3\, m\, e_{,} \sin \xi_{,} + \frac{77}{16} m^2 e_{,} \sin (2\tau - \xi_{,})$$

$$- \frac{11}{16} m^2 e_{,} \sin (2\tau + \xi_{,})$$

$$\frac{a}{\mu} \delta R = \left\{ \frac{1}{12} + \frac{3}{4} + \frac{33}{32} \right\} m^4 + \left\{ \frac{1}{2} + \frac{1}{4} \right\} m^4 \cos 2\tau + \left\{ \left\{ \frac{45}{32} + \frac{45}{16} \right\} m^3 \right.$$

$$\left. + \left\{ \frac{1}{4} + \frac{513}{128} + \frac{75}{64} - \frac{1}{6} - \frac{9}{4} + \frac{3}{4} + \frac{789}{64} + \frac{51}{32} - \frac{99}{32} + \frac{33}{32} \right\} m^4 \right\} e \cos \xi$$

$$+ \left\{ \frac{15}{16} m^3 + \left\{ \frac{171}{64} + \frac{3}{8} - \frac{1}{2} - \frac{3}{4} + \frac{15}{8} \right\} m^4 \right\} e \cos (2\tau - \xi)$$

$$+ \left\{ \frac{25}{32} + \frac{3}{8} - \frac{1}{2} + \frac{1}{4} - \frac{15}{8} \right\} m^4 e \cos (2\tau + \xi)$$

$$+ \left\{ -\frac{3}{4} + \frac{21}{8} - \frac{3}{8} + \frac{1}{4} + \frac{21}{8} - \frac{3}{8} + \frac{231}{64} - \frac{33}{64} + \frac{231}{64} - \frac{33}{64} \right\} m^4 e_{,} \cos \xi_{,}$$

$$+ \left\{ -\frac{9}{4} m^3 + \left\{ \frac{7}{4} - \frac{9}{8} + \frac{3}{4} + \frac{7}{8} \right\} m^4 \right\} e_{,} \cos (2\tau - \xi_{,})$$

$$+ \left\{ \frac{9}{4} m^3 + \left\{ -\frac{1}{4} - \frac{9}{8} + \frac{3}{4} - \frac{1}{8} \right\} m^2 \right\} e_{,} \cos (2\tau + \xi_{,})$$

$$+ \left\{ \frac{3}{4} - \frac{33}{32} \right\} m^4 \cos 4\tau$$

$$+ \left\{ \left\{ \frac{45}{32} - \frac{45}{16} \right\} m^3 + \left\{ \frac{513}{128} - \frac{9}{4} - \frac{789}{64} + \frac{99}{32} \right\} m^4 \right\} e \cos (4\tau - \xi)$$

$$+ \left\{ \frac{75}{64} + \frac{3}{4} - \frac{51}{32} - \frac{33}{32} \right\} m^4 e \cos (4\tau + \xi)$$

$$+ \left\{ \frac{21}{8} + \frac{21}{8} - \frac{231}{64} - \frac{231}{64} \right\} m^4 e_{,} \cos (4\tau - \xi_{,})$$

$$+ \left\{ -\frac{3}{8} - \frac{3}{8} + \frac{33}{64} + \frac{33}{64} \right\} m^4 e_{,} \cos (4\tau + \xi_{,})$$

$$\frac{a}{\mu} \delta R = \frac{179}{96} m^4 + \frac{3}{4} m^4 \cos 2\tau + \left\{ \frac{135}{32} m^3 + \frac{5999}{384} m^2 \right\} e \cos \xi$$

$$+ \left\{ \frac{15}{16} m^3 + \frac{235}{64} m^4 \right\} e \cos (2\tau - \xi)$$

$$- \frac{31}{32} m^4 e \cos (2\tau + \xi) + \frac{163}{16} m^4 e_{,} \cos \xi_{,}$$

$$+\left\{-\frac{9}{4}m^3+\frac{9}{4}m^4\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{\frac{9}{4}m^3-\frac{3}{4}m^4\right\}e_{,}\cos(2\tau+\xi_{,})-\frac{9}{32}m^4\cos 4\tau$$

$$+\left\{-\frac{45}{32}m^3-\frac{957}{128}m^4\right\}e\cos(4\tau-\xi)$$

$$-\frac{45}{64}m^4 e\cos(4\tau+\xi)-\frac{63}{32}m^4 e_{,}\cos(4\tau-\xi_{,})$$

$$+\frac{9}{32}m^4 e_{,}\cos(4\tau+\xi_{,})$$

In the equation which serves to determine the expression for the radius vector, d R signifies the differential of R, only as regards the coordinates of the moon. Hence d R = the differential of R, supposing only n t variable + the differential of R with regard to $\text{n}_{,}$ t, only inasmuch as it is contained in the terms in r, λ and s due to the perturbations; hence d R = the differential of R, supposing only $n\,t$ variable

$$+\left(\frac{\text{d}R}{\text{d}r}\right)\text{d}\,\delta r+\left(\frac{\text{d}R}{\text{d}\lambda}\right)\text{d}\,.\,\delta\lambda+\left(\frac{\text{d}R}{\text{d}s}\right)\text{d}\,.\,\delta s$$

d . δr, d $\delta\lambda$ and d δs being restrained to mean the differentials of those quantities with regard to $\text{n}_{,}t$ only.

$$\left(\frac{\text{d}R}{\text{d}r}\right)\text{d}\,\delta r+\left(\frac{\text{d}R}{\text{d}\lambda}\right)\text{d}\,\delta\lambda=-\text{a}\left(\frac{\text{d}R}{\text{d}\text{a}}\right)\text{d}\,.\,\text{r}\,\delta\frac{1}{r}+\frac{\text{d}R}{\text{d}\tau}\text{d}\,\delta\lambda$$

$$\text{d}\,\text{r}\,\delta\frac{1}{r}=2m^3\sin 2\tau+\frac{15}{4}m^2 e\sin(2\tau-\xi)+\frac{3}{2}m^3 e_{,}\sin\xi_{,}$$

$$+\frac{21}{2}m^3 e_{,}\sin(2\tau-\xi_{,})-\frac{m^3}{2}e_{,}\sin(2\tau+\xi_{,})$$

$$\text{d}\,\delta\lambda=-\frac{11}{4}m^3\cos 2\tau-\frac{15}{2}m^2 e\cos(2\tau-\xi)-3m^2 e_{,}\cos\xi_{,}$$

$$-\frac{231}{16}m^3 e_{,}\cos(2\tau-\xi_{,})+\frac{11}{16}m^3 e_{,}\cos(2\tau-\xi_{,})$$

$$\left(\frac{\text{d}R}{\text{d}r}\right)\text{d}\,\delta r+\left(\frac{\text{d}R}{\text{d}\lambda}\right)\text{d}\,\delta\lambda=\frac{\text{a}}{\mu}R'$$

$$=-\frac{135}{16}m^4 e\sin\xi+\frac{15}{8}m^4 e\sin(2\tau-\xi)$$

$$+\left\{\frac{3}{4}-\frac{63}{8}-\frac{3}{8}-\frac{693}{64}-\frac{33}{64}+\frac{231}{32}+\frac{33}{32}+\frac{21}{4}+\frac{3}{4}\right\}m^5 e_{,}\sin\xi_{,}$$

$$-\frac{9}{4}m^4 e_{,}\sin(2\tau-\xi_{,})-\frac{9}{4}m^4 e_{,}\sin(2\tau+\xi_{,})-\frac{45}{16}m^4 e\sin(4\tau-\xi)$$

$$\frac{1}{\iota} R' = -\frac{135}{16} m^4 e \sin \xi + \frac{15}{8} m^4 e \sin (2\tau - \xi) - \frac{147}{32} m^5 c_{,} \sin \xi_{,}$$

$$- \frac{9}{4} m^4 e_{,} \sin (2\tau - \xi_{,}) - \frac{9}{4} m^4 e_{,} \sin (2\tau + \xi_{,})$$

$$- \frac{45}{16} m^4 e \sin (4\tau - \xi)$$

$$\left(\delta \frac{1}{r}\right)^2 = \left\{\frac{1}{36} + \frac{1}{2}\right\} m^4 + \frac{m^4}{3} \cos 2\tau + \left\{\frac{15}{8} m^3 + \left\{\frac{2}{9} + \frac{187}{32} + \frac{33}{16} \overset{+\frac{95}{16}}{(} \right\} m^4 \right\} e \cos \xi$$

$$+ \left\{\frac{5}{8} m^3 + \left\{\frac{187}{96} + \frac{2}{3}\right\} m^4 \right\} e \cos (2\tau - \xi)$$

$$+ \left\{\frac{33}{48} + \frac{2}{3}\right\} m^4 e \cos (2\tau + \xi)$$

$$+ \left\{-\frac{3}{6} + \frac{7}{2} - \frac{1}{2}\right\} m^2 e_{,} \cos \xi_{,} + \left\{\frac{7}{6} - \frac{3}{2}\right\} m^4 e_{,} \cos (2\tau - \xi_{,})$$

$$+ \left\{-\frac{1}{6} - \frac{3}{2}\right\} m^4 e_{,} \cos (2\tau + \xi_{,}) + \frac{m^4}{2} \cos 4\tau$$

$$+ \left\{\frac{15}{8} m^3 + \left\{\frac{187}{32} \overset{+\frac{95}{16}}{} \right\} m^4 \right\} e \cos 4\tau - \xi)$$

$$+ \frac{33}{16} m^4 e \cos (4\tau + \xi) + \frac{7}{2} m^4 e_{,} \cos (4\tau - \xi_{,}) - \frac{m^4}{2} e_{,} \cos (4\tau + \xi_{,}).$$

$$= \frac{19}{36} m^4 + \frac{m^4}{3} \cos 2\tau + \left\{\frac{15}{8} m^3 + \frac{\cancel{2341}}{288} m^4 \right\} e \cos \xi \quad \boxed{4051}$$

$$+ \left\{\frac{5}{8} m^3 + \frac{251}{96} m^4 \right\} e \cos (2\tau - \xi) + \frac{65}{48} m^4 e \cos (2\tau + \xi)$$

$$+ \frac{5}{2} m^4 e_{,} \cos \xi_{,} - \frac{m^4}{3} e_{,} \cos (2\tau - \xi_{,}) - \frac{5}{3} m^4 e_{,} \cos (2\tau + \xi_{,})$$

$$+ \frac{m^4}{2} \cos 4\tau + \left\{\frac{15}{8} m^3 + \frac{\cancel{187}}{32} m^4 \right\} e \cos (4\tau - \xi) \quad \boxed{377}$$

$$+ \frac{33}{16} m^4 e \cos (4\tau + \xi) + \frac{7}{2} m^4 e_{,} \cos (4\tau - \xi_{,}) - \frac{m^4}{2} \cos (4\tau + \xi_{,})$$

$$r^4 = a^4 (1 - 4 e \cos \xi) \quad \boxed{3443}$$

$$\frac{3}{2} \frac{r^4}{a^2} \left(\delta \frac{1}{r}\right)^2 = \frac{19}{24} m^4 + \frac{m^4}{2} \cos 2\tau + \left\{\frac{45}{16} m^3 + \frac{\cancel{1733}}{192} m^4 \right\} e \cos \xi$$

1095

$$+\left\{\frac{15}{16}m^3+\frac{187}{64}m^4\right\}e\cos(2\tau-\xi)$$

$$+\frac{33}{32}m^4 e\cos(2\tau+\xi)+\frac{15}{4}m^4 e_{\prime}\cos\xi_{\prime}$$

$$-\frac{m^4}{2}e_{\prime}\cos(2\tau-\xi_{\prime})-\frac{5}{2}m^4 e_{\prime}\cos(2\tau+\xi_{\prime})$$

$$+\frac{3}{4}m^4\cos 4\tau+\left\{\frac{45}{16}m^3+\frac{\cancel{465}}{64}m^4\right\}e\cos(4\tau-\xi)$$

$$+\frac{51}{32}m^4 e\cos(4\tau+\xi)+\frac{21}{4}m^4 e_{\prime}\cos(4\tau+\xi_{\prime})$$

$$-\frac{3}{4}m^4 e_{\prime}\cos(4\tau+\xi_{\prime}).$$

I shall now attempt to determine some of the terms in the reciprocal of the radius vector depending upon the second approximation, integrating as before by the method of indeterminate coefficients the equation

$$\frac{d^2.\mathrm{r}^2}{2\,dt^2}-\frac{d^2.\mathrm{r}^3\delta\frac{1}{r}}{dt^2}+\frac{3\,d^2.\mathrm{r}^4\left(\delta\frac{1}{r}\right)^2}{2\,dt^2}-\frac{2\,d^2.\mathrm{r}^5\left(\delta\frac{1}{r}\right)^3}{dt^2}-\frac{\mu}{r}+\frac{\mu}{a}$$

$$+2\int dR+r\left(\frac{dR}{dr}\right)=0$$

$$r_1\left\{(2-2m)^2-1\right\}=(2-2m)^2\,\mathfrak{r}_1-2\left\{\left\{\frac{2}{2-2m}+1\right\}\frac{m_{\prime}\mathrm{a}}{\mu}R_1\right.$$

$$\left.+\frac{m}{(2-2m)}\cdot\frac{m_{\prime}}{\mu}\mathrm{a}\,R'_1\right\}$$

$$(2-2m)^2=4-8m+4m^2 \qquad \mathfrak{r}_1=\frac{m^4}{2} \qquad R'_1=0$$

$$3r_1\left\{1-\frac{8}{3}m+\frac{4}{3}m^2\right\}=2m^4-2\left\{2+m+m^2\right\}\left\{-\frac{3}{4}m^2+\frac{3}{4}m^4\right\}$$

$$=3m^2+\frac{3}{2}m^3+\frac{m^4}{2}$$

$$r_1\left\{1-\frac{8}{3}m+\frac{4}{3}m^2\right\}=m^2+\frac{m^3}{2}+\frac{m^4}{6}$$

$$r_1=\left\{m^2+\frac{m^3}{2}+\frac{m^4}{6}\right\}\left\{1+\frac{8}{3}m+\frac{52}{9}m^2\right\}$$

$$=m^2+\frac{19}{6}m^3+\frac{131}{18}m^4$$

$$r_5\left\{m^2-1\right\}=m^2\,\mathfrak{r}_5-2\left\{\frac{m_{\prime}}{\mu}\mathrm{a}\,R_5+\frac{m_{\prime}}{\mu}\mathrm{a}\,R'_5\right\}$$

$$= -2\left\{\frac{3}{4}m^2 + \frac{163}{16}m^4 + \frac{147}{32}m^4\right\}$$

$$r_5 = \left\{-\frac{3}{2}m^2 + \frac{473}{16}m^4\right\}\left\{1 + m^2\right\} = -\frac{3}{2}m^2 + \frac{449}{16}m^4$$

$$r_6\left\{(2-3m)^2 - 1\right\} = (2-3m)^2\,\mathfrak{r}_6 - 2\left\{\left\{\frac{2}{2-3m} + 1\right\}\frac{m_{\prime}}{\mu}\mathrm{a}\,R_6 + \frac{m}{(2-3m)}\frac{m_{\prime}}{\mu}\mathrm{a}\,R'_6\right\}$$

$$= -2m^4 - 2\left\{\left\{2 + \frac{3}{2}m + \frac{9}{4}m^2\right\}\right.$$

$$\left.\left\{-\frac{21}{8}m^2 - \frac{9}{4}m^3 + \frac{9}{4}m^4\right\} + \frac{9}{8}m^4\right\}$$

$$r_6 = -\frac{2}{3}m^4 - \frac{3}{4}m^4 - \left\{1 + 4m + 13m^2\right\}$$

$$\left\{4 + 3m + \frac{9}{2}m^2\right\}\left\{-\frac{7}{8}m^2 - \frac{3}{4}m^3 + \frac{3}{4}m^4\right\}$$

$$= \frac{7}{2}m^2 + \frac{157}{8}m^3 + \frac{3349}{48}m^4$$

$$r_7\left\{(2-m)^2 - 1\right\} = (2-m)^2\,\mathfrak{r}_7 - 2\left\{\left\{\frac{2}{2-m} + 1\right\}\frac{m_{\prime}}{\mu}\mathrm{a}\,R_7 + \frac{m}{(2-m)}\frac{m_{\prime}}{\mu}\mathrm{a}\,R'_7\right\}$$

$$r_7\left\{3 - 4m + m^2\right\} = -10m^4 - 2\left\{2 + \frac{m}{2} + \frac{m^2}{4}\right\}$$

$$\left\{\frac{3}{8}m^2 + \frac{9}{4}m^3 - \frac{3}{4}m^4 + \frac{9}{8}m^4\right\}$$

$$r_7 = -\frac{10}{3}m^4 - \frac{3}{4}m^4$$

$$-\left\{2 + \frac{m}{2} + \frac{m^2}{4}\right\}\left\{\frac{m^2}{4} + \frac{3}{2}m^3 - \frac{m^4}{2}\right\}\left\{1 + \frac{4}{3}m + \frac{13}{9}m^2\right\}$$

$$= -\frac{m^2}{2} - \frac{91}{24}m^3 - \frac{1265}{144}m^4$$

$$r_{131}\left\{(4-4m)^2 - 1\right\} = (4-4m)^2\,\mathfrak{r}_{131}$$

$$-2\left\{\left\{\frac{4}{4-4m} + 1\right\}\frac{m_{\prime}}{\mu}\mathrm{a}\,R_{131} + \frac{m}{(4-4m)}\frac{m_{\prime}}{\mu}\mathrm{a}\,R'_{131}\right\}$$

$$15\,r_{131} = \frac{16\,.\,3}{4}m^4 + \frac{2\,.\,2\,.\,9}{32}m^4$$

$$r_{131} = \frac{7}{8}m^4$$

$$r_{134}\left\{(4-5m)^2 - 1\right\} - (4-5m)^2\,\mathfrak{r}_{134}$$

$$-2\left\{\left\{\frac{4}{4-5m}+1\right\}\frac{m_{,}a}{\mu}R_{134}+\frac{m}{(4-5m)}\frac{m_{,}}{\mu}a\,R'_{134}\right\}$$

$$15\,r_{134}=\frac{16\,.\,21}{4}m^4+\frac{2\,.\,2\,.\,63}{32}m^4$$

$$r_{134}=\frac{49}{8}m^4$$

$$r_{135}\left\{(4-3m)^2-1\right\}=(4-3m)^2\,\mathrm{r}_{135}$$

$$-2\left\{\left\{\frac{4}{4-3m}+1\right\}\frac{m_{,}a}{\mu}R_{135}+\frac{m}{(4-3m)}\frac{m_{,}}{\mu}\mathrm{a}\,R'_{135}\right\}$$

$$15\,r_{135}=-\frac{16\,.\,3}{4}m^4-\frac{2\,.\,2\,.\,9}{32}m^4$$

$$r_{135}=-\frac{7}{8}m^4$$

The expression for the longitude is to be obtained from the equation

$$\frac{d\lambda}{dt}=\frac{h}{r^2}-\frac{1}{r^2}\int\frac{dR}{d\lambda}dt$$

$$\frac{\mathrm{a}}{\mu}\delta\,.\,\frac{dR}{d\lambda}=-\mathrm{a}\frac{d\,.\left(\frac{dR}{d\lambda}\right)}{d\mathrm{a}}\,\mathrm{r}\,\delta\frac{1}{r}+\frac{d\,.\left(\frac{dR}{d\lambda}\right)}{d\lambda}\delta\lambda$$

$$\frac{\mathrm{a}}{\mu}\frac{dR}{d\lambda}=\frac{3}{2}m^2\sin 2\tau-\frac{9}{2}m^2e\sin(2\tau-\xi)+\frac{3}{2}m^2e\sin(2\tau+\xi)$$

$$+\frac{21}{4}m^2e_{,}\sin(2\tau-\xi_{,})-\frac{3}{4}m^2e_{,}\sin(2\tau+\xi_{,})$$

$$-\frac{\mathrm{a}^2}{\mu}\frac{d\,.\left(\frac{dR}{d\lambda}\right)}{d\mathrm{a}}=-3m^2\sin 2\tau+9m^2e\sin(2\tau-\xi)-3m^2e\sin(2\tau+\xi)$$

$$-\frac{21}{2}m^2e_{,}\sin(2\tau-\xi_{,})+\frac{3}{2}m^2e_{,}\sin(2\tau+\xi_{,})$$

$$\mathrm{r}\,\delta\frac{1}{r}=\frac{m^2}{6}+m^2\cos 2\tau+\frac{m^2}{2}e\cos\xi+\left\{\frac{15}{8}m+\frac{171}{32}m^2\right\}e\cos(2\tau-\xi)$$

$$+\frac{25}{16}m^2e\cos(2\tau+\xi)-\frac{3}{2}m^2e_{,}\cos\xi_{,}+\frac{7}{2}m^2e_{,}\cos(2\tau-\xi_{,})$$

$$-\frac{m^2}{2}e_{,}\cos(2\tau+\xi_{,})$$

$$\frac{\mathrm{a}}{\mu}\,\frac{\mathrm{d}.\dfrac{\mathrm{d}R}{\mathrm{d}\lambda}}{\mathrm{d}\lambda} = 3\,m^2 \cos 2\,\tau - 9\,m^2\, e \cos(2\,\tau - \xi) + 3\,m^2\, e \cos(2\,\tau + \xi)$$

$$+ \frac{21}{2}\,m^2\, e_{,} \cos(2\,\tau - \xi_{,}) - \frac{3}{2}\,m^2\, e_{,} \cos(2\,\tau + \xi_{,})$$

$$\delta\,\lambda = \frac{11}{8}\,m^2 \sin 2\,\tau + \frac{5}{2}\,m^2 e \sin \xi + \left\{\frac{15}{4}\,m + \frac{263}{16}\,m^2\right\} e \sin(2\,\tau - \xi)$$

$$+ \frac{17}{8}\,m^2\, e \sin(2\,\tau + \xi) - 3\,m\, e_{,} \sin \xi_{,} + \frac{77}{16}\,m^2\, e_{,} \sin(2\,\tau - \xi_{,})$$

$$- \frac{11}{16}\,m^2\, e_{,} \sin(2\,\tau + \xi_{,})$$

$$\frac{\mathrm{a}}{\mu}\,\delta\,\frac{\mathrm{d}R}{\mathrm{d}\,\lambda} = -\frac{m^4}{2} \sin 2\,\tau + \Bigg\{ \left\{ -\frac{45}{16} - \frac{45}{8} \right\} m^3 + \bigg\{ -\frac{513}{64} + \frac{75}{32} - \frac{9}{2} - \frac{3}{2} - \frac{789}{32}$$

$$+ \frac{51}{16} - \frac{99}{16} - \frac{33}{16} \bigg\} m^4 \Bigg\} e \sin \xi + \left\{ -\frac{3}{4} + \frac{3}{2} - \frac{15}{4} \right\} m^4\, e \sin(2\,\tau - \xi).$$

$$+ \left\{ -\frac{3}{4} - \frac{1}{2} + \frac{15}{4} \right\} m^4\, e \sin(2\,\tau + \xi) + \bigg\{ -\frac{21}{4} - \frac{3}{4} + \frac{21}{4} + \frac{3}{4} - \frac{231}{32}$$

$$- \frac{33}{32} + \frac{231}{32} + \frac{33}{32} \bigg\} m^4\, e_{,} \sin \xi_{,} + \left\{ \frac{9}{2}\,m^3 + \left\{ \frac{9}{4} - \frac{7}{4} \right\} m^4 \right\} e_{,} \sin(2\,\tau - \xi_{,})$$

$$+ \left\{ -\frac{9}{2}\,m^3 + \left\{ \frac{9}{4} + \frac{1}{4} \right\} m^4 \right\} e_{,} \sin(2\,\tau + \xi_{,}) + \left\{ -\frac{3}{2} + \frac{33}{16} \right\} m^4 \sin 4\,\tau$$

$$+ \left\{ \left\{ -\frac{45}{16} + \frac{45}{8} \right\} m^3 + \left\{ -\frac{513}{64} + \frac{9}{2} + \frac{789}{32} - \frac{99}{16} \right\} m^4 \right\} e\ \mathrm{sln}\ (4\,\tau - \xi)$$

$$+ \left\{ -\frac{75}{32} - \frac{3}{2} + \frac{51}{16} + \frac{33}{16} \right\} m^4\, e \sin(4\,\tau + \xi)$$

$$+ \left\{ -\frac{21}{4} - \frac{21}{4} + \frac{231}{32} + \frac{231}{32} \right\} m^4\, e_{,} \sin(4\,\tau - \xi_{,})$$

$$+ \left\{ \frac{3}{4} + \frac{3}{4} - \frac{33}{32} - \frac{33}{32} \right\} m^4\, e_{,} \sin(4\,\tau + \xi_{,})$$

$$\frac{\mathrm{a}}{\mu}\,\delta\,\frac{\mathrm{d}R}{\mathrm{d}\lambda} = -\frac{m^4}{2} \sin 2\,\tau + \left\{ -\frac{135}{16}\,m^3 - \frac{2649}{64}\,m^4 \right\} e \sin \xi - 3\,m^4\, e \sin(2\,\tau - \xi)$$

$$+ \frac{5}{2}\,m^4\, e \sin(2\,\tau + \xi) + \left\{ \frac{9}{2}\,m^3 + \frac{m^4}{2} \right\} e_{,} \sin(2\,\tau - \xi_{,}$$

$$+ \left\{ -\frac{9}{2}\,m^3 + \frac{5}{2}\,m^4 \right\} e_{,} \sin(2\,\tau + \xi_{,}) + \frac{9}{16}\,m^4 \sin 4\,\tau$$

$$+ \left\{ \frac{45}{16}\,m^3 + \frac{957}{64}\,m^4 \right\} e \sin(4\,\tau - \xi)$$

$$+ \frac{45}{32}\,m^4\, e \sin(4\,\tau + \xi) + \frac{63}{16}\,m^4\, e_{,} \sin(4\,\tau - \xi_{,}) - \frac{9}{16}\,m^4\, e_{,} \sin(4\,\tau + \xi_{,})$$

$$\int \delta \frac{\mathrm{d} R}{\mathrm{d} \lambda} \mathrm{d} t = \frac{m^4}{4} \cos 2\tau + \left\{ \frac{135}{16} m^3 + \frac{2649}{64} m^4 \right\} e \cos \xi$$

$$+ 3 m^4 e \cos (2\tau - \xi) - \frac{5}{6} m^4 e \cos (2\tau + \xi)$$

$$+ \left\{ -\frac{9}{4} m^3 - \frac{29}{8} m^4 \right\} e_{\prime} \cos (2\tau - \xi_{\prime})$$

$$+ \left\{ \frac{9}{4} m^3 - \frac{11}{8} m^4 \right\} e_{\prime} \cos (2\tau + \xi_{\prime}) - \frac{9}{64} m^4 \cos 4\tau$$

$$+ \left\{ -\frac{15}{16} m^3 - \frac{1037}{64} m^4 \right\} e \cos (4\tau - \xi) - \frac{9}{32} m^4 e \cos (4\tau + \xi)$$

$$+ \frac{63}{64} m^4 e_{\prime} \cos (4\tau - \xi_{\prime}) + \frac{9}{64} m^4 e_{\prime} \cos (4\tau + \xi_{\prime})$$

Considering the terms due to the first approximation,

$$\int \frac{\mathrm{d} R}{\mathrm{d} \lambda} \mathrm{d} t = -\frac{3 m^2}{4(1-m)} \cos 2\tau + \frac{9 m^2}{2\left(1 - 2m + \frac{3}{4} m^2\right)} e \cos (2\tau - \xi)$$

$$- \frac{3 m^2}{2\left(3 - 2m - \frac{3}{4} m^2\right)} e \cos (2\tau + \xi) - \frac{21 m^2}{4(2-3m)} e_{\prime} \cos (2\tau - \xi_{\prime})$$

$$+ \frac{3 m^2}{4(2-m)} e_{\prime} \cos (2\tau + \xi_{\prime})$$

$$= \left\{ -\frac{3}{4} m^2 - \frac{3}{4} m^3 - \frac{3}{4} m^4 \right\} \cos 2\tau$$

$$+ \left\{ \frac{9}{2} m^2 + 9 m^3 + \frac{117}{8} m^4 \right\} e \cos (2\tau - \xi)$$

$$+ \left\{ -\frac{m^2}{2} - \frac{m^3}{3} - \frac{17}{72} m^4 \right\} e \cos (2\tau + \xi)$$

$$+ \left\{ -\frac{21}{8} m^2 - \frac{63}{16} m^3 - \frac{189}{32} m^4 \right\} e_{\prime} \cos (2\tau - \xi_{\prime})$$

$$+ \left\{ \frac{3}{8} m^2 + \frac{3}{16} m^3 + \frac{3}{32} m^4 \right\} e_{\prime} \cos (2\tau + \xi_{\prime})$$

Finally,

$$-\int \frac{\mathrm{d} R}{\mathrm{d} \lambda} \mathrm{d} t = \left\{ \frac{3}{4} m^2 + \frac{3}{4} m^3 + \frac{m^4}{2} \right\} \cos 2\tau$$

$$+ \left\{ -\frac{135}{16} m^3 - \frac{2649}{64} m^4 \right\} e \cos \xi$$

$$+ \left\{ -\frac{9}{2} m^2 - 9 m^3 - \frac{141}{8} m^4 \right\} e \cos (2\tau - \xi)$$

$$+\left\{\frac{m^2}{2}+\frac{m^3}{3}+\frac{77}{72}m^4\right\}e\cos(2\tau+\xi)$$

$$+\left\{\frac{21}{8}m^2+\frac{99}{16}m^3+\frac{305}{32}m^4\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{-\frac{3}{8}m^2-\frac{39}{16}m^3+\frac{m^4}{32}\right\}e_{,}\cos(2\tau+\xi_{,})$$

$$+\frac{9}{64}m^4\cos 4\tau+\left\{\frac{15}{16}m^3+\frac{1037}{192}m^4\right\}e\cos(4\tau-\xi)$$

$$+\frac{9}{32}m^4 e\cos(4\tau+\xi)+\frac{63}{64}m^4 e_{,}\cos(4\tau-\xi_{,})-\frac{9}{64}m^4 e_{,}\cos(4\tau+\xi_{,})$$

$$\frac{\mathrm{a}^2}{r^2}=\mathrm{a}^2\left\{\frac{1}{\mathrm{r}^2}+\frac{2}{\mathrm{r}}\,\delta\frac{1}{r}+\left(\delta\frac{1}{r}\right)^2\right\}$$

$$\frac{\mathrm{a}^2}{\mathrm{r}^2}=1+2e\cos\xi$$

$$\mathrm{a}\,\delta\frac{1}{r}=\frac{m^2}{6}+\left\{m^2+\frac{19}{6}m^3+\frac{131}{18}m^4\right\}\cos 2\tau+\frac{2}{3}m^2 e\cos\xi$$

$$+\left\{\frac{15}{8}m+\frac{187}{32}m^2\right\}e\cos(2\tau-\xi)+\frac{33}{16}m^2 e\cos(2\tau+\xi)$$

$$+\left\{-\frac{3}{2}m^2+\frac{449}{16}m^4\right\}e_{,}\cos\xi_{,}+\left\{\frac{7}{2}m^2+\frac{157}{8}m^3+\frac{3349}{48}m^4\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{-\frac{m^2}{2}-\frac{91}{24}m^3-\frac{1265}{144}m^4\right\}e_{,}\cos(2\tau+\xi_{,})+\frac{7}{8}m^4\cos 4\tau$$

$$+\frac{49}{8}m^4 e_{,}\cos(4\tau-\xi_{,})-\frac{7}{8}m^4 e_{,}\cos(4\tau+\xi_{,})$$

For the value of $\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2$, see p. 55.

$$\frac{\mathrm{a}^2}{r^2}=1+\frac{m^2}{3}+\left\{2m^2+\frac{19}{3}m^3+\left\{\frac{131}{9}+\frac{1}{3}\right\}m^4\right\}\cos 2\tau$$

$$+\left\{2+\left\{\frac{4}{3}+\frac{1}{3}\right\}m^2\right\}e\cos\xi+\left\{\frac{15}{4}m+\left\{\frac{187}{16}+1\right\}m^2\right\}e\cos(2\tau-\xi)$$

$$+\left\{\frac{33}{8}+1\right\}m^2 e\cos(2\tau+\xi)+\left\{-3m^2+\left\{\frac{449}{8}+\frac{5}{2}\right\}m^4\right\}e_{,}\cos\xi_{,}$$

$$+\left\{7m^2+\frac{157}{4}m^3+\left\{\frac{3349}{24}-\frac{1}{3}\right\}m^4\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{-m^2-\frac{91}{12}m^3+\left\{-\frac{1265}{72}-\frac{5}{3}\right\}m^4\right\}e_{,}\cos(2\tau+\xi_{,})$$

$$+\left\{\frac{7}{4}+\frac{1}{2}\right\}m^4\cos 4\tau+\left\{\frac{49}{4}+\frac{7}{2}\right\}m^4 e_{,}\cos(4\tau-\xi_{,})$$

$$+\left\{-\frac{7}{4}-\frac{1}{2}\right\}m^4 e_{,}\cos(4\tau+\xi_{,})$$

$$\frac{\mathrm{a}^2}{r^2}=1+\frac{m^2}{3}+\left\{2m^2+\frac{19}{3}m^3+\frac{134}{9}m^4\right\}\cos 2\tau$$

$$+\left\{2+\frac{5}{3}m^2\right\}e\cos\xi+\left\{\frac{15}{4}m+\frac{203}{16}m^2\right\}e\cos(2\tau\quad\xi)$$

$$+\frac{41}{8}m^2 e\cos(2\tau+\xi)+\left\{-3m^2+\frac{469}{8}m^4\right\}e_{,}\cos\xi_{,}$$

$$+\left\{7m^2+\frac{157}{4}m^3+\frac{3341}{24}m^4\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{-m^2-\frac{91}{12}m^3-\frac{1385}{72}m^4\right\}e_{,}\cos(2\tau+\xi_{,})$$

$$+\frac{9}{4}m^4\cos 4\tau+\frac{63}{4}m^4 e_{,}\cos(4\tau-\xi_{,})-\frac{9}{4}m^4 e_{,}\cos(4\tau+\xi_{,})$$

Certain terms multiplied by $m^3 e$ and $m^4 e$ are wanting to complete the last expression. I have not yet attempted to calculate these, as they are affected by the discrepancy which I noticed in p. 50. The following series, however, which results from the multiplication of the last expression with that for $-\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t$, given p. 60 at foot, is not affected by this omission.

$$-\frac{\mathrm{a}^2}{r^2}\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t=\frac{3}{4}m^4+\left\{\frac{3}{4}m^2+\frac{3}{4}m^3+\left\{\frac{1}{2}+\frac{1}{4}\right\}m^4\right\}\cos 2\tau$$

$$+\left\{\left\{-\frac{135}{16}+\frac{45}{32}\right\}m^3+\left\{-\frac{2649}{64}-\frac{9}{2}+\frac{1}{2}+\frac{45}{32}+\frac{609}{128}\right.\right.$$

$$\left.\left.+\frac{123}{64}\right\}m^4\right\}e\cos\xi$$

$$+\left\{\left\{-\frac{9}{2}+\frac{3}{4}\right\}m^2+\left\{-9+\frac{3}{4}\right\}m^3\right.$$

$$\left.+\left\{-\frac{141}{8}-\frac{3}{2}+\frac{1}{2}+\frac{5}{8}\right\}m^4\right\}e\cos(2\tau-\xi)$$

$$+\left\{\left\{\frac{1}{2}+\frac{3}{4}\right\}m^2+\left\{\frac{1}{3}+\frac{3}{4}\right\}m^3\right.$$

$$\left.+\left\{\frac{77}{72}+\frac{1}{6}+\frac{1}{2}+\frac{5}{8}\right\}m^4\right\}e\cos(2\tau+\xi)$$

$$+ \left\{\frac{21}{8} + \frac{21}{8} - \frac{3}{8} - \frac{3}{8}\right\} m^4 e_, \cos \xi_,$$

$$+ \left\{\frac{21}{8} m^2 + \frac{99}{16} m^3 + \left\{\frac{305}{32} + \frac{7}{8} - \frac{9}{8}\right\} m^4\right\} e_, \cos (2\tau - \xi_,)$$

$$+ \left\{-\frac{3}{8} m^2 - \frac{39}{16} m^3 + \left\{\frac{1}{32} - \frac{1}{8} + \frac{9}{8}\right\} m^4\right\} e_, \cos (2\tau + \xi_,)$$

$$+ \left\{\frac{9}{64} + \frac{3}{4}\right\} m^4 \cos 4\tau$$

$$+ \left\{ \left\{\frac{15}{16} + \frac{45}{32}\right\} m^3 \right.$$

$$\left. + \left\{\frac{1037}{192} - \frac{9}{2} + \frac{45}{32} + \frac{609}{128}\right\} m^4 \right\} e \cos (4\tau - \xi)$$

$$+ \left\{\frac{9}{32} + \frac{1}{2} + \frac{123}{64}\right\} m^4 e \cos (4\tau + \xi)$$

$$+ \left\{\frac{63}{64} + \frac{21}{8} + \frac{21}{8}\right\} m^4 e_, \cos (4\tau - \xi_,)$$

$$+ \left\{-\frac{9}{64} - \frac{3}{8} - \frac{3}{8}\right\} m^4 e_, \cos (4\tau + \xi_,)$$

$$\doteq \frac{3}{4} m^4 + \left\{\frac{3}{4} m^2 + \frac{3}{4} m^3 + \frac{3}{4} m^4\right\} \cos 2\tau$$

$$+ \left\{\frac{225}{32} m^3 - \frac{4775}{128} m^4\right\} e \cos \xi$$

$$+ \left\{-\frac{15}{4} m^2 - \frac{33}{4} m^3 - 18\, m^4\right\} e \cos (2\tau - \xi)$$

$$+ \left\{\frac{5}{4} m^2 + \frac{13}{12} m^3 + \frac{85}{36} m^4\right\} e \cos (2\tau + \xi)$$

$$+ \frac{9}{2} m^4 e_, \cos \xi_,$$

$$+ \left\{\frac{21}{8} m^2 + \frac{99}{16} m^3 + \frac{297}{32} m^4\right\} e_, \cos (2\tau - \xi_,)$$

$$+ \left\{-\frac{3}{8} m^2 - \frac{39}{16} m^3 + \frac{33}{32} m^4\right\} e_, \cos (2\tau + \xi_,)$$

$$+ \frac{57}{64} m^4 \cos 4\tau + \left\{\frac{75}{32} m^3 + \frac{2713}{384} m^4\right\} e \cos (4\tau - \xi)$$

$$+ \frac{173}{64} m^4 e \cos (4\tau + \xi) + \frac{399}{64} m^4 e_, \cos (4\tau - \xi_,)$$

$$- \frac{57}{64} m^4 e_, \cos (4\tau + \xi_,)$$

$$\frac{d\lambda}{dt} = \frac{h}{r^2} - \frac{1}{r^2}\int \frac{dR}{d\lambda}\,dt$$

$$h = \sqrt{\mu a}\left\{1 - \frac{m^2}{3}\right\}$$

$$\lambda_1 = \left\{2m^2 + \frac{19}{3}m^3 + \frac{134}{9}m^4 + \frac{3}{4}m^2 + \frac{3}{4}m^3 + \frac{3}{4}m^4 - \frac{2m^4}{3}\right\}\frac{1}{2}(1 + m +$$

$$= \frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4$$

The term in λ_5 multiplied by m^3 depends upon a furthe proximation.

$$\lambda_6 = \left\{7m^3 + \frac{157}{4}m^3 + \frac{3341}{24}m^4 + \frac{21}{8}m^2 + \frac{99}{16}m^3 + \frac{297}{32}m^4 - \frac{7}{3}m^4\right\}$$

$$\frac{1}{2}\left(1 + \frac{3m}{2} + \frac{9}{4}m^2\right)$$

$$= \frac{77}{16}m^2 + \frac{479}{16}m^3 + \frac{7551}{64}m^4$$

$$\lambda_7 = \left\{-m^2 - \frac{91}{12}m^3 - \frac{1385}{72}m^4 - \frac{3}{8}m^2 - \frac{39}{16}m^3 - \frac{21}{32}m^4 + \frac{m^4}{3}\right\}$$

$$\frac{1}{2}\left\{1 + \frac{m}{2} + \frac{m^2}{4}\right\}$$

$$= -\frac{11}{16}m^2 - \frac{257}{48}m^3 - \frac{7337}{576}m^4$$

$$\lambda_{131} = \left\{\frac{9}{4}m^4 + \frac{57}{64}m^4\right\}\frac{1}{4} = \frac{201}{256}m^4$$

$$\lambda_{134} = \left\{\frac{63}{4}m^4 + \frac{399}{64}m^4\right\}\frac{1}{4} = \frac{1407}{256}m^4$$

$$\lambda_{135} = \left\{-\frac{9}{4}m^4 - \frac{57}{64}m^4\right\}\frac{1}{4} = -\frac{201}{256}m^4$$

The relation between the constants h and a is again to b tained from the equations

$$\frac{r^2\,d\lambda}{dt} = h - \int \frac{dR}{d\lambda}\,dt \qquad \text{and}$$

$$\frac{r^2\,d\lambda^2 + dr^2}{dt^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int dR = 0$$

Considering R as a function of a and n,

$$\delta R = -a\,\frac{dR}{da}\,r\,\delta\frac{1}{r} + \frac{dR}{d\lambda}\,\delta\lambda$$

Neglecting the eccentricities, it is sufficient to take

$$r\,\delta\,\frac{1}{r} = -\frac{m^2}{2} + m^2\cos 2\,\tau \qquad \delta\,\lambda = \frac{11}{8}\,m^2\sin 2\,\tau$$

$$-\frac{a^2}{\mu}\frac{\mathrm{d}\,R}{\mathrm{d}\,a} = \frac{m^2}{2} + \frac{3}{2}\,m^2\cos 2\,\tau \qquad \frac{\mathrm{d}\,R}{\mathrm{d}\,\tau} = \frac{3}{2}\,m^2\sin 2\,\tau$$

$$\frac{a}{\mu}\,\delta\,R = -\frac{m^4}{4} + \frac{3}{4}m^4 + \frac{33}{32}m^4 = \frac{49}{32}m^4$$

hence $r_0 = -\frac{m^2}{2} + \frac{49}{16}m^4$, when r is expressed in terms of a and $n\,t$.

$$\frac{1}{r^2}\left\{h^2 - 2\,h\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t + \left\{\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t\right\}^2\right\}$$

$$+\frac{\mathrm{d}\,r^2}{\mathrm{d}\,t^2} - \frac{2\,\mu}{r} + \frac{\mu}{a} + 2\int\mathrm{d}\,R = 0$$

$(\mathrm{d}\,r)^2 = r^4\left(\mathrm{d}\frac{1}{r}\right)^2 = \left(\frac{1}{r}\right)^{-4}\left(\mathrm{d}\,.\,\frac{1}{r}\right)^2 = 2\,n^2\,a^2\,r_1{}^2$, in which term $\frac{\mu}{a^3}$ may be put for n^2.

$\frac{1}{r^2} = \frac{1}{a^2}\{1 + 2\,m^2\cos 2\,\tau\}$ $\qquad$ $\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda} = \frac{3\,m_{\prime}\,a^2}{2\,a_{\prime}^3}\sin 2\,\tau$, h contains no term of the order m^2; when expressed in terms of a.

$$\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t = -\frac{3}{4}\,m^2\cos 2\,\tau$$

$$\frac{2}{r^2}\left\{h\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t - \frac{1}{2}\left\{\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t\right\}^2\right\}$$

$$= 2\frac{\mu}{a}\left\{-\frac{3}{4}\,m^4 - \frac{9}{64}m^4\right\} = -\frac{\mu}{a}\frac{57}{32}m^4$$

$$h^2\left\{1 + 2\,r_0 + r_0{}^2 + \frac{r_1{}^2}{2}\right\} = a\,\mu\left\{1 + 2\,r_0 - 2\,r_1{}^2 - \frac{57}{32}m^4\right\}$$

$$h^2 = a\,\mu\left\{1 + 2\,r_0 - 2\,r_1{}^2 - \frac{57}{32}\,m^4\right\}\left\{1 + 2\,r_0 + 3\,r_0{}^2 - \frac{r_1{}^2}{2}\right\}$$

$$= a\,\mu\left\{1 - \frac{m^4}{4} - \frac{5}{2}\,m^4 - \frac{57}{32}\,m^4\right\}$$

$$= a\,\mu\left\{1 - \frac{145}{32}\,m^4\right\}$$

Considering only the constant term in the equation

$$\frac{d\lambda}{dt} = \frac{h}{r^2} - \frac{1}{r^2}\int \frac{dR}{d\lambda}\,dt$$

$$\frac{d\lambda}{dt} = \sqrt{\frac{\mu}{a^3}}\left\{1 - \frac{145}{64}m^4\right\}\left\{1 + 2r_0 + r_0^2 + \frac{r_1^2}{2} + \frac{3}{4}m^4\right\}$$

$$= \sqrt{\frac{\mu}{a^3}}\left\{1 - \frac{145}{64}m^4 - m^2 + \frac{49}{8}m^4 + \frac{m^4}{4} + \frac{m^4}{2} + \frac{3}{4}m^4\right\}$$

$$= \sqrt{\frac{\mu}{a^3}}\left\{1 - m^2 + \frac{343}{64}m^4\right\}$$

If this quantity be called n and if $n = \sqrt{\frac{\mu}{a^3}}$

$$a = a\left\{1 - m^2 + \frac{343}{64}m^4\right\}^{-\frac{2}{3}} = a\left\{1 + \frac{2}{3}m^2 - \frac{869}{288}m^4\right\}$$

$$\frac{a}{r} = \left\{1 - \frac{m^2}{2} + \frac{49}{16}m^4\right\}\left\{1 + \frac{2}{3}m^2 - \frac{869}{288}m^4\right\}$$

$$= 1 + \frac{m^2}{6} - \frac{83}{288}m^4$$

So that when r is expressed in terms of a and nt,

$$r_0 = \frac{m^2}{6} - \frac{83}{288}m^4$$

M. Plana has

$$1 + \frac{m^2}{6} - \frac{199}{288}m^4$$

I shall now endeavour to obtain the terms in c, proportional to m^3 and m^4, by extending to these terms the method employed in p. 49, for finding the value of c.

$$a\,\delta\frac{1}{r} = \frac{m^2}{6} - \frac{83}{288}m^4 + \left\{\frac{2}{3}m^2 - \frac{869}{288}m^4\right\}e\cos\xi$$

$$r^3 = a^3\left\{1 - 3e\cos\xi\right\} \qquad r^4 = a^4\left\{1 - 4e\cos\xi\right\}$$

$$r^3\delta\frac{1}{r} = a^2\left\{-\frac{m^2}{2} + \frac{83}{96}m^4 + \frac{2}{3}m^2 - \frac{869}{288}m^4\right\}$$

$$= a^2\left\{\frac{m^2}{6} - \frac{155}{72}m^4\right\}e\cos\xi$$

$$\frac{3}{2}r^4\delta\left(\frac{1}{r}\right)^2 = a^2\left\{\frac{45}{16}m^3 + \frac{1733}{192}m^4\right\}e\cos\xi \qquad \text{See p. 55.}$$

$$\frac{r^2}{2} = \frac{r^2}{2} - r^3 \delta \frac{1}{r} + \frac{3}{2} r^4 \left(\delta \frac{1}{r} \right)^2$$

$$= a^2 \left\{ -1 - \frac{m^2}{6} + \frac{155}{72} m^4 + \frac{45}{16} m^3 + \frac{1733}{192} m^4 \right\} e \cos \xi$$

$$= a^2 \left\{ -1 - \frac{m^2}{6} + \frac{45}{16} m^3 + \frac{6439}{576} m^4 \right\} e \cos \xi$$

Hence

$$c^2 \left\{ 1 + \frac{m^2}{6} - \frac{45}{16} m^3 - \frac{6439}{576} m^4 \right\}$$

$$= 1 + \frac{2}{3} m^2 - \frac{869}{288} m^4 - 4 \left\{ \frac{m^2}{2} + \frac{135}{32} m^3 + \frac{5909}{384} m^4 + \frac{135}{32}{}^{*} m^4 \right\}$$

$$c^2 = \left\{ 1 - \frac{4}{3} m^2 - \frac{135}{8} m^3 - \frac{23726}{288} m^4 \right\} \left\{ 1 - \frac{m^2}{6} + \frac{45}{16} m^3 + \frac{6439}{576} m^4 + \frac{m^4}{36} \right\}$$

$$= 1 - \frac{3}{2} m^2 - \frac{225}{16} m^3 - \frac{40869}{1152} m^4$$

$$c = 1 - \frac{3}{4} m^2 - \frac{225}{32} m^3 - \frac{4577}{128} m^4$$

M. Plana has

$$c = 1 - \frac{3}{4} m^2 - \frac{225}{32} m^3 - \frac{4071}{128} m^4 \qquad .$$

The value of c is influenced by former discrepancies. See p. 50, and p. 66.

I shall now show how the term in λ_1 multiplied by $m^2 \frac{a^2}{a_{\prime}^2}$ may be obtained, in order to give an example of the method of arriving at terms of this class, very few of which are sensible, being multiplied by the extremely small quantity $\frac{a^2}{a_{\prime}^2}$.

The only term in r_1 which requires to be taken into account is $\frac{3}{4}(r_{101})^2$, $r_{101} = -\frac{15}{16} m \frac{a}{a_{\prime}}$, the term $-\frac{5}{4} m^2 \frac{a}{a_{\prime}^2}$ in R gives in $2\int d R$, the term $-\frac{5}{8} m^2 \frac{a}{a_{\prime}^2}$, in $r \frac{d R}{d r}$, the term $-\frac{5}{4} m^2 \frac{a}{a_{\prime}^2}$, and in $\int \frac{d R}{d \lambda} d t$, the term $-\frac{5}{16} m^2 \frac{a^2}{a_{\prime}^2}$. Hence,

---

* This term arises from $\delta\, d\, R$, or $\frac{a R'}{\mu}$, see p. 55.

$$r_1\{(2-2m)^2-1\} = (2-2m)^2\frac{3\,.\,225}{4\,.\,256}m^2\frac{a^2}{a_{,}^2}$$

$$+\frac{5}{8}m^2\frac{a^2}{a_{,}^2}+\frac{5}{4}m^2\frac{a^2}{a_{,}^2}$$

$$3\,r_1=\left\{\frac{3\,.\,225}{256}+\frac{15}{8}\right\}m^2\frac{a^2}{a_{,}^2} \qquad r_1=\frac{385}{256}m^2\frac{a^2}{a_{,}^2}$$

$$\lambda_1=\left\{\frac{385}{128}+\frac{225}{512}+\frac{5}{16}\right\}\frac{m^2}{2}\frac{a^2}{a_{,}^2}=\frac{1925}{1024}m^2\frac{a^2}{a_{,}^2}$$

which is the term given by M. Plana.

In order to obtain a further approximation, $\frac{1}{r}+\delta\frac{1}{r}$ and $\lambda+\delta\lambda$ must be substituted in the disturbing function for $\frac{1}{r}$ and λ, and the squares of the quantities $\delta\frac{1}{r}$ and $\delta\lambda$ retained.

Neglecting in R the terms multiplied by $\frac{a^3}{a^4}$ and by s^2, and omitting the factor $m_{,}$

$$R=-\frac{r^2}{4r_{,}^3}\left\{1+3\cos(2\lambda-2\lambda_{,})\right\}$$

$$R=-\frac{(r+\delta r)^2}{4r_{,}^3}\left\{1+3\cos(2\lambda-2\lambda_{,}+2\delta\lambda)\right\}$$

$$=-\frac{\{1+3\cos(2\lambda-2\lambda_{,})\}}{4r_{,}^3}\left\{2r\delta r+\delta r^2\right\}$$

$$+\frac{3r}{r_{,}^3}\sin(2\lambda-2\lambda_{,})\delta r\,\delta\lambda+\frac{3}{2}\frac{r^2}{r_{,}^3}(\cos(2\lambda-2\lambda_{,})(\delta\lambda)^2$$

$$\delta r=-r^2\delta\frac{1}{r}+r^3\left(\delta\frac{1}{r}\right)^2$$

Neglecting the terms multiplied by $\delta\frac{1}{r}$ and $\delta\lambda$,

$$R=-3r^2\frac{\{1+3\cos(2\lambda-2\lambda_{,})\}}{4r_{,}^3}\left(r\delta\frac{1}{r}\right)^2$$

$$-\frac{3r^2}{r_{,}^3}\sin(2\lambda-2\lambda_{,})\left(r\delta\frac{1}{r}\right)\delta\lambda_{,}+\frac{3}{2}\frac{r^2}{r_{,}^3}\cos(2\lambda-2\lambda_{,})(\delta\lambda)^2$$

dR and $r\left(\frac{dR}{dr}\right)$ may be obtained from R as before.

$$\frac{\mathrm{d}R}{\mathrm{d}\lambda} = \frac{3}{2}\,\frac{r^2}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})$$

$$= \frac{3(\mathrm{r} + \delta r)^2}{2r_{,}^3}\Big\{\sin(2\lambda - 2\lambda_{,} + 2\delta\lambda)\Big\}$$

$$= \frac{3(\mathrm{r} + 2\mathrm{r}\delta r)}{2r_{,}^3}\Big\{\sin(2\lambda - 2\lambda_{,})\Big(1 - \frac{4\delta\lambda^2}{2}\Big)$$

$$+ 2\cos(2\lambda - 2\lambda_{,})\delta\lambda\Big\}$$

$$= \frac{3\{2\mathrm{r}\delta r + \delta r^2\}}{r_{,}^3}\sin(2\lambda - 2\lambda_{,}) + \frac{6\cos(2\lambda - 2\lambda_{,})}{r_{,}^3}\,\mathrm{r}\delta r\,\delta\lambda$$

$$- \frac{3\mathrm{r}^2}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})\,(\delta\lambda)^2$$

Neglecting as before the terms multiplied by $\delta\frac{1}{r}$ and $\delta\lambda$,

$$= \frac{9}{2}\,\frac{\mathrm{r}^2}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})\Big(\mathrm{r}\delta\frac{1}{r}\Big)^2 - \frac{6\mathrm{r}^2\cos(2\lambda - 2\lambda_{,})}{r_{,}^3}\Big(\mathrm{r}\delta\frac{1}{r}\Big)\delta\lambda$$

$$- \frac{3\mathrm{r}^2}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})\,(\delta\lambda)^2$$

In order to have the term in $\frac{\mathrm{d}R}{\mathrm{d}\lambda}$, of which the argument is $\xi_{,}$ dependent upon the first power of the disturbing force, and which is multiplied by m^5, it is sufficient to take

$$\frac{\mathrm{a}^2}{\mu}\,\frac{\mathrm{d}.\frac{\mathrm{d}R}{\mathrm{d}\lambda}}{\mathrm{d}a} = -3m^2\sin 2\tau - \frac{21}{2}m^2e_{,}\sin(2\tau - \xi_{,}) + \frac{3}{2}m^2e_{,}\sin(2\tau + \xi_{,})$$

$$r\delta\frac{1}{r} = \frac{19}{6}m^3\cos 2\tau + \frac{157}{8}m^3e_{,}\cos(2\tau - \xi_{,}) - \frac{91}{24}m^3e_{,}\cos(2\tau + \xi_{,})$$

$$\frac{\mathrm{a}}{\mu}\,\frac{\mathrm{d}.\frac{\mathrm{d}R}{\mathrm{d}\lambda}}{\mathrm{d}\lambda} = 3m^2\cos 2\tau + \frac{21}{2}m^2e_{,}\cos(2\tau - \xi_{,}) - \frac{3}{2}m^2e_{,}\cos(2\tau + \xi_{,})$$

$$\delta\lambda = \frac{59}{12}m^3\sin 2\tau + \frac{479}{16}m^3e_{,}\sin(2\tau - \xi_{,}) - \frac{257}{48}m^3e_{,}\sin(2\tau + \xi_{,})$$

These terms give rise to the term in $\frac{\mathrm{d}R}{\mathrm{d}\lambda}$

$$\Big\{-\frac{3.157}{2.8} - \frac{3.91}{2.24} + \frac{3.479}{2.16} - \frac{3.257}{2.48} + \frac{21.19}{4.6} + \frac{3.19}{4.6} + \frac{21.59}{4.12} + \frac{3.59}{4.12}\Big\}m^5e_{,}\sin\xi_{,}$$

$$= -\frac{633}{16} m^5 e_{\prime} \sin \xi_{\prime}$$

In order to have the term in $\frac{dR}{d\lambda}$, of which the argument is $\xi_{\prime}$, multiplied by m^5, and which is dependent upon the cube of the disturbing force, it is sufficient to take

$$\cos(2\lambda - 2\lambda_{\prime}) = \cos 2\tau \qquad r\,\delta\frac{1}{r} = m^2 \cos 2\tau$$

$\delta\lambda = \frac{11}{8} m^2 \sin 2\tau - 3\,m\,e_{\prime} \sin \xi_{\prime}$, putting these values in the equation of line 7, p. 69, I find

$$\delta^2 \frac{dR}{d\lambda} = -\left\{-3 - \frac{33}{8}\right\} 3\,m^5 e_{\prime} \sin \xi_{\prime} = \frac{171}{8} m^5 e_{\prime} \sin \xi_{\prime}$$

whence, finally,

$$\lambda_5 = \left\{\frac{469}{8} + 1 + \frac{9}{2} - \frac{633}{16} + \frac{171}{8}\right\} m^3 = \frac{735}{16} m^3$$

I have calculated many other terms besides those which have been here detailed as examples, but these are sufficient to show how the expressions for the coordinates of the moon may be arrived at directly, by employing the differential equations in which the time or mean longitude is the independent variable, and how a verification may be obtained of M. Plana's final results. Considering the importance of the Lunar Tables, the uncertainty which at present exists with respect to the magnitude of some of the terms, and the extreme complication of the necessary calculations, whichever method be adopted, this verification cannot be looked upon as a problem of mere curiosity.

Finally, through either method the true longitude is expressed in a convergent series of sines of angles, consisting of various combinations of the angles τ, ξ, η and $\xi_{\prime}$ and their multiples, and no others. The reciprocal of the radius vector is expressed in a convergent series of cosines, and the latitude in a convergent series of sines of the same angles, and these two latter quantities contain no term proportional to any power of the time without the periodical terms. The quantities c and g also being rational, it is evident that r and s must vary within given limits, and the stability of the system may be inferred.

The non-sphericity of the earth introduces into the disturbing function the term

$$(M + m)\frac{\delta V}{M}$$

M being the mass of the earth, and m the mass of the moon. See *Mécanique Céleste*, vol. iii. p. 183. V is the quantity denoted by Q, in my *Account of the "Traité sur le flux et reflux de la mer,"* p. 32.

$$Q = \frac{4\pi\rho a^3}{r}\left\{1 - \frac{e^2 + i^2}{2}\right.$$

$$\left. + \frac{a^2\{(e^2 + i^2)x^2 + (i^2 - 2e^2)y^2 + (e^2 - 2i^2)z^2\}}{10r^4}\right\}$$

When $e = i$

$$Q = \frac{4\pi\rho a^3}{r}\left\{1 - e^2 + \frac{a^2e^2\{2x^2 - y^2 - z^2\}}{10r^4}\right\}$$

If q is equal to the ratio of the centrifugal force to the equatorial gravity at the earth's surface, the preceding expression may be put in the form, (See *Account*, *&c.* p. 40,)

$$Q = \frac{M}{r}\left\{1 + \frac{(e^2 - q)}{6r^4}a^2\{2x^2 - y^2 - z^2\}\right\}$$

Substituting

for x, $\grave{r}\sin f\lambda\sin\omega + \grave{r}s\cos\omega$

for y, $\grave{r}\sin f\lambda\cos\omega - \grave{r}s\sin\omega$

and for z, $\grave{r}\cos f\lambda$,

ω being the obliquity of the ecliptic, and $f\lambda$ the longitude of the moon, reckoned from the first point of Aries; the term to be added to R, is

$$-\frac{\mu(e^2 - q)}{2r^3}a^2\left\{\frac{\sin^2 f\lambda\sin^2\omega + 2s\sin f\lambda\sin\omega\cos\omega + \cos^2\omega s^2}{1 + s^2} - \frac{1}{3}\right\}$$

which agrees with the expression of M. Damoiseau, p. 581. In his notation,

$$\frac{e^2}{2} = \alpha\rho, \quad q = \alpha\varphi, \quad a = D, \quad \omega = \lambda, \quad f\lambda = fv$$

Knowing $\frac{e^2}{2}$, (*the compression, l'aplatissement*,) the inequalities due to the quantity δV, may be determined; and conversely, these inequalities obtained by observation, furnish the means of determining the *compression*, which is thus found to be about $\frac{1}{305}$.

The equations

$$\frac{e^2}{2} = K_{(2)} \qquad \frac{j}{2} = \psi$$

$$d\nu + \frac{r'^2 \sin(\lambda' - \nu)}{h^2 \tan\iota}\left\{(1 + s^2)\left(\frac{dR}{ds}\right) - r's\left(\frac{dR}{dr'}\right)\right.$$

$$\left. - \left(\frac{dR}{d\lambda'}\right)\left(\frac{ds}{d\lambda'}\right)\right\} d\lambda' = 0$$

$$d\iota + \frac{r'^2 \cos\iota^2 \cos(\lambda' - \nu)}{h^3}\left\{(1 + s^2)\right\}\left(\frac{dR}{ds}\right) - r's\left(\frac{dR}{dr'}\right)$$

$$\left. - \left(\frac{dR}{d\lambda}\right)\left(\frac{ds}{d\lambda'}\right)\right\} d\lambda' = 0$$

(see p. 10.), serve to verify some of the theorems of Newton in the third volume of the *Principia*.

In fact

$$R = -\frac{m_{,} r'^2}{4 r_{,}^3}\left\{1 + 3\cos(2\lambda' - 2\lambda_{,}) - 2s^2\right\}$$

$$(1 + s^2)\left(\frac{dR}{ds}\right) = m_{,} s (1 + s^2)\frac{r'^2}{r_{,}^3}$$

$$r's\left(\frac{dR}{dr'}\right) = -\frac{m_{,} r'^2}{2 r_{,}^3}\left\{1 + 3\cos(2\lambda' - 2\lambda_{,}) - 2s^2\right\} s$$

$$\left(\frac{dR}{d\lambda}\right) = \frac{3 m_{,} r_{,}^2}{2 r_{,}^3}\sin(2\lambda' - 2\lambda_{,}) \qquad \frac{ds}{d\lambda'} = \tan\iota\cos(\lambda' - \nu)$$

$$d\nu + \frac{m_{,} r'^4 \sin(\lambda' - \nu)}{h^2 r_{,}^3}\left\{\sin(\lambda' - \nu) + \frac{\sin(\lambda' - \nu)}{2}\left\{1 + 3\cos(2\lambda' - \lambda 2_{,})\right\}\right.$$

$$\left. - \frac{3}{2}\cos(\lambda' - \nu)\sin(2\lambda' - 2\lambda_{,})\right\} d\lambda' = 0$$

$$d\nu + \frac{m_{,} r'^4 \sin(\lambda' - \nu)}{h^2 r_{,}^3}\left\{\sin(\lambda' - \nu) + 3\cos(\lambda' - \lambda_{,})\left\{\cos(\lambda' - \lambda_{,})\sin(\lambda_{\backprime} - \nu)\right.\right.$$

$$\left.\left. - \sin(\lambda' - \lambda_{,})\cos(\lambda' - \nu)\right\} - \sin(\lambda' - \nu)\right\} \lambda d' = 0$$

$$-d\nu = \frac{3 m_{,} r'^4}{h^2 r_{,}^3}\sin(\lambda' - \nu)\cos(\lambda' - \lambda_{,})\sin(\lambda'_{,} - \nu)\, d\lambda$$

$$= \frac{3 m_{,} a^3}{\mu\, a_{,}^3}\sin(\lambda - \nu)\cos(\lambda - \lambda_{,})\sin(\lambda_{,} - \nu)\, d\lambda \text{ nearly}$$

$$= \frac{1}{59{\cdot}575}\sin(\lambda - \nu)\cos(\lambda - \lambda_{,})\sin(\lambda_{,} - \nu)\, d\lambda$$

Which agrees with the result of Newton, Prop. XXX. Lib. 3. "Est igitur velocitas nodorum ut IT × PH × AZ, sive ut contentum sub sinubus trium angulorum TPI, PTN et STN...... Sunto enim PK, PH et AZ prædicti tres sinus. Nempe PK sinus distantiæ lunæ a quadraturâ, PH sinus distantiæ lunæ a nodo, et AZ sinus distantiæ nodi a sole, et erit velocitas nodi ut contentum PK × PH × AZ."

Similarly

$$- d\,\iota = \frac{3\, m_{\prime}\, r^4 \cos\iota}{h^2\, r_{\prime}^3} \sin\iota \cos(\lambda - \nu) \cos(\lambda - \lambda_{\prime}) \sin(\lambda_{\prime} - \nu)\, d\,\lambda$$

"Erit angulus G P*g* (seu inclinationis horariæ variatio) ad angulum $33'' \, 16''' \, 3''''$ ut IT × AZ × TG × $\frac{\mathrm{P}p}{\mathrm{PG}}$ ad AT cub." Prop. XXXIV. Lib. 3.

The preceding theorems are not true for the terms multiplied by the higher powers of the ratio of a to $a_{\prime}$, and therefore they do not apply to the Planetary theory. The proof given by Wodehouse (*Treatise on Astronomy*, vol. ii. p. 442.), and noticed by Mr. Whewell, is liable to objection, because the equations which he makes use of to determine the quantities $d\,\iota$ and $d\,\nu$ (in his notation $\delta\,\theta$ and $\delta\,\varphi$,) require the disturbing function to be developed in terms of the mean longitude *explicitly*. The theorems of Newton relating to the motion of the moon's node, and to the variation of the obliquity of her orbit, will ever be ranked amongst his greatest discoveries.

M. Gautier has drawn (*Essai historique sur le problème des trois corps*, p. 53.) a parallel between the methods of Clairaut, d'Alembert and Euler; and it will be seen that their methods do not resemble in any respect those which I have employed. Both Clairaut and d'Alembert, by means of the differential equation of the second order, in which the true longitude is the independent variable, obtained the expression for the reciprocal of the radius vector in terms of cosines of the true longitude. They substituted this value in the differential equation which determines the mean longitude, and obtained by integration the value of the mean motion in terms of the sines of the true longitude. By the reversion of series they then found the true longitude in terms of sines of the mean longitude. The method of Euler is not so simple, but it is remarkable as introducing the employment of three rectangular coordinates and the decomposition of forces in the direction of three rectangular axes.

Although d'Alembert and Clairaut made use of the same differential equations, disguised under a different notation, yet they did not arrive at these in the same manner, nor did they employ the same method of integration. Laplace has pushed the approximations to a much greater extent; but his method coincides in all respects with that of Clairaut.

4. Before the publication of my own investigations I am not aware

that any one had attempted to arrive *directly* at the expressions for the coordinates of the moon in terms of the mean longitude. Lately M. Poisson has published a very important memoir*, in which he has discussed various points of great interest in the Lunar theory, and in which he proposes to arrive directly at the desired expressions by means of the variations of the elliptic constants.

The following are the expressions in the notation of this treatise,

$$\mathrm{d}a = -\frac{2}{an}\frac{\mathrm{d}R}{\mathrm{d}\varepsilon}\mathrm{d}t$$

$$\mathrm{d}\varepsilon = -\frac{\sqrt{1-e^2}}{a^2ne}\left(1-\sqrt{1-e^2}\right)\frac{\mathrm{d}R}{\mathrm{d}e}\mathrm{d}t + \frac{2}{an}\frac{\mathrm{d}R}{\mathrm{d}a}\mathrm{d}t$$

$$\mathrm{d}e = \frac{\sqrt{1-e^2}}{a^2ne}\left(1-\sqrt{1-e^2}\right)\frac{\mathrm{d}R}{\mathrm{d}\varepsilon}\mathrm{d}t + \frac{\sqrt{1-e^2}}{a^2ne}\frac{\mathrm{d}R}{\mathrm{d}\varpi}\mathrm{d}t$$

$$\mathrm{d}\varpi = -\frac{\sqrt{1-e^2}}{a^2ne}\frac{\mathrm{d}R}{\mathrm{d}e}\mathrm{d}t$$

$$\mathrm{d}\nu = -\frac{1}{a^2n\sin\iota\sqrt{1-e^2}}\frac{\mathrm{d}R}{\mathrm{d}\iota}\mathrm{d}t$$

$$\mathrm{d}\iota = \frac{1}{a^2n\sin\iota\sqrt{1-e^2}}\frac{\mathrm{d}R}{\mathrm{d}\nu}\mathrm{d}t$$

See the *Théor. Anal.* vol. i. p. 330, or *The Mechanism of the Heavens*, p. 231.

In these works R is used with a contrary sign to its acceptation in the *Mécanique Céleste*, which I have followed.

See also an excellent paper by Mr. Ivory, in the *Phil. Trans.*, 1832, Part I.

In the notation of M. Poisson in the important paper to which I have referred

$$c = \varepsilon - \varpi \qquad \omega = \varpi - \beta \qquad \alpha = \nu \qquad \gamma = \iota$$

$$\mathrm{d}\beta = \cos\iota\,\mathrm{d}\nu$$

The quantity R may be considered as a function of the quantities ε, β and ϖ, or of the quantities ω, c and ν,

$$\frac{\mathrm{d}R}{\mathrm{d}\varepsilon}\mathrm{d}\varepsilon + \frac{\mathrm{d}R}{\mathrm{d}\beta}\mathrm{d}\beta + \frac{\mathrm{d}R}{\mathrm{d}\varpi}\mathrm{d}\varpi = \frac{\mathrm{d}R}{\mathrm{d}c}\mathrm{d}c + \frac{\mathrm{d}R}{\mathrm{d}\omega}\mathrm{d}\omega + \frac{\mathrm{d}R}{\mathrm{d}\nu}\mathrm{d}\nu$$

$$\mathrm{d}c = \mathrm{d}\varepsilon - \mathrm{d}\varpi \qquad \mathrm{d}\omega = \mathrm{d}\varpi - \mathrm{d}\beta$$

$$\frac{\mathrm{d}R}{\mathrm{d}\varepsilon}\mathrm{d}\varepsilon + \frac{\mathrm{d}R}{\mathrm{d}\beta}\mathrm{d}\beta + \frac{\mathrm{d}R}{\mathrm{d}\varpi}\mathrm{d}\varpi$$

---

* *Mémoire sur le mouvement de la Lune autour de la Terre.*

$$=\frac{\mathrm{d}R}{\mathrm{d}c}\mathrm{d}\varepsilon-\frac{\mathrm{d}R}{\mathrm{d}c}\mathrm{d}\varpi+\frac{\mathrm{d}R}{\mathrm{d}\omega}\mathrm{d}\varpi-\frac{\mathrm{d}R}{\mathrm{d}\omega}\mathrm{d}\beta+\frac{\mathrm{d}R}{\mathrm{d}\nu}\frac{\mathrm{d}\beta}{\cos\iota}$$

whence,

$$\frac{\mathrm{d}R}{\mathrm{d}\varepsilon}=\frac{\mathrm{d}R}{\mathrm{d}c}\qquad\frac{\mathrm{d}R}{\mathrm{d}\varpi}=\frac{\mathrm{d}R}{\mathrm{d}\omega}-\frac{\mathrm{d}R}{\mathrm{d}c}\qquad\frac{\mathrm{d}R}{\mathrm{d}\beta}=\frac{\mathrm{d}R}{\cos\iota\,\mathrm{d}\nu}-\frac{\mathrm{d}R}{\mathrm{d}\omega}$$

$$\mathrm{d}a=-\frac{2}{an}\frac{\mathrm{d}R}{\mathrm{d}\varepsilon}\mathrm{d}t$$

$$\mathrm{d}c=\mathrm{d}\varepsilon-\mathrm{d}\varpi$$

$$=\frac{2}{an}\frac{\mathrm{d}R}{\mathrm{d}a}\mathrm{d}t+\frac{(1-e^2)}{a^2ne}\frac{\mathrm{d}R}{\mathrm{d}e}\mathrm{d}t$$

$$\mathrm{d}e=\frac{\sqrt{1-e^2}}{a^2ne}\{1-\sqrt{1-e^2}\}\frac{\mathrm{d}R}{\mathrm{d}c}$$

$$+\frac{\sqrt{1-e^2}}{a^2ne}\frac{\mathrm{d}R}{\mathrm{d}\omega}-\frac{\sqrt{1-e^2}}{a^2ne}\frac{\mathrm{d}R}{\mathrm{d}c}$$

$$=\frac{\sqrt{1-e^2}}{a^2ne}\frac{\mathrm{d}R}{\mathrm{d}\omega}-\frac{(1-e^2)}{a^2ne}\frac{\mathrm{d}R}{\mathrm{d}c}\mathrm{d}t$$

$$\mathrm{d}\omega=\mathrm{d}\varpi-\mathrm{d}\beta$$

$$=-\frac{\sqrt{1-e^2}}{a^2ne}\frac{\mathrm{d}R}{\mathrm{d}e}\mathrm{d}t+\frac{\cos\iota}{a^2n\sin\iota\sqrt{1-e^2}}\frac{\mathrm{d}R}{\mathrm{d}\iota}\mathrm{d}t$$

$$\mathrm{d}\iota=\frac{1}{a^2n\sin\iota\sqrt{1-e^2}}\frac{\mathrm{d}R}{\mathrm{d}\nu}\mathrm{d}t-\frac{\cos\iota}{a^2n\sin\iota\sqrt{1-e^2}}\frac{\mathrm{d}R}{\mathrm{d}\omega}\mathrm{d}t$$

$$\mathrm{d}\nu=-\frac{1}{a^2n\sin\iota\sqrt{1-e^2}}\frac{\mathrm{d}R}{\mathrm{d}\iota}\mathrm{d}t$$

The latter are the equations employed by M. Poisson.

$$\xi=\mathrm{c}nt+c\qquad\varepsilon-\beta=c+\omega\qquad\eta=\mathrm{g}nt+c+\omega$$

$$\varepsilon-\beta+\nu=c+\omega+\alpha=(\varepsilon^*)\qquad\tau=nt-n_{\prime}t-(\varepsilon)+(\varepsilon_{\prime})$$

Considering R as developed in terms of τ, ξ, $\xi_{\prime}$, &c.,

$$\mathrm{d}a=-\frac{2}{an}\left\{\frac{\mathrm{d}R}{\mathrm{d}\tau}+\frac{\mathrm{d}R}{\mathrm{d}\xi}+\frac{\mathrm{d}R}{\mathrm{d}\eta}\right\}\mathrm{d}t$$

$$\mathrm{d}e=\frac{\sqrt{1-e^2}}{a^2ne}(1-\sqrt{1-e^2})\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}+\frac{\mathrm{d}R}{\mathrm{d}\eta}\right)\mathrm{d}t$$

---

* In M. Poisson's notation.

order, as regards the eccentricities and inclinations, by the method of the variation of the elliptic constants, the development of the disturbing function must be carried one step further. Generally given terms, in the expressions for the coordinates of the disturbed body, may be obtained by whichever method affords greater facility, but it is not allowable to separate the disturbing function into two portions, so as to consider one portion by the method of integrating the differential equations of the second order, and the other by the method of the variation of the elliptic constants, without ascertaining at the same time, that the terms obtained by one method are not already included amongst those obtained by the other. In the former method no inequalities of a given order, as regards the eccentricities and inclinations, arise from terms of a higher order in the disturbing function. So, for example, if only the inequalities were required which are independent of the eccentricities and inclinations, all the terms in the disturbing function multiplied by the eccentricities and inclinations might be neglected. In the second method this is no longer the case, and in order even to obtain the inequalities of the coordinates which are independent of e, the terms in R multiplied by e must not be neglected. So again, by determining the variations of e and ϖ, which arise from the consideration of the term in R, of which the argument is $\tau - \xi + \xi_{\prime}$, the inequality of longitude, of which the argument is $\tau + \xi_{\prime}$, is obtained; which inequality is multiplied by $e_{\prime}$ only, although the term in R from which it was deduced was multiplied by $e\,e_{\prime}$. So, also, by determining the variations of ι and ν which arise from the term in R, of which the argument is $\tau - \eta + \eta_{\prime}$, the inequality of latitude is obtained, of which the argument is $\tau + \eta_{\prime}$, multiplied by $\gamma_{\prime}$ only, although the term from which it was deduced was multiplied by $\gamma\,\gamma_{\prime}$. Generally, in order to arrive at the terms in the longitude multiplied by e^4 and quantities of that order, it would be necessary to carry the development of R to the terms multiplied by e^5, $e^4\,e_{\prime}$, $e^3\,e_{\prime}^2$, &c., and to arrive at the terms in c and g multiplied by e^4, it would even be necessary to obtain the terms in R_0 multiplied by e^6. This circumstance presents another obstacle to the employment, generally, of the method of the variation of the elliptic constants, whether in the lunar or in the planetary theory. On the other hand, certain terms in the inequalities are obtained by a first approximation, through the method of the variation of the elliptic constants, which only arise in the second or even in the subsequent approximations, when the differential equations of the second order are integrated by the method of indeterminate coefficients, as in the case of Arg. 9, &c. Either method, however, is, I think, complete in itself; and both, whether in the lunar or in the planetary theory, properly managed, must lead to results precisely identical in all respects.

By following the analysis of M. Poisson, given in the Memoir already referred to, I have obtained finite expressions for the non-periodical portions of r^2 and r^{-3}. See p. 8.

$$r^2 = a^2 \left\{1 + \frac{3}{2}e^2 + \&c.\right\}$$

$$r^{-3} = a^{-3} \left\{\frac{1}{(1 - e^2)^{\frac{3}{2}}} + \&c.\right\}$$

By the help of these equations I have obtained for the constant part of R (or R_0) in the lunar theory, an expression in finite terms, by developing which, the terms depending upon any powers of the eccentricities may be arrived at.

For this purpose it is sufficient to take

$$R = m_{,} \left\{\frac{r^2}{2 r_{,}^3} - \frac{3 r'^2 r_{,}^2}{2 r_{,}^5} \cos(\lambda' - \lambda_{,})^2\right\} \qquad \text{See p. 25.}$$

$$r' r_{,} \cos(\lambda' - \lambda_{,}) = r' r_{,} \{\cos\lambda' \cos\lambda_{,} + \sin\lambda' \sin\lambda_{,}\}$$

$$\cos\lambda' = \cos(\lambda' - \nu)\cos\nu - \sin(\lambda' - \nu)\sin\nu$$

$$\tan(\lambda' - \nu) = \cos\iota \tan(\lambda - \beta)$$

$$\cos(\lambda' - \nu) = \frac{1}{\sqrt{1 + \cos^2\iota \tan^2(\lambda - \beta)}}$$

$$\sin(\lambda' - \nu) = \frac{\cos\iota \tan(\lambda - \beta)}{\sqrt{1 + \cos^2\iota \tan^2(\lambda - \beta)}}$$

$$\cos\lambda' = \frac{\cos\nu - \cos\iota \tan(\lambda - \beta)\sin\nu}{\sqrt{1 + \cos^2\iota \tan^2(\lambda - \beta)}}$$

$$= \frac{\cos\nu \cos(\lambda - \beta) - \sin(\lambda - \beta)\sin\nu \left\{1 - 2\sin^2\frac{\iota}{2}\right\}}{\sqrt{1 - \sin^2\iota \sin^2(\lambda - \beta)}}$$

$$= \frac{\cos(\lambda - \beta + \nu) + 2\sin^2\frac{\iota}{2}\sin\nu \sin(\lambda - \beta)}{\sqrt{1 - \sin^2\iota \sin^2(\lambda - \beta)}}$$

Similarly,

$$\sin\lambda' = \frac{\sin(\lambda - \beta + \nu) - 2\sin^2\frac{\iota}{2}\cos\nu \sin(\lambda - \beta)}{\sqrt{1 - \sin^2\iota \sin^2(\lambda - \beta)}}$$

$$r' = r\sqrt{1 - \sin^2\iota \sin^2(\lambda - \beta)}$$

therefore,

$$r' r \cos(\lambda' - \lambda_{,}) = r r_{,} \left\{ \cos^2 \frac{\iota}{2} \cos(\lambda - \lambda_{,} - \beta + \nu) \right.$$

$$\left. + \sin^2 \frac{\iota}{2} \cos(\lambda + \lambda_{,} - \beta - \nu) \right\}$$

And it is sufficient to take in this instance,

$$r'^2 r_{,}^2 \cos^2(\lambda' - \lambda_{,})^2 = \frac{r^2 r_{,}^2}{2} \left\{ \cos^4 \frac{\iota}{2} + \sin^4 \frac{\iota}{2} \right\}$$

$$\sin^2 \frac{\iota}{2} = \frac{1 - \cos \iota}{2} \qquad \cos^2 \frac{\iota}{2} = \frac{1 + \cos \iota}{2}$$

therefore,

$$R_0 = - \frac{a^2 \left(1 + \frac{3}{2} e^2\right)}{4 a_{,}^3 \{1 - e_{,}^2\}^{\frac{3}{2}}} \left\{1 - \frac{3}{2} \sin^2 \iota \right\}$$

$$c = 1 - \left(\frac{d \varpi}{d t}\right) + \left(\frac{d \beta}{d t}\right) - \left(\frac{d \nu}{d t}\right) \qquad \text{See p. 77.}$$

$$= 1 - \left(\frac{d \varpi}{d t}\right) - \frac{a(1 - \cos \iota)}{\mu \sin \iota \sqrt{1 - e^2}} \left(\frac{d R}{d \iota}\right)$$

$$= 1 - \frac{3}{4} \frac{(1 - e^2)^{\frac{1}{2}}}{(1 - e_{,}^2)^{\frac{3}{2}}} \left\{1 - \frac{3}{2} \sin^2 \iota \right\} m^2$$

$$- \frac{3 \left(1 + \frac{3}{2} e^2\right)(1 - \cos \iota)}{4 (1 - e_{,}^2)^{\frac{3}{2}} (1 - e^2)^{\frac{1}{2}}} \cos \iota \, m^2$$

$$= 1 - \frac{3}{4} m^2 + \frac{3}{8} m^2 e^2 - \frac{9}{8} m^2 e_{,}^2 + \frac{3}{2} m^2 \gamma^2 + \frac{3}{32} m^2 e^4$$

$$+ \frac{9}{16} m^2 e^2 e_{,}^2 - \frac{45}{32} m^2 e_{,}^4$$

which agrees with the expression of M. Plana, vol. i. p. 485 *.

---

* According to M. Plana,

$$c = 1 - \frac{3}{4} m^2 - \frac{225}{32} m^3 - \frac{4071}{128} m^4 - \frac{265493}{2048} m^5$$

$$- \frac{12822631}{24576} m^6 - \frac{1273925965}{589824} m^7 + \frac{3}{8} m^2 e^2 + \frac{675}{64} m^3 e^2$$

Similarly,

$$g = 1 - \left(\frac{d\,v}{d\,t}\right)$$

$$= 1 + \frac{3\left(1 + \frac{3}{2}e^2\right)\cos\iota}{4\left(1 - e_{\prime}^{2}\right)^{\frac{3}{2}}\left(1 - e^2\right)^{\frac{1}{2}}}m^2$$

$$= 1 + \frac{3}{4}\left(1 + 2e^2 + \frac{9}{8}e^4\right)\left(1 + \frac{3}{2}e_{\prime}^{2} + \frac{15}{8}e_{\prime}^{4}\right)$$

$$\left(1 - \frac{\gamma^2}{2} + \frac{3}{8}\gamma^4\right)m^2$$

$$= 1 + \frac{3}{4}m^2 + \frac{3}{2}m^2e^2 + \frac{9}{8}m^2e_{\prime}^{2} - \frac{3}{8}m^2\gamma^2 + \frac{27}{32}m^2e^4$$

$$+ \frac{9}{4}m^2e^2e_{\prime}^{2} + \frac{45}{32}m^2e_{\prime}^{4} - \frac{3}{4}m^2e^2\gamma^2 - \frac{9}{16}m^2e_{\prime}^{2}\gamma^2$$

$$+ \frac{9}{32}m^2\gamma^4 + \&c.$$

M. Plana has the terms

$$-\frac{21}{64}m^2e^4 + \frac{27}{32}m^2e^2\gamma^2$$

See p. 46. It must not, however, be concluded that these results

---

$$+ \frac{31893}{512}m^4e^2 + \frac{1546665}{4096}m^5e^2 + \frac{3}{2}m^2\gamma^2 + \frac{189}{32}m^3\gamma^2$$

$$+ \frac{3963}{128}m^4\gamma^2 + \frac{336987}{2048}m^5 - \frac{9}{8}m^2e_{\prime}^{2} - \frac{825}{32}m^3e_{\prime}^{2} - \frac{61179}{256}m^4e_{\prime}^{2}$$

$$- \frac{1767849}{1024}m^5e_{\prime}^{2} - \frac{75}{32}m^2e^2\gamma^2 - \frac{189}{32}m^3e^2\gamma^2 + \frac{9}{16}m^2e^2e_{\prime}^{2}$$

$$+ \frac{2475}{64}m^3e^2e_{\prime}^{2} + \frac{9}{4}m^2e_{\prime}^{2}\gamma^2 + \frac{693}{32}m^3e_{\prime}^{2}\gamma^2 - \frac{45}{32}m^2e_{\prime}^{4}$$

$$- \frac{225}{4}m^3e_{\prime}^{4} - \frac{27}{64}m^2\gamma^4 - \frac{2241}{256}m^3\gamma^4 + \frac{3}{32}m^2e^4$$

$$- \frac{45}{32}m^2\frac{a^2}{a_{\prime}^{2}} - \frac{7425}{512}m^3\frac{a^2}{a_{\prime}^{2}}$$

See *Théorie de la Lune*, vol. i. p. 485.

are at variance with each other, until it has been shown that the discrepancy is not removed by the subsequent approximation.

The equation which connects the quantities n and n may in like manner be obtained from the constant part of the expression for $d\varepsilon - d\varpi + d\nu$.

$$d\varepsilon - d\varpi + d\nu = -\frac{\sqrt{1-e^2}}{a^2 n e}\{1 - \sqrt{1-e^2}\}\frac{dR}{de}dt + \frac{2\,dR}{an\,da}dt$$

$$-\frac{(1-\cos\iota)}{a^2 n\sqrt{1-e^2}\sin\iota}\frac{dR}{d\nu}dt$$

$$\lambda' = \left\{1 + \frac{m^2}{(1-e_{\prime}^2)^{\frac{3}{2}}}\left\{\frac{3}{4}(\sqrt{1-e^2} - 1 + e^2)\right.\right.$$

$$\left.\left.-\left(1+\frac{3}{2}e^2\right)\left(1+\frac{3}{4}\right)\frac{(1-\cos\iota)}{\sqrt{1-e^2}}\cos\iota\right\}\right\}nt$$

$$+ \text{ periodic terms}$$

$$= \left\{1 - m^2 - \frac{9}{8}m^2e^2 - \frac{3}{2}m^2e_{\prime}^2 - \frac{3}{4}m^2\gamma^2 + \&c.\right\}nt + \&c.$$

$$\mathrm{n} = n\left\{1 - m^2 - \frac{9}{8}m^2e^2 - \frac{3}{2}m^2e_{\prime}^2 - \frac{3}{4}m^2\gamma^2 + \&c.\right\}$$

which agrees with the result I found, in the *Phil. Trans.* 1834, p. 138, by a different method; but it may be observed, that by the method here used terms are obtained, which by the method of p. 48 arise only from a further approximation.

The terms in c and g which have been found from the expressions for $\frac{d\varpi}{dt}$, $\frac{d\nu}{dt}$, may also be found from the expressions for $\frac{d\varpi}{d\lambda}$, $\frac{d\nu}{d\lambda}$, with equal facility.

$$e\,d\varpi + \{2 + e\cos(\lambda-\varpi)\sin(\lambda-\varpi)r^2\left(\frac{dR}{d\lambda}\right)d\lambda$$

$$-h^2r^2\cos(\lambda-\varpi)\left(\frac{dR}{dr}\right)d\lambda = 0$$

See p. 16. In this equation the square of the inclination is neglected.

It is sufficient to take

$$e\,\mathrm{d}\varpi = r^2 \cos(\lambda - \varpi)\frac{\mathrm{d}R}{\mathrm{d}r}\,\mathrm{d}\lambda$$

$$= -\frac{r^3}{2\,r_{\prime}^{3}}\cos(\lambda - \varpi)\,\mathrm{d}\lambda$$

It is also sufficient to take

$$r_{\prime}^{-3} = a_{\prime}^{-3}(1 - e^2)^{-\frac{3}{2}} \qquad \text{See p. 9.}$$

and in r^3 only the term multiplied by $\cos(\lambda - \varpi)$,

$$r^2\,\mathrm{d}\lambda = a^2 n\sqrt{1 - e^2}\,\mathrm{d}t$$

$$r^3\,\mathrm{d}\lambda = a\sqrt{1 - e^2}\,r^2\,\mathrm{d}v$$

$$\cos(\lambda - \varpi) = \frac{a(\cos v - e)}{r}$$

Let $r^3 = a^3\{A + Be\cos(\lambda - \varpi) + \&\text{c.}\}$

$$\{A + Be\cos(\lambda - \varpi) + \&\text{c.}\} = \sqrt{1 - e^2}\,\frac{r^2}{a^2}\,\mathrm{d}v$$

Multiplying both sides by $\cos(\lambda - \varpi)$,

$$\{A + Be\cos(\lambda - \varpi) + \&\text{c.}\}\cos(\lambda - \varpi)\,\mathrm{d}\lambda$$

$$= \sqrt{1 - e^2}\,(\cos v - e)(1 - e\cos v)\,\mathrm{d}v$$

$$= \sqrt{1 - e^2}\left\{\cos v - \frac{3}{2}e - \left(\frac{e}{2} + e^2\right)\cos 2v\right\}\mathrm{d}v$$

$$= \left\{\frac{B}{2} + \text{a series of cosines of multiples of } (\lambda - \varpi)\right\}\mathrm{d}\lambda$$

Integrating, making $\lambda - \varpi$ and v equal to 180° in the result, I find

$$B = -3\sqrt{1 - e^2}$$

and

$$\mathrm{c} = 1 - \frac{\mathrm{d}\varpi}{\mathrm{d}\lambda} = 1 - \frac{3}{4}\frac{(1 - e^2)^{\frac{1}{2}}}{(1 - e_{\prime}^{2})^{\frac{3}{2}}}m^2 \qquad \text{as before, p. 80.}$$

The part of R which is constant, is

$$= m_{\prime}\left\{-\frac{b_{1,0}}{2\,a_{\prime}} + \frac{a}{2\,a_{\prime}^{2}}\left\{\sin^2\frac{\iota}{2} - \frac{e^2 + e_{\prime}^{2}}{4}\right\}b_{3,1}\right\}$$

whence $\mathrm{c} = 1 - \dfrac{\mathrm{d}\varpi}{\mathrm{d}t}$

$$= 1 - \frac{m_{,} a^2}{4 \mu a_{,}^2} b_{3,1} = 1 - \frac{3}{4} \frac{m_{,}}{\mu} \frac{a^3}{a_{,}^3} - \frac{45}{32} \frac{m_{,}}{\mu} \frac{a^5}{a_{,}^5}$$

$$= 1 - \frac{3}{4} m^2 - \frac{45}{32} m^2 \frac{a^2}{a_{,}^2}$$

$$g = 1 - \frac{d v}{d t}$$

$$= 1 + \frac{m_{,}}{4 \mu} \frac{a^2}{a_{,}^2} b_{3,1} = 1 + \frac{3}{4} \frac{m_{,}}{\mu} \frac{a^3}{a_{,}^3} + \frac{45}{32} \frac{m_{,}}{\mu} \frac{a^5}{a_{,}^5}$$

$$= 1 + \frac{3}{4} m^2 + \frac{45}{32} m^2 \frac{a^2}{a_{,}^2}$$

The following are the numerical values of the constants adopted by M. Plana:

$m = {\cdot}07480130$ $\quad e = {\cdot}05484728$ $\quad e_{,} = {\cdot}01681013$

$\gamma = {\cdot}09005900$ $\quad \frac{a}{a_{,}} = {\cdot}00252064$

My letter a appears to correspond with a $(1 + p)$, or a in M. Plana's notation,

c c

g . . . . g

The letter e of M. Plana's notation corresponds to the quantity I have called e', p. 6. See *Théorie de la Lune*, vol. i. p. 50.

ON THE PERTURBATIONS OF THE PLANETS.

5. The determination of the perturbations of the planets, and of the satellites of Jupiter, flows as an easy corollary from the method already explained of determining the perturbations of the moon.

In the theory of the planetary perturbations the terms depending upon the square of the disturbing force are much less sensible than those which occur in the theory of the moon.

In order to render the Table applicable to the planetary theory, it is only necessary to write the indeterminate i in the arguments, retaining the same indices, thus;

0	0	23	$2\xi+\xi_{,}$	46	$i\tau+3\xi-\xi_{,}$
1	$i\tau$	24	$i\tau-2\xi-\xi_{,}$	47	$2\xi+2\xi_{,}$
2	ξ	25	$i\tau+2\xi+\xi_{,}$	48	$i\tau-2\xi-2\xi_{,}$
3	$i\tau-\xi$	26	$2\xi-\xi_{,}$	49	$i\tau+2\xi+2\xi_{,}$
4	$i\tau+\xi$	27	$i\tau-2\xi+\xi_{,}$	50	$2\xi-2\xi_{,}$
5	$\xi_{,}$	28	$i\tau+2\xi-\xi_{,}$	51	$i\tau-2\xi+2\xi_{,}$
6	$i\tau-\xi_{,}$	29	$\xi+2\xi_{,}$	52	$i\tau+2\xi-2\xi_{,}$
7	$i\tau+\xi_{,}$	30	$i\tau-\xi-2\xi_{,}$	53	$\xi+3\xi_{,}$
8	2ξ	31	$i\tau+\xi+2\xi_{,}$	54	$i\tau-\xi-3\xi_{,}$
9	$i\tau-2\xi$	32	$\xi-2\xi_{,}$	55	$i\tau+\xi+3\xi_{,}$
10	$i\tau+2\xi$	33	$i\tau-\xi+2\xi_{,}$	56	$\xi-3\xi_{,}$
11	$\xi+\xi_{,}$	34	$i\tau+\xi-2\xi_{,}$	57	$i\tau-\xi+3\xi_{,}$
12	$i\tau-\xi-\xi_{,}$	35	$3\xi_{,}$	58	$i\tau+\xi-3\xi_{,}$
13	$i\tau+\xi+\xi_{,}$	36	$i\tau-3\xi_{,}$	59	4ξ
14	$\xi-\xi_{,}$	37	$i\tau+3\xi_{,}$	60	$i\tau-4\xi_{,}$
15	$i\tau-\xi+\xi_{,}$	38	4ξ	61	$i\tau+4\xi_{,}$
16	$i\tau+\xi-\xi_{,}$	39	$i\tau-4\xi$	62	2η
17	$2\xi_{,}$	40	$i\tau+4\xi$	63	$i\tau-2\eta$
18	$i\tau-2\xi_{,}$	41	$3\xi+\xi_{,}$	64	$i\tau+2\eta$
19	$i\tau+2\xi_{,}$	42	$i\tau-3\xi-\xi_{,}$	146	η
20	3ξ	43	$i\tau+3\xi+\xi_{,}$	147	$i\tau-\eta$
21	$i\tau-3\xi$	44	$3\xi-\xi_{,}$	148	$i\tau+\eta$
22	$i\tau+3\xi$	45	$i\tau-3\xi+\xi_{,}$		

Two inequalities only of the order of the cubes of the eccentricities have yet been considered.

If nt, $n_{,}t$ represent the mean motions of Jupiter and Saturn, the following are the arguments of one of the inequalities in question, called the great inequality of Jupiter and Saturn:

$5\tau-3\xi$ or $2nt-5n_{,}t+3\varpi$ .... Arg. 21,5
$4\tau-2\xi-\xi_{,}$ or $2nt-5n_{,}t+2\varpi+\varpi_{,}$ 24,4
$3\tau-\xi-2\xi_{,}$ or $2nt-5n_{,}t+\varpi+2\varpi_{,}$ 33,3
$3\tau-\xi-2\eta_{,}$ or $2nt-5n_{,}t+\varpi+2\nu_{,}$
$2\tau-\xi_{,}-2\eta_{,}$ or $2nt-5n_{,}t+\varpi_{,}+2\nu_{,}$

If nt, $n_{,}t$ represent the mean motions of Mercury and the Earth, the other inequality arises from the arguments

$4\tau-3\xi$ or $nt-4n_{,}t+3\varpi$ .... Arg. 21,4
$3\tau-2\xi-\xi_{,}$ or $nt-4n_{,}t+2\varpi+\varpi_{,}$ 24,3
$2\tau-\xi-2\xi_{,}$ or $nt-4n_{,}t+\varpi+2\varpi_{,}$ 33,2
$2\tau-\xi-2\eta_{,}$ or $nt-4n_{,}t+\varpi+2\nu_{,}$
$\tau-\xi_{,}-2\eta_{,}$ or $nt-4n_{,}t+\varpi_{,}+2\nu_{,}$

The terms depending upon the fifth powers of the eccentricities are sensible in the great inequality of Jupiter. The only other inequality in which terms of this order have been found to be sensible, occurs in the theory of the Earth and Venus, and was discovered by Professor Airy.

If nt, $n_{,}t$ represent the mean motions of Venus and the Earth, this inequality arises from the arguments,

$13\tau - 5\xi$ or $8nt - 13n_{,}t + 5\varpi$
$12\tau - 4\xi - \xi_{,}$ or $8nt - 13n_{,}t + 4\varpi + \varpi_{,}$
$11\tau - 3\xi - 2\xi_{,}$ or $8nt - 13n_{,}t + 3\varpi + 2\varpi_{,}$
$10\tau - 2\xi - 3\xi_{,}$ or $8nt - 13n_{,}t + 2\varpi + 3\varpi_{,}$
$9\tau - \xi - 4\xi_{,}$ or $8nt - 13n_{,}t + \varpi + 4\varpi_{,}$
$8\tau - 5\xi_{,}$ or $8nt - 13n_{,}t + 5\varpi_{,}$
$13\tau - \xi - 4\eta$ or $8nt - 13n_{,}t + \varpi + 4\nu$
$12\tau - \xi_{,} - 4\eta$ or $8nt - 13n_{,}t + \varpi_{,} + 4\nu$
$13\tau - 3\xi - 2\eta$ or $8nt - 13n_{,}t + 3\varpi + 2\nu$
$12\tau - 2\xi - \xi_{,} - 2\eta$ or $8nt - 13n_{,}t + 2\varpi + \varpi_{,} + 2\nu$
$11\tau - \xi - 2\xi_{,} - 2\eta$ or $8nt - 13n_{,}t + \varpi + 2\varpi_{,} + 2\nu$
$10\tau - 3\xi_{,} - 2\eta$ or $8nt - 13n_{,}t + 3\varpi_{,} + 2\nu$

The care with which Professor Airy has determined the numerical amount of this inequality (see *Phil. Trans.* 1832, part 1.), is no less remarkable than its detection. This discovery by Professor Airy forms the only direct contribution towards the perfection of the Planetary Tables by an Englishman since the publication of the Tables of the Planets by Halley, in 1749.

These inequalities become sensible, as is well known, from the small divisors, $2n-5n_{,}$, $n-4n_{,}$, $8n-13n_{,}$, respectively, introduced by integration.

I now proceed, first to obtain the development of the disturbing function, and then show how the inequalities of the reciprocal of the radius vector and of the longitude may be obtained by means of the equations I have already employed in the lunar theory.

Let

$$\left\{1 - \frac{2a}{a_{,}}\cos\tau + \frac{a^2}{a_{,}^2}\right\}^{-\frac{n}{2}} = \frac{1}{2}b_{n,0} + b_{n,1}\cos\imath + b_{n,2}\cos 2\tau + \&c.$$

Then

$$\frac{1}{2}b_{1,0} = 1 + \left(\frac{1}{2}\right)^2\frac{a^2}{a_{,}^2} + \left(\frac{3}{2.4}\right)^2\frac{a^4}{a_{,}^4} + \left(\frac{3.5}{2.4.6}\right)^2\frac{a^6}{a_{,}^6} + \&c.$$

$$b_{1,1} = \frac{a}{a_{,}} + \frac{1.3}{2.4}\,\frac{a^3}{a_{,}^3} + \frac{1.3.3.5}{2.4.4.6}\,\frac{a^5}{a_{,}^5} + \&c.$$

$$b_{1,2} = \frac{3}{4}\,\frac{a^2}{a_{,}^2} + \frac{1.3.5}{2.4.6}\,\frac{a^4}{a_{,}^4} + \frac{1.3.3.5.7}{2.4.4.6.8}\,\frac{a^6}{a_{,}^6} + \&c.$$

$$b_{1,3} = \frac{3.5}{4.6}\,\frac{a^3}{a_{,}^3} + \frac{1.3.5.7}{2.4.6.8}\,\frac{a^5}{a_{,}^5} + \frac{1.3.3.5.7.9}{2.4.4.6.8.10}\,\frac{a^7}{a_{,}^7} + \&c.$$

$$\frac{1}{2}b_{3,0} = 1 + \left(\frac{3}{2}\right)^2\frac{a^2}{a_{,}^2} + \left(\frac{3.5}{2.4}\right)^2\frac{a^4}{a_{,}^4} + \left(\frac{3.5.7}{2.4.6}\right)^2\frac{a^6}{a_{,}^6} + \&c.$$

$$b_{3,1} = \frac{3a}{a_{,}} + \frac{3.3.5}{2.4}\,\frac{a^3}{a_{,}^3} + \frac{3.5.3.5.7.9}{2.4.4.6}\,\frac{a^5}{a_{,}^5} + \&c.$$

$$b_{3,2} = \frac{3.5}{4}\frac{a^2}{a_{\prime}^2} + \frac{3.3.5.7}{2.4.6}\frac{a^4}{a_{\prime}^4} + \frac{3.5.3.5.7}{2.4.4.6.8}\frac{a^6}{a_{\prime}^6} + \&c.$$

$$b_{3,3} = \frac{3.5.7}{4.6}\frac{a^3}{a_{\prime}^3} + \frac{3.3.5.7.9}{2.4.6.8}\frac{a^5}{a_{\prime}^5} + \frac{3.5.3.5.7.9.11}{2.4.4.6.8.10}\frac{a^7}{a_{\prime}^7} + \&c.$$

$$\frac{1}{2}b_{5,0} = 1 + \left(\frac{5}{2}\right)^2\frac{a^2}{a_{\prime}^2} + \left(\frac{5.7}{2.4}\right)^2\frac{a^4}{a_{\prime}^4} + \left(\frac{5.7.9}{2.4.6}\right)^2\frac{a^6}{a_{\prime}^6} + \&c.$$

$$b_{5,1} = 5\frac{a}{a_{\prime}} + \frac{5.5.7}{2.4}\frac{a^3}{a_{\prime}^3} + \frac{5.7.5.7.9}{2.4.4.6}\frac{a^5}{a_{\prime}^5} + \&c.$$

$$b_{5,2} = \frac{5.7}{4}\frac{a^2}{a_{\prime}^2} + \frac{5.5.7.9}{2.4.6}\frac{a^4}{a_{\prime}^4} + \frac{5.7.5.7.9.11}{2.4.4.6.8}\frac{a^6}{a_{\prime}^6} + \&c.$$

$$b_{5,3} = \frac{5.7.9}{4.6}\frac{a^3}{a_{\prime}^3} + \frac{5.5.7.9.11}{2.4.6.8}\frac{a^5}{a_{\prime}^5} + \frac{5.7.5.7.9.11.13}{2.4.4.6.8.10}\frac{a^7}{a_{\prime}^7}$$

These well known expressions for the quantities b are here given for the convenience of reference. The following equations of condition obtain, of which the first three are given in the *Mécanique Céleste*, vol. i. p. 268. See also the *Mechanism of the Heavens*, p. 243. In the notation of the *Méc. Cél.* $b_{n,i} = b_{\frac{n}{2}}^{(i)}$

$$\frac{(2i-n)}{2}b_{n,i} = (i-1)\frac{(a^2+a_{\prime}^2)}{a_{\prime}^2}b_{n,i-1} - \frac{(2i-4+n)}{2}\frac{a}{a_{\prime}}b_{n,i-2}$$

$$b_{n,i} = \frac{(a^2+a_{\prime}^2)}{a_{\prime}^2}b_{n+2,i} - \frac{a}{a_{\prime}}b_{n+2,i-1} - \frac{a}{a_{\prime}}b_{n+2,i+1}$$

$$\left(1-\frac{a^2}{a_{\prime}^2}\right)^2 b_{n,i} = \frac{(2i-2+n)}{(n-2)}\left(1+\frac{a^2}{a_{\prime}^2}\right)b_{n-2,i} - \frac{2(2i-n+4)}{(n-2)}\frac{a}{a_{\prime}}b_{n-2,i+1}$$

$$i\,b_{n,i} = \frac{n\,a}{2\,a_{\prime}}\left\{b_{n+2,i-1} - b_{n+2,i+1}\right\}$$

Since

$$\left\{1-\frac{2a}{a_{\prime}}\cos\tau + \frac{a^2}{a_{\prime}^2}\right\}^{-\frac{n}{2}} = \frac{1}{2}b_{n,0} + b_{n,1}\cos\tau + b_{n,2}\cos 2\tau + \&c.$$

Differentiating with regard to τ,

$$\frac{n\,a}{a_{\prime}}\sin\tau\left\{\frac{1}{2}b_{n+1,0} + b_{n+1,1}\cos\tau + b_{n+2,2}\cos 2\tau + \&c.\right\} = b_{n,1}\sin\tau + 2\,b_{n,2}\sin 2\tau + \&c.$$

and equating coefficients, the last equation is obtained.

Also,

$$\frac{a\,\mathrm{d}\,b_{n,i}}{\mathrm{d}\,a} = -\frac{n\,a}{a_{,}}\left\{\frac{a}{a_{,}}\,b_{n+2,i} - \frac{1}{2}\,b_{n+2,i+1} - \frac{1}{2}\,b_{n+2,i+1}\right\}$$

$$\frac{a_{,}\,\mathrm{d}\,b_{n,i}}{\mathrm{d}\,a_{,}} = -\frac{a\,\mathrm{d}\,b_{n,i}}{\mathrm{d}\,a}$$

$b_{n,i}$ is the same as $b_{\frac{s}{2}}^{(i)}$ in the notation of the *Méc. Cél.*

The following are particular instances of the general equations above, and are inserted here for the convenience of reference.

$$\frac{3}{2}\frac{a}{a_{,}}\,b_{1,2} = \frac{(a^2+a_{,}^2)}{a_{,}^2}\,b_{1,1} - \frac{a}{2\,a_{,}}\,b_{1,0}$$

$$\frac{7}{2}\frac{a}{a_{,}}\,b_{1,4} = \frac{3\,(a^2+a_{,}^2)}{a_{,}^2}\,b_{1,3} - \frac{5}{2}\frac{a}{a_{,}}\,b_{1,2}$$

$$\frac{5}{2}\frac{a}{a_{,}}\,b_{1,3} = \frac{2\,(a^2+a_{,}^2)}{a_{,}^2}\,b_{1,2} - \frac{3}{2}\frac{a}{a_{,}}\,b_{1,1}$$

$$\frac{9}{2}\frac{a}{a_{,}}\,b_{1,5} = \frac{4\,(a^2+a_{,}^2)}{a_{,}^2}\,b_{1,4} - \frac{7}{2}\frac{a}{a_{,}}\,b_{1,3}$$

$$\frac{a}{2a_{,}}\,b_{3,2} = \frac{(a^2+a_{,}^2)}{a_{,}^2}\,b_{3,1} - \frac{3\,a}{2a_{,}}\,b_{3,0}$$

$$\frac{5}{2}\frac{a}{a_{,}}\,b_{3,4} = \frac{3\,(a^2+a_{,}^2)}{a_{,}^2}\,b_{3,3} - \frac{7}{2}\frac{a}{a_{,}}\,b_{3,2}$$

$$\frac{3}{2}\frac{a}{a_{,}}\,b_{3,3} = \frac{2\,(a^2+a_{,}^2)}{a_{,}^2}\,b_{3,2} - \frac{5}{2}\frac{a}{a_{,}}\,b_{3,1}$$

$$\frac{7}{2}\frac{a}{a_{,}}\,b_{3,5} = \frac{4\,(a^2+a_{,}^2)}{a_{,}^2}\,b_{3,4} - \frac{9}{2}\frac{a}{a_{,}}\,b_{3,3}$$

$$-\frac{a}{2a_{,}}\,b_{5,2} = \frac{(a^2+a_{,}^2)}{a_{,}^2}\,b_{5,1} - \frac{5\,a}{2a_{,}}\,b_{5,0}$$

$$\frac{3}{2}\frac{a}{a_{,}}\,b_{5,4} = \frac{3\,(a^2+a_{,}^2)}{a_{,}^2}\,b_{5,3} - \frac{9}{2}\frac{a}{a_{,}}\,b_{5,2}$$

$$\frac{a}{2a_{,}}\,b_{5,3} = \frac{2\,(a^2+a_{,}^2)}{a_{,}^2}\,b_{5,2} - \frac{7}{2}\frac{a}{a_{,}}\,b_{5,1}$$

$$\frac{5}{2a_{,}}\,b_{5,5} = \frac{4\,(a^2+a_{,}^2)}{a_{,}^2}\,b_{5,4} - \frac{11}{2}\frac{a}{a_{,}}\,b_{5,3}$$

$$b_{1,0} = \frac{a^2+a_{,}^2}{a_{,}^2}\,b_{3,0} - 2\,\frac{a}{a_{,}}\,b_{3,1}$$

$$b_{1,2} = \frac{a^2+a_{,}^2}{a_{,}^2}\,b_{3,2} - \frac{a}{a_{,}}\,b_{3,1} - \frac{a}{a_{,}}\,b_{3,3}$$

$$b_{1,1} = \frac{a^2+a_{,}^2}{a_{,}^2}\,b_{3,1} - \frac{a}{a_{,}}\,b_{3,0} - \frac{a}{a_{,}}\,b_{3,2}$$

$$b_{1,3} = \frac{a^2+a_{,}^2}{a_{,}^2}\,b_{3,3} - \frac{a}{a_{,}}\,b_{3,2} - \frac{a}{a_{,}}\,b_{3,4}$$

$$b_{3,0} = \frac{a^2+a_{,}^2}{a_{,}^2}\,b_{5,0} - 2\,\frac{a}{a_{,}}\,b_{5,1}$$

$$b_{3,2} = \frac{a^2+a_{,}^2}{a_{,}^2}\,b_{5,2} - \frac{a}{a_{,}}\,b_{5,1} - \frac{a}{a_{,}}\,b_{5,3}$$

$$b_{3,1} = \frac{a^2+a_{,}^2}{a_{,}^2}\,b_{5,1} - \frac{a}{a_{,}}\,b_{5,0} - \frac{a}{a_{,}}\,b_{5,2}$$

$$b_{3,3} = \frac{a^2+a_{,}^2}{a_{,}^2}\,b_{5,3} - \frac{a}{a_{,}}\,b_{5,2} - \frac{a}{a_{,}}\,b_{5,4}$$

$$b_{1,1} = \frac{a}{2\,a_{,}}\left\{b_{3,0} - b_{3,2}\right\}$$

$$3\,b_{1,3} = \frac{a}{2\,a_{,}}\left\{b_{3,2} - b_{3,4}\right\}$$

$$b_{3,1} = \frac{3}{2}\frac{a}{a_{,}}\left\{b_{5,0} - b_{5,2}\right\}$$

$$3\,b_{3,3} = \frac{3}{2}\frac{a}{a_{,}}\left\{b_{5,2} - b_{5,4}\right\}$$

$$2\,b_{1,2} = \frac{a}{2\,a_{,}}\left\{b_{3,1} - b_{3,3}\right\}$$

$$4\,b_{1,4} = \frac{a}{2\,a_{,}}\left\{b_{3,2} - b_{3,4}\right\}$$

$$2\,b_{3,2} = \frac{3}{2}\frac{a}{a_{,}}\left\{b_{5,1} - b_{5,3}\right\}$$

$$4\,b_{3,4} = \frac{3}{2}\frac{a}{a_{,}}\left\{b_{5,3} - b_{5,5}\right\}$$

$$\frac{1\,.\,b_{1,0}}{\mathrm{d}\,a} = -\frac{a}{a_{\prime}}\left\{\frac{a}{a_{\prime}}b_{3,0} - b_{3,1}\right\} \qquad \frac{a\,\mathrm{d}\,.\,b_{3,0}}{\mathrm{d}\,a} = -3\frac{a}{a_{\prime}}\left\{\frac{a}{a_{\prime}}b_{5,0} - b_{5,1}\right\}$$

$$\frac{1\,.\,b_{1,1}}{\mathrm{d}\,a} = -\frac{a}{a_{\prime}}\left\{\frac{a}{a_{\prime}}b_{3,1} - \frac{1}{2}b_{3,0} - \frac{1}{2}b_{3,2}\right\} \qquad \frac{a\,\mathrm{d}\,.\,b_{3,1}}{\mathrm{d}\,a} = -3\frac{a}{a_{\prime}}\left\{\frac{a}{a_{\prime}}b_{5,1} - \frac{1}{2}b_{5,0} - \frac{1}{2}b_{5,1}\right\}$$

$$\frac{1\,.\,b_{1,2}}{\mathrm{d}\,a} = -\frac{a}{a_{\prime}}\left\{\frac{a}{a_{\prime}}b_{3,2} - \frac{1}{2}b_{3,1} - \frac{1}{2}b_{3,3}\right\} \qquad \frac{a\,\mathrm{d}\,.\,b_{5,0}}{\mathrm{d}\,a} = -5\frac{a}{a_{\prime}}\left\{\frac{a}{a_{\prime}}b_{7,0} - b_{7,1}\right\}$$

$$\frac{1\,.\,b_{1,3}}{\mathrm{d}\,a} = -\frac{a}{a_{\prime}}\left\{\frac{a}{a_{\prime}}b_{3,3} - \frac{1}{2}b_{3,2} - \frac{1}{2}b_{3,4}\right\} \qquad \frac{a\,\mathrm{d}\,.\,b_{5,1}}{\mathrm{d}\,a} = -5\frac{a}{a_{\prime}}\left\{\frac{a}{a_{\prime}}b_{7,1} - \frac{1}{2}b_{7,0} - \frac{1}{2}b_{7,2}\right\}$$

The quantities $b_{1,0}$, $b_{1,1}$, from which all the other quantities b_3, b_5, &c. depend, may be obtained at once from Table IX. in the *Exercises de calcul intégral*, by M. Legendre, vol. iii. See also vol. i. p. 171 of the same work.

$$\left(1 + \frac{a}{a_{\prime}}\right)b_{1,0} = \frac{4}{\pi}\int\frac{\mathrm{d}\,\phi}{\Delta} \qquad \left(1 + \frac{a}{a_{\prime}}\right)b_{1,1} = \frac{2}{\pi}\int - \frac{\mathrm{d}\,\phi\cos 2\,\phi}{\Delta}$$

the integrals being taken from $\phi = 0$ to $\phi = \frac{1}{2}\pi$.

$$\Delta = \surd(1 - c^2\sin^2\phi)$$

$c^2 = \dfrac{4\,a\,a_{\prime}}{(a + a_{\prime})^2} = \dfrac{4\,\alpha}{(1 + \alpha)^2}$, α* being $= \dfrac{a}{a_{\prime}}$ as in the notation of the *Mécanique Céleste.*

$$b_{1,0} = \frac{4}{\pi(1 + \alpha)}\mathrm{F}^1 \qquad b_{1,1} = \frac{.\,2}{\pi(1 + \alpha)}\left\{\frac{2}{c^2}(\mathrm{F}^1 - \mathrm{E}^1) - \mathrm{F}^1\right\}$$

In the theory of Jupiter disturbed by Saturn, $\alpha = {\cdot}54531725$; and hence in this instance if $c = \sin\theta$, $\theta = 72° 53' 17''$.

By interpolation, I find from Table IX. p. 424,

$\mathrm{F}\,(72° 53' 18'') = 2{\cdot}6460986$

and $b_{1,0} = 2{\cdot}180214$, which differs but slightly from the value of $b_{1,0}$ given by Laplace, viz. $2{\cdot}1802348$.

Taking the orbit of the planet m for the plane $x\,y$,

$$R = m_{\prime}\left\{\frac{r\,r_{\prime}'\cos(\lambda - \lambda_{\prime}')}{r_{\prime}^3} - \frac{1}{\{r^2 - 2\,r\,r_{\prime}'\cos(\lambda - \lambda_{\prime}') + r_{\prime}^2\}^{\frac{1}{2}}}\right\}$$

If $r^2 - 2\,r\,r_{\prime}'\cos(\lambda - \lambda_{\prime}') + r_{\prime}^2 = a^2 - 2\,a^2\,a_{\prime}\cos\tau + a_{\prime}^2 + E$, E being a function of the eccentricities and inclinations,

$$R = \text{terms independent of } b$$

$$- \frac{m_{\prime}}{\{a^2 - 2\,a\,a_{\prime}\cos\tau + a_{\prime}^2\}^{\frac{1}{2}}}$$

---

* ϱ in the notation of Woodhouse's *Astronomy*, vol. ii. p. 287.

$$+\frac{m_{\prime}E}{2\{a^2-2aa_{\prime}\cos\tau+a_{\prime}^2\}^{\frac{3}{2}}}$$

$$-\frac{3m_{\prime}E^2}{2.4\{a^2-2aa_{\prime}\cos\tau+a_{\prime}^2\}^{\frac{5}{2}}}+\&c.$$

and putting for E its value

R = terms independent of b

$$-\frac{m_{\prime}}{a_{\prime}}\left\{\frac{1}{2}b_{1,0}+b_{1,1}\cos\tau+b_{1,2}\cos 2\tau+\&c.\right\}$$

$$+\frac{m_{\prime}}{2a_{\prime}^3}\left\{\frac{1}{2}b_{3,0}+b_{3,1}\cos\tau+b_{3,2}\cos 2\tau+\&c.\right\}$$

$$\left\{a^2\left\{\frac{3}{2}e^2-2e\cos\xi-\frac{e^2}{2}\cos 2\xi\right\}-2aa_{\prime}\left\{\left(-\sin^2\frac{\iota_{\prime}}{2}-\frac{e^2+e_{\prime}^2}{2}\right)\cos\tau\right.\right.$$

$$-\frac{3}{2}e\cos(\tau-\xi)+\frac{e}{2}\cos(\tau+\xi)+\frac{e}{2}\cos(\tau+\xi_{\prime})-\frac{3}{2}e_{\prime}\cos(\tau-\xi_{\prime})$$

$$+\frac{e^2}{8}\cos(\tau-2\xi)+\frac{3}{8}e^2\cos(\tau+2\xi)-\frac{3}{4}ee_{\prime}\cos(\tau-\xi-\xi_{\prime})$$

$$-\frac{3}{4}ee_{\prime}\cos(\tau+\xi+\xi_{\prime})+\frac{9}{4}ee_{\prime}\cos(\tau-\xi+\xi_{\prime})+\frac{ee_{\prime}}{4}\cos(\tau+\xi-\xi_{\prime})$$

$$\left.+\frac{3}{8}e_{\prime}^2\cos(\tau-2\xi_{\prime})+\frac{e_{\prime}^2}{8}\cos(\tau+2\xi_{\prime})+\sin^2\frac{\iota_{\prime}}{2}\cos(2\tau-\eta_{\prime})\right\}$$

$$\left.+a_{\prime}^2\left\{\frac{3}{2}e_{\prime}^2-2e_{\prime}\cos\xi_{\prime}-\frac{e_{\prime}^2}{2}\cos 2\xi_{\prime}\right\}+\&c.\right]$$

$$-\frac{3}{2.4}\frac{m_{\prime}}{a_{\prime}^5}\left\{\frac{1}{2}b_{5,0}+b_{5,1}\cos\tau+b_{5,2}\cos 2\tau+\&c.\right\}$$

$$\left\{2a^2e\cos\xi-3aa_{\prime}e\cos(\tau-\xi_{\prime})+aa_{\prime}e\cos(\tau+\xi)\right.$$

$$\left.+aa_{\prime}e_{\prime}\cos(\tau+\xi_{\prime})-3aa_{\prime}e_{\prime}\cos(\tau+\xi_{\prime})+2a_{\prime}^2e_{\prime}\cos\xi_{\prime}\right\}^2+\&c.$$

By developing this value of R, an expression is obtained which is complicated, but admits of many reductions in consequence of the equations of condition between the quantities b.

The first term of R for instance

$$= m_{,}\left\{-\frac{b_{1,0}}{2a_{,}}+\frac{3(a^2e^2+a_{,}^2e_{,}^2)}{2.4\,a_{,}^3}b_{3,0}+\frac{a\,a_{,}}{2\,a_{,}^3}\left(\sin^2\frac{\iota_{,}}{2}+\frac{(e^2+e_{,}^2)}{2}\right)b_{3,1}\right.$$

$$-\frac{3}{4.4\,a_{,}^5}\left(2\,a^4e^2+5\,a^2a_{,}^2(e^2+e_{,}^2)+2\,a_{,}^4e_{,}^2\right)b_{5,0}$$

$$\left.+\frac{3.2}{2.4\,a_{,}^5}(a^2e^2+a_{,}^2e_{,}^2)\,a\,a_{,}b_{5,1}+\frac{1.3.3}{2.4.2}\,\frac{a^2a_{,}^2}{a_{,}^5}(e^2+e_{,}^2)\,b_{5,2}\right\}$$

$$= m_{,}\left\{-\frac{b_{1,0}}{2a_{,}}+\frac{3(a^2e^2+a_{,}^2e_{,}^2)}{2.4\,a_{,}^3}b_{3,0}+\frac{a\,a_{,}}{2\,a_{,}^3}\left(\sin^2\frac{\iota_{,}}{2}+\left(\frac{e^2+e_{,}^2}{2}\right)\right)b_{3,1}\right.$$

$$-\frac{3.2}{2.4}\,\frac{(a^2+a_{,}^2)}{a_{,}^2}\,\frac{(a^2e^2+a_{,}^2e_{,}^2)}{2\,a_{,}^3}b_{5,0}-\frac{3.3}{4.4}\,\frac{a^2a_{,}^2}{a_{,}^5}(e^2+e_{,}^2)\,b_{5,0}$$

$$\left.+\frac{3.2}{2.4}\,\frac{(a^2e^2+a_{,}^2e_{,}^2)}{a_{,}^3}\,\frac{a}{a_{,}}b_{5,1}+\frac{3.3}{2.4.2}\,\frac{a^2a_{,}^2}{a_{,}^5}(e^2+e_{,}^2)\,b_{5,2}\right\}$$

and since

$$b_{3,0}=\frac{(a^2+a_{,}^2)}{a_{,}^2}b_{5,0}-2\,\frac{a}{a_{,}}\,b_{5,1}\qquad\text{See p. 24.}$$

$$b_{3,1}=\frac{3\,a}{2\,a_{,}}\left\{b_{5,0}-b_{5,2}\right\}$$

this term reduces itself to

$$m_{,}\left\{-\frac{b_{1,0}}{2\,a_{,}}+\frac{a\,a_{,}}{2\,a_{,}^3}\left(\sin^2\frac{\iota_{,}}{2}+\frac{e^2+e_{,}^2}{2}\right)b_{3,1}-\frac{3}{2.4}\,\frac{a\,a_{,}}{a_{,}^3}(e^2+e_{,}^2)\,b_{3,1}\right\}$$

$$= m_{,}\left\{-\frac{b_{1,0}}{2\,a_{,}}+\frac{a}{2\,a_{,}^2}\left(\sin^2\frac{\iota_{,}}{2}-\frac{e^2+e_{,}^2}{4}\right)b_{3,1}\right\}$$

Again, the coefficient of $\cos(\varpi-\varpi_{,})$, or $\cos(\tau-\xi+\xi_{,})$

$$= m_{,}\left\{-\frac{9}{8}\,\frac{a}{a_{,}^3}b_{3,0}-\frac{a}{8\,a_{,}^2}b_{3,2}+\frac{3.6}{2.4}\,\frac{(a^2+a_{,}^2)}{2\,a_{,}^5}\,a\,a_{,}b_{5,0}\right.$$

$$\left.-\frac{3.7}{2.4.2}\,\frac{a^2a_{,}^2}{a_{,}^5}b_{5,1}-\frac{3}{2.4}\,\frac{(a^2+a_{,}^2)}{a_{,}^5}\,a\,a_{,}b_{5,2}-\frac{3}{2.4.2}\,\frac{a^2a_{,}^2}{a_{,}^5}b_{5,3}\right\}e\,e_{,}$$

$$= m_{,}\left\{-\frac{9}{8}\,\frac{a}{a_{,}^2}b_{3,0}-\frac{a}{8\,a_{,}^2}b_{3,2}+\frac{3.6}{2.4}\,\frac{a}{a_{,}^2}\left\{\frac{(a^2+a_{,}^2)}{2\,a_{,}^2}b_{5,0}-\frac{a}{a_{,}}\,b_{5,1}\right\}\right.$$

$$\left.-\frac{3}{8}\,\frac{a}{a_{,}^2}\left\{\frac{(a^2+a_{,}^2)}{a_{,}^2}b_{5,2}-\frac{a}{a_{,}}b_{5,1}-\frac{a}{d_{,}}b_{5,3}\right\}+\frac{9\,a^2}{16\,a_{,}^3}\left\{b_{5,1}-b_{5,3}\right\}\right\}$$

$$= m_{,}\left\{-\frac{9}{8}\,\frac{a}{a_{,}^2}b_{3,0}-\frac{a}{8\,a_{,}^2}b_{3,2}+\frac{9}{8}\,\frac{a}{a_{,}^2}b_{3,0}-\frac{3}{8}\,\frac{a}{a_{,}^2}b_{3,2}+\frac{2.3}{8}\,\frac{a}{a_{,}^2}b_{3,2}\right\}e\,e_{,}$$

$$= m_{,}\frac{a}{4\,a_{,}^2}e\;e\;b_{3,2}$$

So that the part of R which is independent of $n\,t$ and $n_{,}\,t$,

$$= m_{,}\left\{-\frac{b_{1,0}}{2a_{,}}+\frac{a}{2a_{,}^2}\left(\sin^2\frac{\iota_{,}}{2}-\frac{e^2+e_{,}^2}{4}\right)b_{3,1}+\frac{a}{4a_{,}^2}b_{3,2}ee_{,}\cos(\varpi-\varpi_{,})\right\}$$

$$= m_{,}\left\{-\frac{b_{1,0}}{2a_{,}} + \frac{a}{2a_{,}^2}\left(\sin^2\frac{\iota_{,}}{2} - \frac{e^2+e_{,}^2}{4}\right)b_{3,1}\right.$$

$$\left. - \left\{\frac{3a}{4a_{,}^2}b_{3,0} - \frac{a^2+a_{,}^2}{2a_{,}^3}b_{3,1}\right\} e e_{,} \cos(\varpi - \varpi_{,})\right\}$$

In the general case, when ι_1, ι_2 are the inclinations of the orbits of the planets P and $P_{,}$ to any plane, the direction of which is arbitrary,

$$\cos\iota_{,} = \cos\iota_1\cos\iota_2 + \sin\iota_1\sin\iota_2\cos(\nu_1 - \nu_2)$$

$$\sin^2\frac{\iota_{,}}{2} = \frac{1 - \cos\iota_1\cos\iota_2 - \sin\iota_1\sin\iota_2\cos(\nu_1 - \nu_2)}{2}$$

The part of R which is independent of $n\,t$, $n_{,}\,t$

$$= m_{,}\left\{-\frac{b_{1,0}}{2a_{,}} + \frac{a}{4a_{,}^2}\left\{1 - \cos\iota_1\cos\iota_2 - \sin\iota_1\sin\iota_2\cos(\nu_1 - \nu_2)\right.\right.$$

$$\left.\left. - \frac{e^2+e_{,}^2}{2}\right\}b_{3,1} - \left\{\frac{3a}{4a_{,}^2}b_{3,0} - \frac{(a^2+a_{,}^2)}{2a_{,}^3}b_{3,1}\right\} e e_{,}\cos(\varpi - \varpi_{,})\right\}$$

$$\cos\iota = \frac{1}{\sqrt{1+\tan^2\iota}} = 1 - \tfrac{1}{2}\tan^2\iota, \quad \sin\iota = \frac{\tan\iota}{\sqrt{1+\tan^2\iota}} = \tan\iota \text{ nearly.}$$

$$= m_{,}\left\{-\frac{b_{1,0}}{2a_{,}} + \frac{a}{8a_{,}^2}\left\{(\tan^2\iota_1 + \tan^2\iota_2 - 2\tan\iota_1\tan\iota_2\cos(\nu_1 - \nu_2)\right.\right.$$

$$\left.\left. - e^2 - e_{,}^2\right\}b_{3,1} - \left\{\frac{3a}{4a_{,}^2}b_{3,0} - \frac{(a^2+a_{,}^2)}{2a_{,}^3}b_{3,1}\right\} e e_{,}\cos(\varpi - \varpi_{,})\right\}$$

$$= m_{,}\left\{-\frac{b_{1,0}}{2a_{,}} + \frac{a}{8a_{,}^2}\left\{(\tan\iota_1\cos\nu_1 - \tan\iota_2\cos\nu_2)^2\right.\right.$$

$$\left. + (\tan\iota_1\sin\nu_1 - \tan\iota_2\sin\iota_2)^2 - e^2 - e_{,}^2\right\}b_{3,1}$$

$$\left. - \left\{\frac{3a}{4a_{,}^2}b_{3,0} - \frac{(a^2+a_{,}^2)}{2a_{,}^3}b_{3,1}\right\} e e_{,}\cos(\varpi - \varpi_{,})\right\}$$

and if $\tan\iota\sin\nu = p$, $\tan\iota\cos\nu = q$, this quantity

$$= m_{,}\left\{-\frac{b_{1,0}}{2a_{,}} + \frac{a}{8a_{,}^2}\left\{(p_1 - p_2)^2 + (q_1 - q_2)^2 - e^2 - e_{,}^2\right\}b_{3,1}\right.$$

$$-\left\{\frac{3\,a}{4\,a_{,}^{2}}b_{3,0}-\frac{a^{2}+a_{,}^{2}}{2\,a_{,}^{3}}b_{3,1}\right\}e\,e_{,}\cos(\varpi-\varpi_{,})\Bigg\}$$

which evidently agrees with the result given by M. de Pontécoulant, *Théor. anal.* vol. i. p. 363. All the other coefficients of terms multiplied by the squares and products of the eccentricities are susceptible of reductions similar to those in the two preceding pages, and finally;

$$R=m_{,}\frac{a}{a_{,}^{2}}\left\{\cos^{2}\frac{\iota_{,}}{2}-\frac{e^{2}+e_{,}^{2}}{2}\right\}\cos\tau$$

$$-\frac{3\,m_{,}}{2}\frac{a}{a_{,}^{2}}e\cos(\tau-\xi)+\frac{m_{,}a}{2\,a_{,}^{2}}e\cos(\tau+\xi)$$

$$+\frac{2\,m_{,}a}{a_{,}^{2}}e_{,}\cos(\tau-\xi_{,})+\frac{m_{,}a}{8\,a_{,}^{2}}e^{2}\cos(\tau+\xi_{,})$$

$$+\frac{3\,m_{,}a}{8\,a_{,}^{2}}e^{2}\cos(\tau+2\,\xi)-\frac{3\,m_{,}a}{a_{,}^{2}}e\,e_{,}\cos(\tau-\xi-\xi_{,})$$

$$+\frac{m_{,}a}{a_{,}^{2}}e\,e_{,}\cos(\tau+\xi-\xi_{,})+\frac{27}{8}\frac{m_{,}a}{a_{,}^{2}}e_{,}^{2}\cos(\tau-2\,\xi_{,})$$

$$+\frac{m_{,}a}{8\,a_{,}^{2}}e_{,}^{2}\cos(\tau+2\,\xi_{,})+\frac{m_{,}a}{a_{,}^{2}}\sin^{2}\frac{\iota_{,}}{2}\cos(\tau+2\,\eta_{,})$$

$$+m_{,}\Sigma\left\{-\frac{b_{1,i}}{2\,a_{,}}+\frac{a}{4\,a_{,}^{2}}\sin^{2}\frac{\iota_{,}}{2}\Big(b_{3,i-1}+b_{3,i+1}\Big)\right.$$

$$\left.+\frac{a\,(e^{2}+e_{,}^{2})}{16\,a_{,}^{2}}\Big((3\,i-1)\,b_{3,i-1}-(3\,i+1)\,b_{3,i+1}\Big)\right\}\cos i\,\tau$$

$$+m_{,}\Sigma\left\{-\frac{a}{4\,a_{,}^{2}}b_{3,i-1}-\frac{a^{2}}{2\,a_{,}^{3}}b_{3,i}+\frac{3\,a}{4\,a_{,}^{2}}b_{3,i+1}\right\}e\cos(i\,\tau+\xi)$$

$$+m_{,}\Sigma\left\{\frac{3}{4}\frac{a}{a_{,}^{2}}b_{3,i-1}-\frac{1}{2\,a_{,}}b_{3,i}-\frac{a}{4\,a_{,}^{2}}b_{3,i+1}\right\}e_{,}\cos(i\,\tau+\xi_{,})$$

$$+m_{,}\Sigma\left\{-\frac{(2+i)}{16}\frac{a}{a_{,}^{2}}b_{3,i-1}-\frac{(1+i)}{2}\frac{a^{2}}{a_{,}^{3}}b_{3,i}\right.$$

$$\left.+\frac{(8+9\,i)}{16}\frac{a}{a_{,}^{2}}b_{3,i+1}\right\}e^{2}\cos(i\,\tau+2\,\xi)$$

$$+ m_{,} \Sigma \Big\{ \frac{(3+9i)}{8} \frac{a}{a_{,}^{2}} b_{3,i-1} - \frac{i}{a_{,}} b_{3,i}$$

$$- \frac{(1+i)}{8} \frac{a}{a_{,}^{2}} b_{3,i+1} \Big\} e\, e_{,} \cos (i\tau + \xi + \xi_{,})$$

$$+ m_{,} \Sigma \Big\{ - \frac{(1+3i)}{8} \frac{a}{a_{,}^{2}} b_{3,i-1}$$

$$+ \frac{3(1+i)}{8} \frac{a}{a_{,}^{2}} b_{3,i+1} \Big\} e\, e_{,} \cos (i\tau + \xi - \xi_{,})$$

$$+ m_{,} \Sigma \Big\{ \frac{(8-9i)}{16} \frac{a}{a_{,}^{2}} b_{3,i-1} - \frac{(1-i)}{2a_{,}} b_{3,i}$$

$$- \frac{(2-i)}{16} \frac{a}{a_{,}^{2}} b_{3,1+i} \Big\} e_{,}^{2} \cos (i\tau + 2\xi_{,})$$

$$- m_{,} \Sigma \frac{a}{2a_{,}^{2}} b_{3,i-1} \sin^{2} \frac{\iota_{,}}{2} \cos (i\tau + 2\eta_{,})$$

i being every whole number, positive, negative and zero, and observing that $b_{m,n} = b_{m,-n}$. The preceding expression differs in form from that given in the *Mécanique Céleste*, and in the *Théorie analytique du Système du Monde*, vol. i. p. 463, and is more simple. M. de Pontécoulant has given two examples of reductions applied to the terms upon which the *secular inequalities* depend; but these reductions may be extended to all the terms, and then the expression is obtained which I have given above.

In order to prevent the confusion which arises from too great generality, and for the convenience of reference, I shall now write at length the development of R, which results from the last expression, by giving some particular values to the indeterminate i.

$$R = m_{,} \Big\{ - \frac{b_{1,0}}{2a_{,}} + \frac{a}{2a_{,}^{2}} \Big\{ \sin^{2} \frac{\iota_{,}}{2} - \frac{e^{2} + e_{,}^{2}}{4} \Big\} b_{3,1}$$

$$+ \Big\{ \frac{a}{a_{,}^{2}} \Big(\cos^{2} \frac{\iota_{,}}{2} - \frac{e^{2} + e_{,}^{2}}{2} \Big) - \frac{b_{1,1}}{a_{,}} + \frac{a}{2a_{,}^{2}} \sin^{2} \frac{\iota_{,}}{2} (b_{3,0} + b_{3,2})$$

$$+ \frac{a}{a_{,}^{2}} \Big(\frac{e^{2} + e_{,}^{2}}{8} \Big) (2b_{3,0} - 4b_{3,2}) \Big\} \cos \tau$$

$$+ \Big\{ - \frac{b_{1,2}}{a_{,}} + \frac{a}{2a_{,}^{2}} \sin^{2} \frac{\iota_{,}}{2} (b_{3,1} + b_{3,3}) + \frac{a}{a_{,}^{2}} \frac{(e^{2} + e_{,}^{2})}{8} (5b_{3,1} - 7b_{3,3}) \Big\} \cos 2\tau$$

$$+ \Big\{ - \frac{b_{1,3}}{a_{,}} + \frac{a}{2a_{,}^{2}} \sin^{2} \frac{\iota_{,}}{2} (b_{3,2} + b_{3,4}) + \frac{a}{a_{,}^{2}} \frac{(e^{2} + e_{,}^{2})}{8} (8b_{3,2} - 10b_{3,4}) \Big\} \cos 3\tau$$

$$+ \Big\{ - \frac{a^{2}}{2a_{,}^{3}} b_{3,0} + \frac{a}{2a_{,}^{2}} b_{3,1} \Big\} e \cos \xi$$

$$+\left\{-\frac{3a}{2a_{,}^{2}}+\frac{3a}{4a_{,}^{2}}b_{3,0}-\frac{a^{2}}{2a_{,}^{3}}b_{3,1}-\frac{a}{4a_{,}^{2}}b_{3,2}\right\}e\cos(\tau-\xi)$$

$$+\left\{\frac{3a}{4a_{,}^{2}}b_{3,1}-\frac{a^{2}}{2a_{,}^{3}}b_{3,2}-\frac{a}{4a_{,}^{2}}b_{3,3}\right\}e\cos(2\tau-\xi)$$

$$+\left\{\frac{3a}{4a_{,}^{2}}b_{3,2}-\frac{a^{2}}{2a_{,}^{3}}b_{3,3}-\frac{a}{4a_{,}^{2}}b_{3,4}\right\}e\cos(3\tau-\xi)$$

$$+\left\{\frac{a}{2a_{,}^{2}}-\frac{a}{4a_{,}^{2}}b_{3,0}-\frac{a^{2}}{2a_{,}^{3}}b_{3,1}+\frac{3a}{4a_{,}^{2}}b_{3,2}\right\}e\cos(\tau+\xi)$$

$$+\left\{-\frac{a}{4a_{,}^{2}}b_{3,1}-\frac{a^{2}}{2a_{,}^{3}}b_{3,2}+\frac{3a}{4a_{,}^{2}}b_{3,3}\right\}e\cos(2\tau+\xi)$$

$$+\left\{-\frac{a}{4a_{,}^{2}}b_{3,2}-\frac{a^{2}}{2a_{,}^{3}}b_{3,3}+\frac{3a}{4a_{,}^{2}}b_{3,4}\right\}e\cos(3\tau+\xi)$$

$$+\left\{-\frac{1}{2a_{,}}b_{3,0}+\frac{a}{2a_{,}^{2}}b_{3,1}\right\}e_{,}\cos\xi_{,}$$

$$+\left\{\frac{2a}{a_{,}^{2}}-\frac{a}{4a_{,}^{2}}b_{3,0}-\frac{1}{2a_{,}}b_{3,1}+\frac{3a}{4a_{,}^{2}}b_{3,2}\right\}e_{,}\cos(\tau-\xi_{,})$$

$$+\left\{-\frac{a}{4a_{,}^{2}}b_{3,1}-\frac{1}{2a_{,}}b_{3,2}+\frac{3a}{4a_{,}^{2}}b_{3,3}\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{-\frac{a}{4a_{,}^{2}}b_{3,2}-\frac{1}{2a_{,}}b_{3,3}+\frac{3a}{4a_{,}^{2}}b_{3,3}\right\}e_{,}\cos(3\tau-\xi_{,})$$

$$+\left\{\frac{3a}{4a_{,}^{2}}b_{3,0}-\frac{1}{2a_{,}}b_{3,1}-\frac{a}{4a_{,}^{2}}b_{3,2}\right\}e_{,}\cos(\tau+\xi_{,})$$

$$+\left\{\frac{3a}{4a_{,}^{2}}b_{3,1}-\frac{1}{2a_{,}}b_{3,2}-\frac{a}{4a_{,}^{2}}b_{3,3}\right\}e_{,}\cos(2\tau+\xi_{,})$$

$$+\left\{\frac{3a}{4a_{,}^{2}}b_{3,2}-\frac{1}{2a_{,}}b_{3,3}-\frac{a}{4a_{,}^{2}}b_{3,4}\right\}e_{,}\cos(3\tau+\xi_{,})$$

$$+\left\{-\frac{a^{2}}{2a_{,}^{3}}b_{3,0}+\frac{3a}{8a_{,}^{2}}b_{3,1}\right\}e^{2}\cos 2\xi$$

$$+\left\{\frac{a}{8a_{,}^{2}}-\frac{a}{16a_{,}^{2}}b_{3,0}-\frac{a}{16a_{,}^{2}}b_{3,2}\right\}e^{2}\cos(\tau-2\xi)$$

$$+\left\{-\frac{5a}{8a_{,}^{2}}b_{3,1}+\frac{a^{2}}{2a_{,}^{3}}b_{3,2}\right\}e^{2}\cos(2\tau-2\xi)$$

$$+\left\{-\frac{19a}{16a_{,}^{2}}b_{3,2}+\frac{a^{2}}{a_{,}^{3}}b_{3,3}+\frac{a}{16a_{,}^{2}}b_{3,4}\right\}e^{2}\cos(3\tau-2\xi)$$

$$+\left\{\frac{3a}{8a_{,}^{2}}-\frac{3a}{16a_{,}^{2}}b_{3,0}-\frac{a^{2}}{a_{,}^{3}}b_{3,1}+\frac{17a}{16a_{,}^{2}}b_{3,2}\right\}e^{2}\cos(\tau+2\xi)$$

$$+\left\{-\frac{a}{4a_{,}^{2}}b_{3,1}-\frac{3a^{2}}{2a_{,}^{3}}b_{3,2}+\frac{13a}{8a_{,}^{2}}b_{3,3}\right\}e^{2}\cos(2\tau+2\xi)$$

$$+\left\{-\frac{5a}{16a_{,}^{2}}b_{3,2}-\frac{2a^{2}}{a_{,}^{3}}b_{3,3}+\frac{35a}{16a_{,}^{2}}b_{3,4}\right\}e^{2}\cos(3\tau+2\xi)$$

$$+\frac{a}{4a_{,}^{2}}b_{3,1}\,ee_{,}\cos(\xi+\xi_{,})$$

$$+\left\{-\frac{3a}{a_{,}^{2}}+\frac{1}{a_{,}}b_{3,1}-\frac{3a}{4a_{,}^{2}}b_{3,2}\right\}ee_{,}\cos(\tau-\xi-\xi_{,})$$

$$+\left\{\frac{a}{8a_{,}^{2}}b_{3,1}+\frac{2}{a_{,}}b_{3,2}-\frac{15a}{8a_{,}^{2}}b_{3,3}\right\}ee_{,}\cos(2\tau-\xi-\xi_{,})$$

$$+\left\{\frac{a}{4a_{,}^{2}}b_{3,2}+\frac{3}{a_{,}}b_{3,3}-\frac{3a}{a_{,}^{2}}b_{3,4}\right\}ee_{,}\cos(3\tau-\xi-\xi_{,})$$

$$+\left\{\frac{3a}{2a_{,}^{2}}b_{3,0}-\frac{1}{a_{,}}b_{3,1}-\frac{a}{4a_{,}^{2}}b_{3,2}\right\}ee_{,}\cos(\tau+\xi+\xi_{,})$$

$$+\left\{\frac{21a}{8a_{,}^{2}}b_{3,1}-\frac{2}{a_{,}}b_{3,2}-\frac{3a}{8a_{,}^{2}}b_{3,3}\right\}ee_{,}\cos(2\tau+\xi+\xi_{,})$$

$$+\left\{\frac{15a}{4a_{,}^{2}}b_{3,2}-\frac{3}{a_{,}}b_{3,3}-\frac{a}{2a_{,}^{2}}b_{3,4}\right\}ee_{,}\cos(3\tau+\xi+\xi_{,})$$

$$+\frac{a}{4a_{,}^{2}}b_{3,1}\,ee_{,}\cos(\xi-\xi_{,})$$

$$+\frac{a}{4a_{,}^{2}}b_{3,2}\,ee_{,}\cos(\tau-\xi+\xi_{,})$$

$$+\left\{-\frac{3a}{8a_{,}^{2}}b_{3,1}+\frac{5a}{8a_{,}^{2}}b_{3,3}\right\}ee_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\left\{-\frac{3a}{4a_{,}^{2}}b_{3,2}+\frac{a}{a_{,}^{2}}b_{3,4}\right\}ee_{,}\cos(3\tau-\xi+\xi_{,})$$

$$+\left\{\frac{a}{a_{,}^{2}}-\frac{a}{2a_{,}^{2}}b_{3,0}+\frac{3a}{4a_{,}^{2}}b_{3,2}\right\}ee_{,}\cos(\tau+\xi-\xi_{,})$$

$$+\left\{-\frac{7a}{8a_{,}^{2}}b_{3,1}+\frac{9a}{8a_{,}^{2}}b_{3,3}\right\}ee_{,}\cos(2\tau+\xi-\xi_{,})$$

$$+\left\{-\frac{5a}{4a_{,}^{2}}b_{3,2}+\frac{3a}{2a_{,}^{2}}b_{3,4}\right\}ee_{,}\cos(3\tau+\xi-\xi_{,})$$

$$+\left\{-\frac{1}{2a_{,}}b_{3,0}+\frac{3a}{8a_{,}^{2}}b_{3,1}\right\}e_{,}^{2}\cos 2\xi_{,}$$

$$+\left\{\frac{27a}{8a_{,}^{2}}-\frac{3a}{16a_{,}^{2}}b_{3,0}-\frac{1}{a_{,}}b_{3,1}+\frac{17a}{16a_{,}^{2}}b_{3,2}\right\}e_{,}^{2}\cos(\tau-2\xi_{,})$$

$$+\left\{-\frac{a}{4a_{,}^{2}}b_{3,1}-\frac{3}{2a_{,}}b_{3,2}+\frac{13a}{8a_{,}^{2}}b_{3,3}\right\}e_{,}^{2}\cos(2\tau-2\xi_{,})$$

$$+\left\{-\frac{5a}{16a_{,}^{2}}b_{3,2}-\frac{2}{a_{,}}b_{3,3}+\frac{35a}{16a_{,}^{2}}b_{3,4}\right\}e_{,}^{2}\cos(3\tau-2\xi_{,})$$

$$+\left\{\frac{a}{8a_{,}^{2}}-\frac{a}{16a_{,}^{2}}b_{3,0}-\frac{a}{16a_{,}^{2}}b_{3,2}\right\}e_{,}^{2}\cos(\tau+2\xi_{,})$$

$$+\left\{-\frac{5a}{8a_{,}^{2}}b_{3,1}+\frac{1}{2a_{,}}b_{3,2}\right\}e_{,}^{2}\cos(2\tau+2\xi_{,})$$

$$+\left\{-\frac{19\,a}{16\,a_{,}^{2}}b_{3,2}+\frac{1}{a_{,}}b_{3,3}+\frac{a}{16\,a_{,}^{2}}b_{3,4}\right\}e_{,}^{2}\cos\left(3\tau+2\xi_{,}\right)$$

$$-\frac{m_{,}a}{2\,a_{,}^{2}}b_{3,1}\sin^{2}\frac{\iota_{,}}{2}\cos 2\,\eta_{,}-\frac{m_{,}a}{2\,a_{,}^{2}}b_{3,2}\sin^{2}\frac{\iota_{,}}{2}\cos\left(\tau-2\,\eta_{,}\right)$$

$$-\frac{m_{,}a}{2\,a_{,}^{2}}b_{3,3}\sin^{2}\frac{\iota_{,}}{2}\cos\left(2\,\tau-2\,\eta_{,}\right)$$

$$+m_{,}\left\{\frac{a}{a_{,}^{2}}-\frac{a}{2\,a_{,}^{2}}b_{3,1}\right\}\sin^{2}\frac{\iota_{,}}{2}\cos\left(\tau+2\,\eta_{,}\right)$$

$$-\frac{m_{,}a}{2\,a_{,}^{2}}b_{3,1}\sin^{2}\frac{\iota_{,}}{2}\cos\left(2\,\tau+2\,\eta_{,}\right)$$

The terms in $r\,\frac{\mathrm{d}R}{\mathrm{d}r}$ multiplied by e and $e_{,}$ admit of similar reductions. Considering only these terms,

$$r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{3\,m_{,}}{2}\frac{a}{a_{,}^{2}}e\cos\left(\tau-\xi\right)+\frac{m_{,}a}{2\,a_{,}^{2}}e\cos\left(\tau+\xi\right)$$

$$+\frac{2\,m_{,}a}{a_{,}^{2}}e_{,}\cos\left(\tau-\xi_{,}\right)$$

$$+m_{,}\Sigma\left\{-\frac{i}{4}\frac{a}{a^{2}}b_{3,i-1}+\frac{(1+2\,i)}{2}\frac{a^{2}}{a_{,}^{3}}b_{3,i}-\frac{3\,i}{4}\frac{a}{a_{,}^{2}}b_{3,i-1}\right\}e\cos\left(i\,\tau+\xi\right)$$

$$+m_{,}\Sigma\left\{-\frac{3\,(1+i)}{4}\frac{a}{a_{,}^{2}}b_{3,i-1}+\frac{i\,a}{a_{,}}b_{3,i}+\frac{(1-i)}{4}b_{3,i+1}\right\}e_{,}\cos\left(i\,\tau+\xi_{,}\right)$$

If $a > a_{,}$, and

$$\left\{1-\frac{a_{,}}{a}\cos\tau+\frac{a_{,}^{2}}{a^{2}}\right\}^{-\frac{1}{2}}=\frac{1}{2}b_{1,0}+b_{1,1}\cos\tau+b_{1,2}\cos 2\,\tau+\text{\&c.}$$

$$\left\{1-\frac{a_{,}}{a}\cos\tau+\frac{a_{,}^{2}}{a^{2}}\right\}^{-\frac{3}{2}}=\frac{1}{2}b_{3,0}+b_{3,1}\cos\tau+b_{3,2}\cos 2\,\tau+\text{\&c.}$$

the value of R may easily be inferred from the value which it has in the former case. Considering only the terms multiplied by the eccentricities,

$$r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{3m_{,}}{2}\frac{a}{a_{,}^{2}}e\cos\left(\tau-\xi\right)+\frac{m_{,}}{2}\frac{a}{a_{,}^{2}}e\cos\left(\tau+\xi\right)$$

$$+\left\{-\frac{5\,a}{16\,a_{,}^{2}}b_{3,2}-\frac{2\,a^{2}}{a_{,}^{3}}b_{3,3}+\frac{35\,a}{16\,a_{,}^{2}}b_{3,4}\right\}e^{2}\cos\left(3\,\tau+2\,\xi\right)$$

$$+\frac{a}{4\,a_{,}^{2}}b_{3,1}\,e\,e_{,}\cos\left(\xi+\xi_{,}\right)$$

$$+\left\{-\frac{3\,a}{a_{,}^{2}}+\frac{1}{a_{,}}b_{3,1}-\frac{3\,a}{4\,a_{,}^{2}}b_{3,2}\right\}e\,e_{,}\cos\left(\tau-\xi-\xi_{,}\right)$$

$$+\left\{\frac{a}{8\,a_{,}^{2}}b_{3,1}+\frac{2}{a_{,}}b_{3,2}-\frac{15\,a}{8\,a_{,}^{2}}b_{3,3}\right\}e\,e_{,}\cos\left(2\,\tau-\xi-\xi_{,}\right)$$

$$+\left\{\frac{a}{4\,a_{,}^{2}}b_{3,2}+\frac{3}{a_{,}}b_{3,3}-\frac{3\,a}{a_{,}^{2}}b_{3,4}\right\}e\,e_{,}\cos\left(3\,\tau-\xi-\xi_{,}\right)$$

$$+\left\{\frac{3\,a}{2\,a_{,}^{2}}b_{3,0}-\frac{1}{a_{,}}b_{3,1}-\frac{a}{4\,a_{,}^{2}}b_{3,2}\right\}e\,e_{,}\cos\left(\tau+\xi+\xi_{,}\right)$$

$$+\left\{\frac{21\,a}{8\,a_{,}^{2}}b_{3,1}-\frac{2}{a_{,}}b_{3,2}-\frac{3\,a}{8\,a_{,}^{2}}b_{3,3}\right\}e\,e_{,}\cos\left(2\,\tau+\xi+\xi_{,}\right)$$

$$+\left\{\frac{15\,a}{4\,a_{,}^{2}}b_{3,2}-\frac{3}{a_{,}}b_{3,3}-\frac{a}{2\,a_{,}^{2}}b_{3,4}\right\}e\,e_{,}\cos\left(3\,\tau+\xi+\xi_{,}\right)$$

$$+\frac{a}{4\,a_{,}^{2}}b_{3,1}\,e\,e_{,}\cos\left(\xi-\xi_{,}\right)$$

$$+\frac{a}{4\,a_{,}^{2}}b_{3,2}\,e\,e_{,}\cos\left(\tau-\xi+\xi_{,}\right)$$

$$+\left\{-\frac{3\,a}{8\,a_{,}^{2}}b_{3,1}+\frac{5\,a}{8\,a_{,}^{2}}b_{3,3}\right\}e\,e_{,}\cos\left(2\,\tau-\xi+\xi_{,}\right)$$

$$+\left\{-\frac{3\,a}{4\,a_{,}^{2}}b_{3,2}+\frac{a}{a_{,}^{2}}b_{3,4}\right\}e\,e_{,}\cos\left(3\,\tau-\xi+\xi_{,}\right)$$

$$+\left\{\frac{a}{a_{,}^{2}}-\frac{a}{2\,a_{,}^{2}}b_{3,0}+\frac{3\,a}{4\,a_{,}^{2}}b_{3,2}\right\}e\,e_{,}\cos\left(\tau+\xi-\xi_{,}\right)$$

$$+\left\{-\frac{7\,a}{8\,a_{,}^{2}}b_{3,1}+\frac{9\,a}{8\,a_{,}^{2}}b_{3,3}\right\}e\,e_{,}\cos\left(2\,\tau+\xi-\xi_{,}\right)$$

$$+\left\{-\frac{5\,a}{4\,a_{,}^{2}}b_{3,2}+\frac{3\,a}{2\,a_{,}^{2}}b_{3,4}\right\}e\,e_{,}\cos\left(3\,\tau+\xi-\xi_{,}\right)$$

$$+\left\{-\frac{1}{2\,a_{,}}b_{3,0}+\frac{3\,a}{8\,a_{,}^{2}}b_{3,1}\right\}e_{,}^{2}\cos 2\,\xi_{,}$$

$$+\left\{\frac{27\,a}{8\,a_{,}^{2}}-\frac{3\,a}{16\,a_{,}^{2}}b_{3,0}-\frac{1}{a_{,}}\,b_{3,1}+\frac{17\,a}{16\,a_{,}^{2}}b_{3,2}\right\}e_{,}^{2}\cos\left(\tau-2\,\xi_{,}\right)$$

$$+\left\{-\frac{a}{4\,a_{,}^{2}}b_{3,1}-\frac{3}{2\,a_{,}}b_{3,2}+\frac{13\,a}{8\,a_{,}^{2}}b_{3,3}\right\}e_{,}^{2}\cos\left(2\,\tau-2\,\xi_{,}\right)$$

$$+\left\{-\frac{5\,a}{16\,a_{,}^{2}}b_{3,2}-\frac{2}{a_{,}}b_{3,3}+\frac{35\,a}{16\,a_{,}^{2}}b_{3,4}\right\}e_{,}^{2}\cos\left(3\,\tau-2\,\xi_{,}\right)$$

$$+\left\{\frac{a}{8\,a_{,}^{2}}-\frac{a}{16\,a_{,}^{2}}b_{3,0}-\frac{a}{16\,a_{,}^{2}}b_{3,2}\right\}e_{,}^{2}\cos\left(\tau+2\,\xi_{,}\right)$$

$$+\left\{-\frac{5\,a}{8\,a_{,}^{2}}b_{3,1}+\frac{1}{2\,a_{,}}b_{3,2}\right\}e_{,}^{2}\cos\left(2\,\tau+2\,\xi_{,}\right)$$

$$+\left\{-\frac{19\,a}{16\,a_\prime^2}b_{3,2}+\frac{1}{a_\prime}b_{3,3}+\frac{a}{16\,a_\prime^2}b_{3,4}\right\}e_\prime^2\cos(3\tau+2\xi_\prime)$$

$$-\frac{m_\prime a}{2\,a_\prime^2}b_{3,1}\sin^2\frac{\iota_\prime}{2}\cos 2\,\eta_\prime-\frac{m_\prime a}{2\,a_\prime^2}b_{3,2}\sin^2\frac{\iota_\prime}{2}\cos(\tau-2\,\eta_\prime)$$

$$-\frac{m_\prime a}{2\,a_\prime^2}b_{3,3}\sin^2\frac{\iota_\prime}{2}\cos(2\tau-2\,\eta_\prime)$$

$$+m_\prime\left\{\frac{a}{a_\prime^2}-\frac{a}{2\,a_\prime^2}b_{3,1}\right\}\sin^2\frac{\iota_\prime}{2}\cos(\tau+2\,\eta_\prime)$$

$$-\frac{m_\prime a}{2\,a_\prime^2}b_{3,1}\sin^2\frac{\iota_\prime}{2}\cos(2\tau+2\,\eta_\prime)$$

The terms in $r\dfrac{\mathrm{d}R}{\mathrm{d}r}$ multiplied by e and $e_\prime$ admit of similar reductions. Considering only these terms,

$$r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{3\,m_\prime}{2}\frac{a}{a_\prime^2}e\cos(\tau-\xi)+\frac{m_\prime a}{2\,a_\prime^2}e\cos(\tau+\xi)$$

$$+\frac{2\,m_\prime a}{a_\prime^2}e_\prime\cos(\tau-\xi_\prime)$$

$$+m_\prime\Sigma\left\{-\frac{i}{4}\frac{a}{a^2}b_{3,i-1}+\frac{(1+2\,i)}{2}\frac{a^2}{a_\prime^3}b_{3,i}\right.$$

$$\left.-\frac{3\,i}{4}\frac{a}{a_\prime^2}b_{3,i-1}\right\}e\cos(i\tau+\xi)$$

$$+m_\prime\Sigma\left\{-\frac{3\,(1+i)}{4}\frac{a}{a_\prime^2}b_{3,i-1}+\frac{i\,a}{a_\prime}b_{3,i}\right.$$

$$\left.+\frac{(1-i)}{4}b_{3,i+1}\right\}e_\prime\cos(i\tau+\xi_\prime)$$

If $a > a_\prime$, and

$$\left\{1-\frac{a_\prime}{a}\cos\tau+\frac{a_\prime^2}{a^2}\right\}^{-\frac{1}{2}}=\frac{1}{2}b_{1,0}+b_{1,1}\cos\tau+b_{1,2}\cos 2\,\tau+\text{\&c.}$$

$$\left\{1-\frac{a_\prime}{a}\cos\tau+\frac{a_\prime^2}{a^2}\right\}^{-\frac{3}{2}}=\frac{1}{2}b_{3,0}+b_{3,1}\cos\tau+b_{3,2}\cos 2\,\tau+\text{\&c.}$$

the value of R may easily be inferred from the value which it has in the former case. Considering only the terms multiplied by the eccentricities,

$$r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{3\,m_\prime}{2}\frac{a}{a_\prime^2}e\cos(\tau-\xi)+\frac{m_\prime}{2}\frac{a}{a_\prime^2}e\cos(\tau+\xi)$$

$$+ (6i + 6)\frac{a^2 \mathrm{d}^2 R_1}{\mathrm{d} a^2} - \frac{a^3 \mathrm{d}^3 R_1}{\mathrm{d} a^3}\Big\}$$

Changing the sign of i, I find

$$R_{22} = -\frac{1}{48}\Big\{(26i - 30i^2 + 8i^3) R_1 + (9 - 27i + 12i^2) a\frac{\mathrm{d} R_1}{\mathrm{d} a}$$

$$+ (6i - 6)\frac{a^2 \mathrm{d}^2 R_1}{\mathrm{d} a^2} + \frac{a^3 \mathrm{d}^3 R_1}{\mathrm{d} a^3}\Big\}$$

which agrees with the expression given by Burckhardt for $(M^{(0)})$, *Mémoires de l'Institut*, 1808, *Second Semestre*, p. 39.

Similarly,

$$2 R_{51} = -\frac{a \mathrm{d} R_{33}}{2 \mathrm{d} a} - i R_{33} - \frac{3 a \mathrm{d} R_{19}}{4 \mathrm{d} a} - \frac{5i}{4} R_{19}$$

$$R_{33} = -\frac{a \mathrm{d} R_{19}}{2 \mathrm{d} a} - i R_{19}$$

$$2 R_{51} = \frac{a^2 \mathrm{d}^2 R_{19}}{4 \mathrm{d} a^2} + \frac{(2i - 1)}{2}\frac{a \mathrm{d} R_{19}}{\mathrm{d} a} + \frac{(4i^2 - 5i)}{4} R_{19}$$

$$2 R_{19} = \frac{a_{\prime}^2 \mathrm{d}^2 R_1}{4 \mathrm{d} a_{\prime}^2} + \frac{(2i - 1)}{2}\frac{a_{\prime} \mathrm{d} R_1}{\mathrm{d} a_{\prime}} + \frac{(4i^2 - 5i)}{4} R_1$$

If $i = 2$

$$2 R_{51} = \frac{a^2 \mathrm{d}^2 R_{19}}{4 \mathrm{d} a^2} + \frac{3}{2}\frac{a \mathrm{d} R_{19}}{\mathrm{d} a} + \frac{3}{2} R_{19}$$

I shall now give the details of the calculation of R_{51}, for $i = 2$, as another example of the remarkable reductions which may be procured in consequence of the equations of condition which obtain between the quantities b. By the expression for R, p. 96, I find the coefficient of $e_{\prime}^2 \cos (2\tau + 2\xi_{\prime})$ or

$$R_{19} = -\frac{5a}{8a_{\prime}^2} b_{3,1} + \frac{1}{2a_{\prime}} b_{3,2}$$

$$\frac{a \mathrm{d} R_{19}}{\mathrm{d} a} = -\frac{5a}{8a_{\prime}^2} b_{3,1} + \frac{15a^2}{8a_{\prime}^3}\Big\{\frac{a}{a_{\prime}} b_{5,1} - \frac{1}{2} b_{5,0} - \frac{1}{2} b_{5,2}\Big\}$$

$$-\frac{3a}{2a_{\prime}^2}\Big\{\frac{a}{a_{\prime}} b_{5,2} - \frac{1}{2} b_{5,1} - \frac{1}{2} b_{5,3}\Big\}$$

$$= -\frac{5a}{8a_{\prime}^2} b_{3,1} + \frac{15a}{8a_{\prime}^2}\Big\{\frac{a^2 + a_{\prime}^2}{a_{\prime}^2} b_{5,1} - \frac{a}{a_{\prime}} b_{5,0} - \frac{a}{a_{\prime}} b_{5,2}\Big\}$$

$$-\frac{3\,a}{4\,a_{,}^{2}}\left\{b_{5,1}-b_{5,3}\right\}-\frac{3}{8}\,\frac{a}{a_{,}^{2}}b_{5,1}+\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}-\frac{9}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}$$

$$=\frac{5\,a}{4\,a_{,}^{2}}b_{3,1}-\frac{1}{a_{,}}\,b_{3,2}-\frac{3\,a}{8\,a_{,}^{2}}b_{5,1}+\frac{15\,a^{2}}{16\,a_{,}^{3}}b_{5,0}-\frac{9\,a^{2}}{16\,a_{,}^{3}}b_{5,2}$$

$$R_{33}=\frac{3}{16}\,\frac{a}{a_{,}^{2}}b_{5,1}-\frac{15}{32}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}+\frac{9}{32}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}+\frac{5}{8}\,\frac{a}{a_{,}^{2}}b_{3,1}-\frac{1}{2\,a_{,}}\,b_{3,2}$$

$$\iota\frac{\mathrm{d}\,R_{33}}{\mathrm{d}\,a}=\frac{3}{16}\,\frac{a}{a_{,}^{2}}b_{5,1}-\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}+\frac{9}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}-\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}\left\{\frac{a}{a_{,}}\,b_{7,1}-\frac{1}{2}\,b_{7,0}-\frac{1}{2}\,b_{7,2}\right\}$$

$$+\frac{75}{16}\,\frac{a^{3}}{a_{,}^{4}}\left\{\frac{a}{2\,a_{,}}\,b_{7,0}-\frac{1}{2}\,b_{7,1}\right\}-\frac{45}{32}\,\frac{a^{3}}{a_{,}^{4}}\left\{\frac{a}{a_{,}}\,b_{7,2}-\frac{1}{2}\,b_{7,1}-\frac{1}{2}\,b_{7,3}\right\}$$

$$-\frac{5}{4}\,\frac{a}{a_{,}^{2}}b_{3,1}+\frac{1}{a_{,}}\,b_{3,2}-\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}+\frac{3}{8}\,\frac{a}{a_{,}^{2}}b_{5,1}+\frac{9}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}$$

$$=\frac{3}{16}\,\frac{a}{a_{,}^{2}}b_{5,1}-\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}+\frac{9}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}+\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}\left\{\frac{a^{2}+a_{,}^{2}}{2\,a_{,}^{2}}\,b_{7,0}-\frac{a}{a_{,}}\,b_{7,1}\right\}$$

$$+\frac{15}{32}\,\frac{a^{2}}{a_{,}^{3}}\left\{\frac{a^{2}+a_{,}^{2}}{a_{,}^{2}}\,b_{7,2}-\frac{a}{a_{,}}\,b_{7,1}-\frac{a}{a_{,}}\,b_{7,3}\right\}+\frac{15}{8}\,\frac{a^{4}}{a_{,}^{5}}\,b_{7,0}-\frac{15}{8}\,\frac{a^{4}}{a_{,}^{5}}b_{7,2}$$

$$-\frac{75}{64}\,\frac{a^{3}}{a_{,}^{4}}b_{7,1}+\frac{75}{64}\,\frac{a^{3}}{a_{,}^{4}}b_{7,3}-\frac{5}{4}\,\frac{a}{a_{,}^{2}}b_{3,1}+\frac{1}{a_{,}}\,b_{3,2}-\frac{15}{16}\frac{a^{2}}{a_{,}^{3}}\,b_{5,0}+\frac{3}{8}\,\frac{a}{a_{,}^{2}}b_{5,1}$$

$$+\frac{9}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}$$

$$=-\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}+\frac{3}{16}\,\frac{a^{2}}{a_{,}^{2}}b_{5,1}+\frac{9}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}+\frac{15}{32}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}+\frac{15}{32}\,\frac{a^{2}}{a_{,}^{3}}\,b_{5,2}$$

$$+\frac{3}{4}\,\frac{a^{3}}{a_{,}^{4}}b_{5,1}-\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}-\frac{5}{4}\,\frac{a}{a_{,}^{2}}b_{3,1}+\frac{1}{a_{,}}\,b_{3,2}-\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}+\frac{3}{8}\,\frac{a}{a_{,}^{2}}\,b_{5,1}$$

$$+\frac{9}{16}\,\frac{a^{2}}{a_{,}^{2}}b_{5,2}$$

$$=-\frac{45}{32}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}+\frac{9}{16}\frac{a}{a_{,}^{2}}b_{5,1}+\frac{3}{4}\,\frac{a^{3}}{a_{,}^{4}}b_{5,1}+\frac{21}{32}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}-\frac{5}{4}\,\frac{a}{a_{,}^{2}}b_{3,1}+\frac{1}{a_{,}}b_{3,2}$$

Finally,

$$2\,R_{51}=\frac{45}{64}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}-\frac{9}{32}\,\frac{a}{a_{,}^{2}}b_{5,1}-\frac{3}{8}\,\frac{a^{3}}{a_{,}^{4}}b_{5,1}-\frac{21}{64}\,\frac{a^{2}}{a_{,}^{3}}\,b_{5,2}+\frac{5}{8}\,\frac{a}{a_{,}^{2}}b_{3,1}-\frac{1}{2\,a_{,}}b_{3,2}$$

$$+\frac{15}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}-\frac{3}{8}\,\frac{a}{a_{,}^{2}}b_{5,1}-\frac{9}{16}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}-\frac{5}{4}\,\frac{a}{a_{,}^{2}}b_{3,1}+\frac{1}{a_{,}}\,b_{3,2}$$

$$-\frac{15}{16}\,\frac{a}{a_{,}^{2}}b_{3,1}+\frac{3}{4\,a_{,}}b_{3,2}+\frac{9}{32}\,\frac{a}{a_{,}^{2}}b_{5,1}-\frac{45}{64}\,\frac{a^{2}}{a_{,}^{3}}b_{5,0}+\frac{27}{64}\,\frac{a^{2}}{a_{,}^{3}}b_{5,2}$$

$$+\frac{25}{16}\frac{a}{a_{,}^{2}}b_{3,1}-\frac{5}{4\,a_{,}}b_{3,2}$$

$$=-\frac{3}{8}\,\frac{a}{a_{,}^{2}}\left\{\frac{a^{2}+a_{,}^{2}}{a_{,}^{2}}\,b_{5,1}-\frac{5}{2}\,\frac{a}{a_{,}}\,b_{5,0}\right\}-\frac{15}{32}\frac{a^{2}}{a_{,}^{3}}b_{5,2}$$

$$=-\frac{9}{32}\frac{a^{2}}{a_{,}^{3}}b_{5,2}$$

Hence the disturbing function contains the term

$$-\frac{9\,m_{\prime}}{64}\,\frac{a^2}{a_{\prime}^3}\,b_{5,2}\,e^2\,e_{\prime}^2\cos(2\,\tau-2\,\xi+2\,\xi_{\prime})\qquad\text{or}$$

$$-\frac{9\,m_{\prime}}{64}\,\frac{a^2}{a_{\prime}^3}\,b_{5,2}\,e^2\,e_{\prime}^2\cos(2\,\varpi-2\,\varpi_{\prime})$$

If the coefficients corresponding to the *n*th argument in the expressions for $\frac{a}{r}$ and λ be called r_n and λ_n, the Table which is given in the Appendix may also be used (when the square of the disturbing force is neglected,) for the integration of the equations

$$\frac{\mathrm{d}^2 . r^2}{2\,\mathrm{d}\,t^2}-\frac{\mu}{r}+\frac{\mu}{a}+2\int\mathrm{d}\,R+r\frac{\mathrm{d}\,R}{\mathrm{d}\,r}=0$$

and

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,t}=\frac{h}{r^2}-\frac{1}{r^3}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t$$

$$-\frac{\mathrm{d}^2\,\mathrm{r}^3\,\delta\,\frac{1}{r}}{\mathrm{d}\,t^2}-\mu\,\delta\frac{1}{r}+2\int\mathrm{d}\,R+\frac{r\,\mathrm{d}\,R}{\mathrm{d}\,r}=0$$

$$-r_0+\frac{a\,m_{\prime}\,\mathrm{d}\,R_0}{\mu\,\mathrm{d}\,a}=0$$

$$\left\{\frac{i\,(n-n_{\prime})}{n}\right\}^2 r_1-r_1+\frac{2\,n\,a}{(n-n_{\prime})}\,\frac{m_{\prime}}{\mu}R_1+\frac{a^2\,m_{\prime}\,\mathrm{d}\,R_1}{\mu\,\mathrm{d}\,a}=0$$

$$\left\{\frac{i\,(n-n_{\prime})-n}{n}\right\}^2\left\{r_3+\frac{3}{2}\,r_1\right\}-r_3+\frac{2\,(i-1)\,n\,a}{(i\,(n-n_{\prime})-n)}\,\frac{m_{\prime}}{\mu}R_3$$

$$+\frac{a^2\,m_{\prime}\,\mathrm{d}\,R_3}{\mu\,\mathrm{d}\,a}=0$$

$$\left\{\frac{i\,(n-n_{\prime})+n}{n}\right\}^2\left\{r_4+\frac{3}{2}\,r_1\right\}-r_4+\frac{2\,(i+1)\,n\,a}{(i\,(n-n_{\prime})+n)}\,\frac{m_{\prime}}{\mu}R_4$$

$$+\frac{a^2\,m_{\prime}\,\mathrm{d}\,R_4}{\mu\,\mathrm{d}\,a}=0$$

$$\frac{n_{\prime}^2}{n^2}r_5-r_5+\frac{a^2\,m_{\prime}\,\mathrm{d}\,R_5}{\mu\,\mathrm{d}\,a}=0$$

$$\left\{\frac{i\,(n-n_{\prime})-n_{\prime}}{n}\right\}^2 r_6-r_6+\frac{2\,i\,n\,a}{(i\,(n-n_{\prime})-n_{\prime})}\,\frac{m_{\prime}}{\mu}R_6+\frac{a^2\,m_{\prime}\,\mathrm{d}\,R_6}{\mu\,\mathrm{d}\,a}=0$$

$$\left\{\frac{i\,(n-n_{\prime})+n_{\prime}}{n}\right\}^2 r_7-r_7+\frac{2\,i\,n\,a}{(i\,(n-n_{\prime})+n_{\prime})}\,\frac{m_{\prime}}{\mu}R_7+\frac{a^2\,m_{\prime}\,\mathrm{d}\,R_7}{\mu\,\mathrm{d}\,a}=0$$

Except for $r_{7,1}$; or the term of which the argument is $\tau + \xi_{,}$, which must be separately considered.

If $\frac{\mathrm{d}R}{\mathrm{d}z} = m_{,} Z$, the inequalities of latitude may be determined from the equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d} t^2} + \frac{\mu z}{r^3} + m_{,} Z = 0$$

which gives if $z = a\gamma \sin\eta + a\gamma z_{147} \sin(i\tau - \eta) + a\gamma z_{148} \sin(i\tau + \eta)$

$$-g^2 + 3r_0 + 1 + \frac{m_{,}}{\mu} a^2 Z_{146} = 0$$

$$-\left\{\frac{i(n-n_{,}) - n}{n}\right\}^2 z_{147} + \frac{3r_i}{2} + z_{147} + \frac{m_{,}}{\mu} a^2 Z_{147} = 0$$

$$-\left\{\frac{i(n-n_{,}) + n}{n}\right\}^2 z_{148} + \frac{3r^i}{2} + z_{148} + \frac{m_{,}}{\mu} a^2 Z_{148} = 0$$

$$\frac{\mathrm{d}R}{\mathrm{d}z} = \frac{m_{,} a\gamma}{2a_{,}^3}\left\{b_{3,0} \sin\eta - b_{3,1}\sin(\tau - \eta) - b_{3,2}\sin(2\tau - \eta) \right.$$
$$\left. + b_{3,1}\sin(\tau + \eta) + b_{3,2}\sin(2\tau + \eta) + \&c.\right\}$$

If $z = 0$, that is if the longitudes are reckoned upon the orbit of m, then if $a < a_{,}$,

$$\frac{\mathrm{d}R}{\mathrm{d}z} = \frac{m_{,}\gamma_{,}\sin\eta_{,}}{a_{,}^3}\left\{1 - \frac{1}{2}b_{3,0} - b_{3,1}\cos\tau - b_{3,2}\cos 2\tau - \&c.\right\}$$

$$= \frac{m_{,}\gamma_{,}}{a_{,}^2}\left\{\left(1 - \frac{1}{2}b_{3,0}\right)\sin\eta_{,} + \frac{1}{2}b_{3,1}\sin(\tau - \eta_{,}) \right.$$
$$+ \frac{1}{2}b_{3,2}\sin(2\tau - \eta_{,}) - \frac{1}{2}b_{3,1}\sin(\tau + \eta_{,})$$
$$\left. - \frac{1}{2}b_{3,2}\sin(2\tau + \eta_{,}) + \&c.\right\}$$

$$\lambda = \mathrm{n}t + \varepsilon$$
$$+ \left\{2r_1 - \frac{na}{(n - n_{,})}\frac{m_{,}}{\mu}R_1\right\}\frac{n}{i(n - n_{,})}\sin i\tau$$

[1]

$$+ \left\{2r_3 + r_1 - \frac{ina}{(i(n - n_{,}) - n)}\frac{m_{,}}{\mu}R_3\right.$$

$$\left.- \frac{n a}{(n - n_{\prime})} R_1 \right\} \frac{n e}{(i (n - n_{\prime}) - n)} \sin (i \tau - \xi)$$

[3]

$$+ \left\{ 2 r_4 + r_1 - \frac{i n a}{(i (n - n_{\prime}) + n)} \frac{m_{\prime}}{\mu} R_4 \right.$$

$$\left.- \frac{n a}{(n - n_{\prime})} \frac{m_{\prime}}{\mu} R_1 \right\} \frac{n e}{(i (n - n_{\prime}) + n)} \sin (i \tau + \xi)$$

[4]

$$+ \frac{2 n r_5}{n_{\prime}} e_{\prime} \sin \xi_{\prime}$$

[5]

$$+ \left\{ 2 r_6 - \frac{i n a}{(i (n - n_{\prime}) - n_{\prime})} \frac{m_{\prime}}{\mu} R_6 \right\} \frac{n e_{\prime}}{(i (n - n_{\prime}) - n_{\prime})} \sin (i \tau - \xi_{\prime})$$

[6]

$$+ \left\{ 2 r_7 - \frac{i n a}{(i (n - n_{\prime}) + n_{\prime})} \frac{m_{\prime}}{\mu} R_7 \right\} \frac{n e_{\prime}}{(i (n - n_{\prime}) + n_{\prime})} \sin (i \tau + \xi_{\prime})$$

[7]

i being any positive whole number; excluding the argument $\tau + \xi_{\prime}$ which must be considered separately.

If we call the quantity R_n which corresponds to $i = m$, $R_{n,m}$, then neglecting the squares of the eccentricities, when $a < a_{\prime}$

$$R_0 = - \frac{b_{1,0}}{2 a_{\prime}}, \quad R_{1,1} = \frac{a}{a_{\prime}^2} - \frac{b_{1,1}}{a_{\prime}}, \quad R_{1,2} = - \frac{b_{1,2}}{a_{\prime}}, \quad R_{1,3} = - \frac{b_{1,3}}{a_{\prime}}$$

$$r_{1,1} = \frac{m_{\prime}}{\mu} \frac{n^2}{(2 n - n_{\prime}) n_{\prime}} \left\{ \frac{2 n}{(n - n_{\prime})} \left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}} b_{1,1} \right) \right.$$

$$\left. + \frac{a^2}{a_{\prime}^3} + \frac{a^2}{a_{\prime}^2} \left(\frac{a}{a_{\prime}} b_{3,1} - \frac{1}{2} b_{3,0} - \frac{1}{2} b_{3,2} \right) \right\}$$

$$\lambda_{1,1} = \frac{n}{(n - n_{\prime})} \left\{ 2 r_{1,1} - \frac{m_{\prime} n}{\mu (n - n_{\prime})} \left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}} b_{1,1} \right) \right\}$$

When $a > a_{\prime}$, if

$$\frac{1}{2} b_{1,0} = 1 + \left(\frac{1}{2} \right)^2 \frac{a_{\prime}^2}{a^2} + \left(\frac{3}{2 \cdot 4} \right)^2 \frac{a_{\prime}^4}{a^4} + \&c.$$

$$b_{1,1} = \frac{a_{\prime}}{a} + \frac{3}{2 \cdot 4} \frac{a_{\prime}^3}{a^3} + \frac{3 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6} \frac{a_{\prime}^5}{a^5} + \&c.$$

$$r_{1,1} = \frac{m_{\prime} n^2}{\mu (2 n - n_{\prime}) n_{\prime}} \left\{ \frac{2 n}{(n - n_{\prime})} \left(\frac{a^2}{a_{\prime}^2} - b_{1,1} \right) \right.$$

$$+\frac{a^2}{a_{\prime}^2}+b_{1,1}-\frac{a_{\prime}}{a}\left(\frac{a_{\prime}}{a}b_{3,2}-\frac{1}{2}b_{3,0}-\frac{1}{2}b_{3,2}\right)\Bigg\}$$

$$\lambda_{1,1}=\frac{n}{(n-n_{\prime})}\left\{2\,r_{1,1}-\frac{m_{\prime}\,n}{\mu\,(n-n_{\prime})}\left(\frac{a^2}{a_{\prime}^2}-b_{1,1}\right)\right\}$$

In the planetary theory, as in that of the moon, (see p. 48.)

$$h^2=\mu\,a\,(1-e^2) \qquad n=\sqrt{\frac{\mu}{a^3}} \qquad n\,\{1+2\,r_0\}=\mathrm{n}$$

$$r_0=-\frac{a^2\,\mathrm{d}\,R_0}{\mathrm{d}\,a}=\frac{m_{\prime}}{\mu}\left\{\frac{a^3}{2\,a_{\prime}^3}b_{3,0}-\frac{a^2}{2\,a_{\prime}^2}b_{3,1}\right\}$$

If $\sqrt{\frac{\mu}{a^3}}\{1+2\,r_0\}=\sqrt{\frac{\mu}{\mathrm{a}^3}}$

$$a=\mathrm{a}\left\{1+\frac{4}{3}\,r_0\right\}$$

$$\frac{\mathrm{a}}{r}=1-\frac{m_{\prime}a^3}{6\,\mu\,a_{\prime}^3}b_{3,0}+\frac{m_{\prime}a^2}{6\,\mu\,a_{\prime}^2}b_{3,1}+\&\text{c.}$$

If r is considered as a function of a and $\mathrm{n}\,t$, so that

$$\xi=\mathrm{c}\,\mathrm{n}\,t+\epsilon-\varpi$$

$$\frac{\mathrm{a}}{r}=1-\frac{m_{\prime}a^3}{6\,\mu\,a_{\prime}^3}b_{3,0}+\frac{m_{\prime}a^2}{6\,\mu\,a_{\prime}^2}b_{3,1}$$

$$+\,e\left\{1-\frac{2\,m_{\prime}a^3}{3\,\mu\,a_{\prime}^3}b_{3,0}+\frac{2\,m_{\prime}a^2}{3\,\mu\,a_{\prime}^2}b_{3,1}\right\}\cos\xi+\&\text{c.}$$

If the coefficient corresponding to the argument n in the development of

$$2\int\mathrm{d}\,R+\frac{r\,\mathrm{d}\,R}{\mathrm{d}\,r}$$

be called q_n, then the coefficient of $\cos\xi$ or $\cos\,(c\,n\,t+\epsilon-\varpi)$ may be called q_2. By the expression given p. 96, the coefficient of $\cos\xi$ in the development of R or $m_{\prime}\,R_2$ is

$$m_{\prime}\left\{-\frac{a^2}{2\,a_{\prime}^3}b_{3,0}+\frac{a}{2\,a_{\prime}^2}b_{3,1}\right\}$$

and by the expression of p. 97 the coefficient of $\cos\xi$ in the development of $r\frac{\mathrm{d}\,R}{\mathrm{d}\,r}$, is

$+ m_{,}\dfrac{a^2}{2a_{,}^3}b_{3,0}$ whence $q_2 = -\dfrac{a^2}{2a_{,}^2}b_{3,0} + \dfrac{a}{a_{,}^2}b_{3,1}$

If $\quad -\dfrac{m_{,}a^3}{6\mu a_{,}^3}b_{3,0} + \dfrac{m_{,}a^2}{6\mu a_{,}^2}b_{8,1}$, be denoted by r_0,

and $-\dfrac{2m_{,}a^2}{3\mu a_{,}^3}b_{3,0} + \dfrac{2m_{,}a^2}{3\mu a_{,}^2}b_{3,1}$ be denoted by r_2,

$$\frac{\mathrm{a}}{r} = 1 + \mathrm{r}_0 + e(1 + \mathrm{r}_2)\cos\xi + \&c.$$

$$\frac{r^2}{\mathrm{a}^2} = 1 - 2\mathrm{r}_0 - 2e(1 + \mathrm{r}_2 - 3\mathrm{r}_0)\cos\xi + \&c.$$

$$\mathrm{c}^2\{1 + \mathrm{r}_2 - 3\mathrm{r}_0\} = 1 + \mathrm{r}_2 + \frac{m_{,}}{\mu}\left\{\frac{a^3}{2a_{,}^3}b_{3,0} - \frac{a^2}{a_{,}^2}b_{3,1}\right\}$$

$$\mathrm{c}^2 = 1 - \frac{m_{,}a^2}{2\mu a_{,}^2}b_{3,1} \qquad \mathrm{c} = 1 - \frac{m_{,}a^2}{4\mu a_{,}^2}b_{3,1}$$

This value of c agrees, of course, with the value of c found in the lunar theory (p. 49 and p. 83).

As in the lunar theory,

$$\lambda = \mathrm{n}t + \frac{2(1 - r_0)}{\mathrm{c}}e\sin\xi + \&c.$$

$$= \mathrm{n}t + 2\left\{1 - \frac{m_{,}}{\mu}\left\{\frac{a^3}{2a_{,}^3}b_{3,0} - \frac{a^2}{2a_{,}^2}b_{3,1} - \frac{m_{,}a^2}{4\mu a_{,}^2}b_{3,1}\right\}\right\}e\sin\xi$$

$$= \mathrm{n}t + 2\left\{1 - \frac{m_{,}}{\mu}\left\{\frac{a^3}{2a_{,}^3}b_{3,0} - \frac{3a^2}{4a_{,}^2}b_{3,1}\right\}\right\}e\sin\xi + \&c.$$

If $e\left\{1 - \dfrac{m_{,}}{\mu}\left\{\dfrac{a^3}{2a_{,}^3}b_{3,0} - \dfrac{3a^2}{4a_{,}^2}b_{3,1}\right\}\right\}$ be called e, then

$$\lambda = \mathrm{n}t + 2\mathrm{e}\sin\xi + \&c.$$

The quantity e results immediately from observation, and

$$\frac{\mathrm{a}}{r} = 1 - \frac{m_{,}a^3}{6\mu a_{,}^3}b_{3,0} + \frac{m_{,}a^2}{6\mu a_{,}^2}b_{3,1}$$

$$+ \mathrm{e}\left\{1 - \frac{m_{,}a^3}{6\mu a_{,}^3}b_{3,0} - \frac{m_{,}a^2}{12\mu a_{,}^2}b_{3,1}\right\}\cos\xi + \&c$$

$$\frac{r}{\mathrm{a}} = 1 + \frac{m_{,}a^3}{6\,\mu\,a_{,}^3}b_{3,0} - \frac{m_{,}a^2}{6\,\mu\,a_{,}^2}b_{3,1}$$

$$- \mathrm{e}\left\{1 + \frac{m_{,}a^3}{6\,\mu\,a_{,}^3}b_{3,0} - \frac{5\,m_{,}a^2}{12\,\mu\,a_{,}^2}b_{3,1}\right\}\cos\xi$$

These are the expressions when $a < a_{,}$ that is, when a planet m is *disturbed* by a superior planet $m_{,}$.

If when $a < a_{,}$,

$\frac{a_{,}}{r_{,}} = 1 + r_{,0} + e_{,}\cos\xi_{,} + \&c.$, I find

$$r_{,0} = \frac{m}{\mu_{,}}\left\{\frac{1}{2}b_{3,0} - \frac{a}{2\,a_{,}}b_{3,1}\right\} \qquad \mathrm{c}_{,} = 1 - \frac{m\,a}{4\,\mu\,a_{,}}b_{3,1}$$

$$\xi_{,} = \mathrm{c}_{,}\mathrm{n}\,t + \varepsilon_{,} - \varpi_{,} \qquad \lambda_{,} = \mathrm{n}\,t + 2\,\mathrm{e}_{,}\sin\xi_{,} + \&c.$$

$$\frac{\mathrm{a}_{,}}{r_{,}} = 1 - \frac{m}{6\,\mu}b_{3,0} + \frac{m\,a}{6\,\mu\,a_{,}}b_{3,1}$$

$$+ \mathrm{e}_{,}\left\{1 - \frac{m}{6\,\mu}b_{3,0} - \frac{m\,a}{12\,\mu\,a_{,}}b_{3,1}\right\}\cos\xi_{,} + \&c.$$

The preceding results are in accordance with those given by Laplace. *Méc. Cél.*, vol. i. p. 279.

As in the lunar, so in the planetary theory, in some cases to obtain even the first term it is necessary to carry further the development of the disturbing function, that is to say, the expression for R, which results from the substitution of the elliptic values merely of the coordinates of the planets, is insufficient. Thus $r_{7,1}$ being of the order of the terms which belong to the elliptic value of r, and not being proportional to $m_{,}$, $q_{7,1}$ contains the term

$$r_{7,1}\left\{-\frac{a^2}{2\,a_{,}^3}b_{3,0} + \frac{a}{a_{,}^2}b_{3,1}\right\}$$

as is evident from the expression for q_2. Therefore,

$$q_{7,1} = -\frac{a}{2\,a_{,}^2}b_{3,2} + r_{7,1}\left\{-\frac{a^2}{2\,a_{,}^3}b_{3,0} + \frac{a}{a_{,}^2}b_{3,1}\right\}$$

$$r_{7,1}\left\{\frac{\{\mathrm{n}-\mathrm{n}_{,}+\mathrm{c}_{,}\mathrm{n}_{,}\}^2}{\mathrm{n}^2}(1-3\,\mathrm{r}_0) - 1 - \frac{a^3}{2\,a_{,}^3}b_{3,0} + \frac{a^2}{a_{,}^2}b_{3,1}\right\} = \frac{m_{,}a^2}{2\,a_{,}^2}b_{3,2}$$

$$\mathrm{c}_{,} = 1 - \frac{m\,a}{4\,\mu\,a_{,}}b_{3,1} \qquad \mathrm{r}_0 = -\frac{m_{,}a^3}{6\,\mu\,a_{,}^3}b_{3,0} + \frac{m_{,}a^2}{6\,\mu\,a_{,}^2}b_{3,1}$$

$$r_{7,1} = \frac{m_{\prime} a^2 b_{3,2}}{a_{\prime}^2 \mu \left\{ \frac{a^2 m_{\prime}}{a_{\prime}^2 \mu} - \frac{m a n_{\prime}}{\mu a_{\prime} n} \right\} b_{3,1}} = \frac{m_{\prime} a b_{3,2}}{a_{\prime} \left\{ \frac{a}{a_{\prime}} m_{\prime} - \frac{m n_{\prime}}{n} \right\} b_{3,1}}$$

$r_{7,1}$ coincides with the quantity I have denoted by r_{105} in the lunar theory (see p. 41). If, as in that theory, $c_{\prime}$ be made equal to unity,

$$r_{7,1} = \frac{b_{3,2}}{b_{3,1}} = \frac{5 a}{4 a_{\prime}}$$

The last result is the same as that of M. Plana.

If s_{187} denote the coefficient of $\gamma_{\prime} \sin (\tau + \eta_{\prime})$ in the expression for s the tangent of the latitude, $\frac{dR}{dz}$ contains the term

$$\frac{m_{\prime} a}{2 a_{\prime}^2} b_{3,0} s_{187} \gamma_{\prime} \sin (\tau + \eta_{\prime})$$

Integrating the equation

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} + \left(\frac{dR}{dz}\right) = 0$$

by the method of indeterminate coefficients it is easy to obtain the equation

$$s_{187} \left\{ \frac{(\mathrm{n} - \mathrm{n}_{\prime} + \mathrm{g}_{\prime} \mathrm{n}_{\prime})^2}{\mathrm{n}^2} - 1 - 3 \mathrm{r}_0 \right\} = \frac{m_{\prime} a^3}{2 \mu a_{\prime}^3} s_{187} b_{3,0} + \frac{m_{\prime} a^2}{\mu a_{\prime}^2} b_{3,1}$$

$$\mathrm{g}_{\prime} = 1 - \frac{m a}{\mu a_{\prime}} b_{3,1}$$

$$s_{187} \left\{ \left\{ 1 + \frac{m n_{\prime} a}{\mu n a_{\prime}} b_{3,1} \right\}^2 - 1 + \frac{m_{\prime} a^3}{2 \mu a_{\prime}^3} b_{3,0} - \frac{m_{\prime} a^2}{2 \mu a_{\prime}^2} b_{3,1} \right\} = \frac{m_{\prime} a^2}{\mu a_{\prime}^2} b_{3,1}$$

$$s_{187} = \frac{m_{\prime} \frac{a}{a_{\prime}}}{m_{\prime} \frac{a}{a_{\prime}} - m \frac{n_{\prime}}{n}}$$

The quantity $r_{7,1}$ is denoted by f' in the *Méc. Cél.*, vol. i. p. 279, and by $-f'$ in the *Théor. Anal.*, vol. i. p. 503; but the value to which I have arrived is, apparently, not in accordance with those given either by Laplace or by M. de Pontécoulant. A similar expression, to the same degree of approximation, for $r_{7,1}$ may be obtained from the consideration of the *secular inequalities* of the eccentricity and of the longitude of the perihelion, so that this inequality may be rejected altogether, both in the radius vector and in the longitude, provided a certain secular variation be introduced into those elliptic constants. The expression for s_{187} may be arrived at in the same way; and this inequality may also be superseded by introducing certain variations in the longitude of the node and the inclination of the orbit to a fixed plane.

According to the *Mécanique Célestc*, vol. i. p. 279,

$$\frac{\delta r}{\mathrm{a}} = \frac{m'}{6} a^2 \left(\frac{\mathrm{d} A^{(0)}}{\mathrm{d} a}\right) - m' f e \cos(nt + \varepsilon - \varpi) - m' f' e' \cos(nt + \varepsilon - \varpi')$$

$$+ \frac{1}{2} m' C n t e \sin(nt + \varepsilon - \varpi) + \frac{1}{2} m' D n t e' \sin(nt + \varepsilon - \varpi_{\prime})$$

$$C = a^2 \left(\frac{\mathrm{d} A^{(0)}}{\mathrm{d} a}\right) + \frac{1}{2} a^3 \left(\frac{\mathrm{d}^2 A^{(0)}}{\mathrm{d} a^2}\right)$$

$$D = a A^{(1)} - a^2 \left(\frac{a' A^{(0)}}{\mathrm{d} a}\right) - \frac{1}{2} a^3 \left(\frac{\mathrm{d}^2 A^{(1)}}{\mathrm{d} a^2}\right)$$

$$f = \frac{2}{3} a^2 \left(\frac{\mathrm{d} A^{(0)}}{\mathrm{d} a}\right) + \frac{1}{4} a^3 \left(\frac{\mathrm{d}^2 A^{(0)}}{\mathrm{d} a^2}\right)$$

$$f' = \frac{1}{4} \left\{ a A^{(1)} - a^2 \left(\frac{\mathrm{d} A^{\prime 1)}}{\mathrm{d} a}\right) - {}^{*} a^3 \left(\frac{\mathrm{d}^2 A^{(1)}}{\mathrm{d} a^2}\right) \right\}$$

The theory given in this passage of the *Mécanique Céleste* amounts to making,

$$\frac{C}{2} = -\frac{\mu}{2 m_{\prime}} r_0 - \frac{a}{2} q_2 = -\frac{a^2}{4 a_{\prime}^2} b_{3,1}$$

$$\frac{D}{2} = a q_{7,1} = -\frac{a^2}{a_{\prime}^2} b_{3,2}$$

$$f = -\frac{\mu}{6 m_{\prime}} r_0 - \frac{a}{2} q_2 = \frac{a^3}{6 a_{\prime}^3} b_{3,0} - \frac{5 a^2}{12 a_{\prime}^2} b_{3,1}$$

$$f' = -\frac{a}{2} q_{7,1} + \frac{a}{2} R_{7,1} = \frac{3}{8} \frac{a^2}{a_{\prime}^2} b_{3,0} - \frac{a}{4 a_{\prime}} b_{3,1} + \frac{a^2}{8 a_{\prime}^2} b_{3,2}$$

These results coincide with those to which I have arrived, as regards the quantities C and f; this will be rendered evident by the following details:

The expression for $\frac{r}{\mathrm{a}}$ given p. 107, line 1, may be put in the form

$$\frac{r}{\mathrm{a}} = 1 + \frac{1}{3} r_0 - \mathrm{e} \left\{ 1 - \frac{1}{6} r_0 - \frac{m_{\prime}}{2 \mu} a q_2 \right\} \cos(nt + \varepsilon - \varpi)$$

$$- e \left\{ \frac{1}{2} r_0 - \frac{m_{\prime}}{2 \mu} a q_2 \right\} n t \sin(nt + \varepsilon - \varpi)$$

---

[*] In the *Mécanique Céleste* the sign is +, but it should evidently be —. This correction is given by Laplace, *Méc. Cél.*, vol. iii. p. 60.

In the notation of the *Mécanique Céleste,*

$$r_0 = \frac{a^2}{2}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right) \qquad a\,q_2 = -\frac{a^3}{2}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right) - \frac{3\,a^2}{2}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)$$

$$\frac{\mu}{6m_{,}}r_0 + \frac{a}{2}q_2 = \frac{a^2}{12}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right) - \frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right) - \frac{3}{4}a^2\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)$$

$$= -\frac{2\,a^2}{3}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right) - \frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right) = -f$$

$$\frac{\mu}{2\,m_{,}}r_0 + \frac{a}{2}q_2 = \frac{a^2}{4}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right) - \frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right) - \frac{3}{4}a^2\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)$$

$$= -\frac{a^2}{2}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right) - \frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right) = -\frac{C}{2} = \frac{a^2}{4\,a_{,}^2}b_{3,1}$$

$$a\,q_{7,1} = -\frac{1}{2}\left\{a^2 a'\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}a\,\mathrm{d}\,a'}\right) + 2\,a^2\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right)\right.$$

$$\left. + 2\,a_{,}a\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a'}\right) + 4\,a\,A^{(1)}\right\}$$

$$= -\left\{\frac{1}{2}2\,a\,A^{(1)} - 2\,a^2\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right) - a^3\left(\frac{\mathrm{d}^2\,A^{(1)}}{\mathrm{d}\,a^2}\right)\right\}$$

$$R_{7,1} = -\frac{1}{2}a'\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a'}\right) - A^{(1)}$$

$$= \frac{a}{2}\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right) - \frac{1}{2}A^{(1)}$$

$$\frac{a}{2}q_{7,1} - \frac{a}{2}R_{7,1} = -\frac{a\,A^{(1)}}{2} + \frac{a^2}{2}\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right) + \frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(1)}}{\mathrm{d}\,a^2}\right)$$

$$+ \frac{a\,A^{(1)}}{4} - \frac{a^2}{4}\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right)$$

$$= -\frac{a\,A^{(1)}}{4} + \frac{a^2}{4}\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right) + \frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(1)}}{\mathrm{d}\,a^2}\right) = -f'$$

$$-\frac{1}{2}a\,q_{7,1} = \frac{1}{2}\left\{a\,A^{(1)} - a^2\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right) - \frac{a^3}{2}\left(\frac{\mathrm{d}^2\,A^{(1)}}{\mathrm{d}\,a^2}\right)\right\}$$

$$= -\frac{D}{2} = \frac{a^2}{2\,a_{,}^2}b_{3,2}$$

The expression given by Laplace, *Méc. Cél.*, vol. i. p. 279, may be put in the form,

$$\frac{\delta r}{\mathrm{a}} = \frac{m_{,}}{\mu}\left\{\frac{a^3}{6\,a_{,}^3}b_{3,0} - \frac{a^2}{6\,a_{,}^2}b_{3,1}\right\}$$

$$-\frac{m_{,}}{\mu}\left\{\frac{a^3}{6\,a_{,}^3}b_{3,0} - \frac{5\,a^2}{12\,a_{,}^2}b_{3,1}\right\} e\cos(nt + \varepsilon - \varpi)$$

$$+\frac{m_{,}}{\mu}\left\{\frac{a^3}{a_{,}^3}b_{3,0} - \frac{5\,a^2}{4\,a_{,}^2}b_{3,1}\right\} ent\sin(nt + \varepsilon - \varpi)$$

$$-\frac{m_{,}}{\mu}\left\{\frac{3}{3}\frac{a^2}{a_{,}^2}b_{3,0} - \frac{a}{4\,a_{,}}b_{3,1} + \frac{a^2}{8\,a_{,}^2}b_{3,2}\right\} e_{,}\cos(nt + \varepsilon - \varpi_{,})$$

$$-\frac{m_{,}}{\mu}\frac{a^2}{a_{,}^2}b_{3,2}\,e_{,}\,nt\sin(nt + \varepsilon - \varpi_{,})$$

$$-r_1\cos i\tau - \{r_3 - r_1\}\,e\cos(i\tau - \xi) - \{r_4 - r_1\}\,e\cos(i\tau + \xi)$$

$$-r_5\,e_{,}\cos\xi_{,} - r_6\,e_{,}\cos(i\tau - \xi_{,}) - r_7\,e_{,}\cos(i\tau + \xi_{,})$$

i being any positive whole number.

In Laplace's notation, when $a < a_{,}$,

$$\frac{a}{2a_{,}}b_{3,0} - \frac{1}{2}b_{3,1} = -\frac{\mathrm{d}\,b_{\frac{1}{2}}^{(0)}}{2\,\mathrm{d}\,\alpha}, \quad \frac{a}{a_{,}}b_{3,1} - \frac{1}{2}b_{3,0} - \frac{1}{2}b_{3,2} = -\frac{\mathrm{d}\,b_{\frac{1}{2}}^{(1)}}{\mathrm{d}\,\alpha}$$

$$\frac{a}{a_{,}}b_{3,2} - \frac{1}{2}b_{3,1} - \frac{1}{2}b_{3,3} = -\frac{\mathrm{d}\,b_{\frac{1}{2}}^{(2)}}{\mathrm{d}\,\alpha},$$

$$\frac{a}{a_{,}}b_{3,3} - \frac{1}{2}b_{3,2} - \frac{1}{2}b_{3,4} = -\frac{\mathrm{d}\,b_{\frac{1}{2}}^{(3)}}{\mathrm{d}\,\alpha}$$

$$3\left\{\frac{a}{2\,a_{,}}b_{5,0} - \frac{1}{2}b_{5,1}\right\} = -\frac{\mathrm{d}\,b_{\frac{3}{2}}^{(0)}}{2\mathrm{d}\alpha}$$

$$3\left\{\frac{a}{a_{,}}b_{5,1} - \frac{1}{2}b_{5,0} - \frac{1}{2}b_{5,2}\right\} = -\frac{\mathrm{d}\,.\,b_{\frac{3}{2}}^{(1)}}{\mathrm{d}\,\alpha}$$

$$3\left\{\frac{a}{a_{,}}b_{5,2} - \frac{1}{2}b_{5,1} \quad \frac{1}{2}b_{5,3}\right\} = -\frac{\mathrm{d}\,.\,b_{\frac{3}{2}}^{(2)}}{\mathrm{d}\,\alpha}$$

$$3\left\{\frac{a}{a_{,}}b_{5,3} - \frac{1}{2}b_{5,2} - \frac{1}{2}b_{5,4}\right\} = \frac{\mathrm{d}\,.\,b_{\frac{3}{2}}^{(3)}}{\mathrm{d}\,\alpha}$$

The numerical values of these quantities are given for the pal planets in the third volume of the *Mécanique Céleste*.

6. The following numerical examples will serve to expl expressions given above. When $a < a_{\prime}$

$$\frac{r}{a} = 1 + \frac{m_{\prime}}{\mu}\left\{\frac{a^3}{6\,a_{\prime}^3}b_{3,0} - \frac{a^2}{6\,a_{\prime}^2}b_{3,1}\right\}$$

$$-\,e\left\{1 + \frac{m_{\prime}}{\mu}\left\{\frac{a^3}{6\,a_{\prime}^3}b_{3,0} - \frac{5}{12}\frac{a^2}{a_{\prime}^2}b_{3,1}\right\}\right\}\cos(n\,t + \varepsilon - \varpi$$

$$-\,r_1\cos\left(i\,(n\,t - n_{\prime}\,t) + \varepsilon - \varepsilon_{\prime}\right)\quad\text{See p. 107.}$$

$$r_{1,1} = \frac{m_{\prime}}{\mu}\,\frac{n^2}{(2\,n - n_{\prime})\,n_{\prime}}\left\{\frac{2\,n}{(n - n_{\prime})}\left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}}b_{1,1}\right)\right.$$

$$\left. + \frac{a^2}{a_{\prime}^2} + \frac{a^2}{a_{\prime}^2}\left(\frac{a}{a_{\prime}}b_{3,1} - \frac{1}{2}b_{3,0} - \frac{1}{2}b_{3,2}\right)\right\}\quad\text{See p. 107}$$

$$\lambda_{1,1} = \frac{n}{(n - n_{\prime})}\left\{2\,r_{1,1} - \frac{m_{\prime}\,n}{\mu\,(n - n)}\left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}}b_{1,1}\right)\right\}$$

In the theory of Jupiter disturbed by Saturn,

$\frac{a}{a_{\prime}} = \cdot 54531725$, $\quad b_{1,1} = \cdot 6206406$, $\quad \frac{1}{2}b_{3,0} - \frac{4\cdot 358387}{2} - 2\cdot 179193$

$b_{3,1} = 3\cdot 185493$ $\quad b_{3,2} = 2\cdot 082131$ $\quad \frac{m_{\prime}}{\mu} - \frac{1}{3359\cdot 4}$

$n = 337210\cdot 78$ $\quad n_{\prime} = 135792\cdot 34$

$a = 5\cdot 20116636$ $\quad e = \cdot 0480767$

See *Mécanique Céleste*, vol. iii. p. 61 & 81.

Whence

$$\begin{array}{rl} \log.\frac{a^2}{a_{\prime}^2} = & 9\cdot 4732982 \\ \log.\,b_{3,1} = & 0\cdot 5031766 \\ \log.\,n = & \underline{5\cdot 5279013} \\ & 5\cdot 5043761 \\ & \underline{4\cdot 1283217} \end{array} \qquad \begin{array}{rl} \log.\,3359\cdot 4 = & 3\cdot 5262617 \\ \log.\,4 = & \underline{0\cdot 6020600} \\ & \underline{\underline{4\cdot 1283217}} \end{array}$$

$\underline{\underline{1\cdot 3760544}} = \log.\,23''\cdot 771 = \frac{m_{\prime}\,a^2\,n}{4\,\mu\,a_{\prime}^2}\,b_{3,1}$, or (4,5) in the nota the *Mécanique Céleste*, p. 87.

$$\log.\frac{a^2}{a_{\prime}^2} = 9{\cdot}4732982$$

$$\log.b_{3,2} = 0{\cdot}3185081$$

$$\log.n = 5{\cdot}5279013$$

$$5{\cdot}3197076$$

$$4{\cdot}1283217$$

$\underline{\underline{1{\cdot}1913859}} = 15''{\cdot}537 = \frac{m_{\prime}a^2 n}{4\mu a_{\prime}^2}b_{3,2}$ or $\boxed{4,5}$ in the notation of the *Mécanique Céleste*, p. 87.

$$\frac{a^3}{2a_{\prime}^3}b_{3,0} = {\cdot}353381, \quad \frac{a^3}{2a_{\prime}^3}b_{3,1} = {\cdot}473636 \quad \frac{a^3}{2a_{\prime}^3}b_{3,0} - \frac{a^2}{2a_{\prime}^2}b_{3,1} = -{\cdot}120255$$

$$\log.{\cdot}120255 = 9{\cdot}0801033$$

$$\log.5{\cdot}20116636 = 0{\cdot}7161007$$

$$9{\cdot}7962040$$

$$\log.\frac{3\mu}{m_{\prime}} = 4{\cdot}0033829$$

$\underline{\underline{5{\cdot}7928211}} = \log.{\cdot}0000620613$ minus

Laplace has . . . . . . ·0000620566 minus

See *Mécanique Céleste*, vol. iii. p. 121, line 5.

$$\log.\left\{\frac{5}{12}\frac{a^2}{a_{\prime}^2}b_{3,1} - \frac{a^3}{6a_{\prime}^3}b_{3,0}\right\} = 9{\cdot}4423277$$

$$\log.\text{e} = 8{\cdot}6819347$$

$$\log.a = 0{\cdot}7161007$$

$$8{\cdot}8403631$$

$$\log.\frac{\mu}{m_{\prime}} = 3{\cdot}5262617$$

$5{\cdot}3141014 = \log.{\cdot}0000206111 +$

Laplace has . . . . ·0000206111 the sign omitted, *Mécanique Céleste*, vol. iii. p. 122, line 28.

Calculation of $r_{1,1}$

$$\log.n_{\prime} = 5{\cdot}1328751$$

$$\log.n = 5{\cdot}5279013$$

$\underline{\underline{9{\cdot}6049738}} = \log.{\cdot}4026928$

$$1 - \frac{n_{\prime}}{n} = {\cdot}5973072$$

$$\frac{a}{a_{\prime}}b_{3,1} - \frac{1}{2}b_{3,0} - \frac{1}{2}b_{3,2} = -1{\cdot}48315$$

$$\log.1{\cdot}48315 = 0{\cdot}1711851$$

$$\log.\frac{a^2}{a_{\prime}^2} = 9{\cdot}4732982$$

$\underline{\underline{9{\cdot}6444333}} = \log.{\cdot}441045$

$$\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}}b_{1,1} = -{\cdot}041075$$

$$\log.{\cdot}041075 = 8{\cdot}6135776$$

$$\log.\frac{n-n_{\prime}}{n} = 0{\cdot}7761940$$

$8{\cdot}8373836 = \log.{\cdot}0687676$

$$2$$

$$-{\cdot}1375352$$

$$-{\cdot}441045$$

$$-{\cdot}5785802$$

$$\frac{a^2}{a^2} = {\cdot}297371$$

$$\underline{\underline{-{\cdot}281209}}$$

log. 1·5973072 = 0·2033884
log. ·4026928 = 9·6049738
log. 3359·4 = 3·5262617
3·3346239

$r_{1,1}$ = − ·0001301383

log. ·281209 = 9·4490293
3·3346259
6·1144054 = log. ·0001301383
log. a = ·7161007
6·8305061 = log. ·000676871 +
Laplace has (p. 121) . . . ·000676876 +

Calculation of the coefficient of sin $(n t - n_{\prime} t + \epsilon - \epsilon_{\prime})$ in the value of λ, or $\lambda_{1,1}$. See p. 105.

$2 r_{1,1}$ = − ·0002602766
·0000204702
log. ·0002398064 = 6·3798601
log. $\frac{n - n_{\prime}}{n}$ = 9·7761940
6.6036661
5·8038801*
2·4075461 = log. 255″·591 minus.

log. $\frac{n}{n - n_{\prime}} \left\{ \frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}} b_{1,1} \right\}$ = 8·8373836
log. 3359·4 = 3·5262617
5·3111219 = log. ·0000204702

Laplace has . . . . 255·5917, the sign omitted, p. 120.

In the notation of the *Mécanique Céleste*, vol. iii. p. 120,

$$n = n^{\text{iv}}, \qquad n_{\prime} = n^{\text{v}}.$$

When $a_{\prime} < a$

$$r_{1,1} = \frac{m_{\prime} n^2}{\mu (2 n - n_{\prime}) n_{\prime}} \left\{ \frac{2 n}{(n - n_{\prime})} \left(\frac{a^2}{a_{\prime}^2} - b_{1,1} \right) \right.$$
$$\left. + \frac{a^2}{a_{\prime}^2} + b_{1,1} - \frac{a_{\prime}}{a} \left(\frac{a}{a} b_{3,1} - \frac{1}{2} b_{3} \quad \frac{1}{2} b_{3,2} \right) \right\}$$

$$\lambda_{1,1} = \frac{n}{(n - n_{\prime})} \left\{ 2 r_{1,1} - \frac{m_{\prime} n}{\mu (n - n_{\prime})} \left(\frac{a^2}{a_{\prime}^2} - b_{1,1} \right) \right\}$$

In the theory of Saturn disturbed by Jupiter,

$\frac{a_{\prime}}{a}$ = ·54531725 $b_{1,1}$ = ·6206406 $\frac{m_{\prime}}{\mu} = \frac{1}{1067·09}$

$\frac{a_{\prime}}{a} b_{3,1} - \frac{1}{2} b_{3,0} - \frac{1}{2} b_{3,2}$ = − 1·483154 a = 9·5378709

---

* The logarithm 5·8038801 is added to convert the coefficient into centesimal seconds; the logarithm 5·3144251 must be added to convert it into sexagesimal seconds.

Calculation of $r_{1,1}$:

$$
\begin{array}{l}
\log. n_{\prime} = 5{\cdot}5279013 \\
\log. n = 5{\cdot}1328751 \\
\hline
0{\cdot}3950262 = \log. 2{\cdot}48328 \\
1 - \frac{n_{\prime}}{n} = -\ 1{\cdot}48328
\end{array}
$$

$$
\begin{array}{l}
\frac{a^2}{a_{\prime}^{2}} - b_{1,1} = 2{\cdot}74216 \\
\log. 2{\cdot}74216 = 0{\cdot}4380928 \\
\log. \frac{n - n_{\prime}}{n} = 0{\cdot}1712231 \\
\hline
0{\cdot}2668697 = \log. 1{\cdot}84872 \\
\qquad\qquad\qquad\qquad 2 \\
\hline
\qquad\qquad\qquad\quad 3{\cdot}69744
\end{array}
$$

$$
\begin{array}{l}
\log. 1{\cdot}48315 = 0{\cdot}1711851 \\
\log. \frac{a_{\prime}}{a} = 9{\cdot}7366492 \\
\hline
9{\cdot}9078343 = \log. \cdot 80879 \\
\frac{a_{\prime}^{2}}{a^2} = 3{\cdot}36280 \\
b_{1,1} = \cdot 62064 \\
\hline
4{\cdot}79223 \\
3{\cdot}69744 \\
\hline
\log. 1{\cdot}09479 = 0{\cdot}0393310 \\
\qquad\qquad\qquad 3{\cdot}1074262 \\
\hline
\log. r_{1,1} = 6.9319048 \\
r_{1,1} = -\ \cdot 00085488
\end{array}
$$

$$
\begin{array}{l}
\log. 2 - \frac{n_{\prime}}{n} = 9{\cdot}6841988 \\
\log. \frac{n_{\prime}}{n} = 0{\cdot}3950262 \\
\log. 1067{\cdot}09 = 3{\cdot}0282012 \\
\hline
\qquad\qquad\qquad 3{\cdot}1074262
\end{array}
$$

$$
\begin{array}{l}
\log. r_{1,1} = 6{\cdot}9119048 \\
\log. a = 0{\cdot}9794518 \\
\hline
7.9113568 = \log. \cdot 00815374
\end{array}
$$

Laplace has . . . . ·00815384 p. 85.

Calculation of $\lambda_{1,1}$,

$$
\begin{array}{l}
\log. \frac{n}{n - n_{\prime}} \left(\frac{a^2}{a_{\prime}^{2}} - b_{1,1}\right) = 0{\cdot}2668697 \\
\log. 1067{\cdot}09 = 3{\cdot}0282012 \\
\hline
9{\cdot}2386685 = \log. \cdot 00173248
\end{array}
$$

$$
\begin{array}{l}
2\, r_{1,1} = -\ \cdot 00170976 \\
\qquad\quad \cdot 00173248 \\
\hline
\log. \cdot 00002272 = 5.3564083 \\
\log. \frac{n - n_{\prime}}{n} = 0{\cdot}1712231 \\
\hline
5{\cdot}1851852 \\
5{\cdot}8038801 \\
\hline
0{\cdot}9890653 = \log. 9''{\cdot}7513
\end{array}
$$

Laplace has 9″·742382 (p. 134.) the sign omitted, which should be +

CORRECTIONS.

P. 21, line 24, *for* $\frac{n-1}{2}$ *read* $\frac{q-1}{2}$

P. 33, line 19, *for* $\cos(2\tau + \xi_{,} + 2\eta)$ *read* $\cos(2\tau + \xi_{,} - 2\eta)$

P. 35, line 26, *for* $g n t + \varepsilon - \varpi$ *read* $g n t + \varepsilon - \beta$

P. 65, line 15, *for* $\left\{1 + 2r_0 + 3r_0^2 - \frac{r_1^2}{2}\right\}$

read $\left\{1 - 2r_0 + 3r_0^2 - \frac{r_1^2}{2}\right\}$

P. 76, line 19, *for* $-\frac{e}{2an}$ *read* $-\frac{e}{2a^2n}$

P. 77, line 14, *for* $\frac{d\varepsilon}{d\nu}$ *read* $\frac{d\varepsilon}{dt}$

PRINTED BY RICHARD TAYLOR,
RED LION COURT, FLEET STREET.

TABLE.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	35	
20	22.	8						2														20
	−21		10	− 9					4	− 3												
21																						21
	−20	9	− 8					3	− 2											1		
22	20	10		8				4		2										1		22
23	25.	11			8			5			2											23
	−24		13	−12		10	− 9		7	− 6		4	− 3									
24																						24
	−23	12	−11		9	− 8		6	− 5		3	− 2										
25	23	13		11	10		8	7		5	4		2									25
26	28.	14												2								26
	−27		16	−15	8	− 9	10	− 5	6	− 7					4	− 3						
27					9				5						2							27
	−26	15	−14				8	− 7						3								
28	26	16		14		8		6						4		2						28
					10					− 5												
29	31.	17			11						5						2					29
	−30		19	−18		13	−12					7	− 6					4	− 3			
30			18.																			30
	−29	18	−17		12	−11					6	− 5					3	− 2				
31	29	19		17	13		11				7		5				4		2			31
32	34.																					32
	−33	−17	18	−19	14	−15	16							− 5	6	− 7	2	− 3	4			
33			17		15										5		3					33
	−32	19					−14							7					− 2			
34	32	18				14								6				2				34
				−17	16											− 5	4					
35	37.				17												5					35
	−36					19	−18											7	− 6			
36																						36
	−35				18	−17											6	− 5			1	
37	35				19		17										7		5		1	37
38		20						8												2		38
			22	−21					10	− 9												
39																						39
		21	−20					9	− 8											3		
40		22		20				10		8										4		40
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	35	

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	35	
41		23			20			11			8									5		41
			25	−24					13	−12		10	−9									
42																						42
		24	−23		21			12	−11		9	−8								6		
43		25		23	22			13		11	10		8							7		43
44		26						14						8								44
			28	−27	20				16	−15					10	−9				−5		
45					21																	45
		27	−26					15	−14					9	−8					7		
46		28		26				16		14				10		8				6		46
					22																	
47		29			23			17			11						8					47
			31	−30					19	−18		13						10	−9			
48																						48
		30	−29		24			18	−17		12	−11					9	−8				
49		31		29	25			19		17	13		11				10		8			49
50		32												14								50
			34	−33	26			−17	18	−19					16	−15	8	−9	10			
51					27				17								9					51
		33	−32					19						15	−14				−8			
52		34		32				18						16		14		8				52
					28					−17							10					
53		35			29						17						11				2	53
			37	−36								19	−18					13	−12			
54																						54
		36	−35		30						18	−17					12	−11			3	
55		37		35	31						19		17				13		11		4	55
56																						56
		−35	36	−37	32									−17	18	−19	14	−15	16		2	
57			35		33										17		15				3	57
		37												19					−14			
58		36												18				14				58
				−35	34											−17	16				4	
59					35												17				5	59
																		19	−18			
60																						60
					36												18	−17			6	
61					37												19		17		7	61
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	35	

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
62	64. -63	66. -65	68. -67	70. -69	72. -71	....	....	....	....	....	....	....	....	....	....	....	....	62
63	...... -62	67 69	65	...... -66	73 75	....	....	....	....	....	....	....	....	....	....	....	....	63
64	62	68 70	66	...... -65	74 76	....	....	....	....	....	....	....	....	....	....	....	....	64
65	69. -68	...... -62	 63	...... -64	....	....	....	....	....	....	....	....	....	....	....	....	....	65
66	70. -67	62	 64	...... -63	....	....	....	....	....	....	....	....	....	....	....	....	....	66
67	76 -66	 63	...... -62	....	....	....	....	....	....	....	....	....	....	....	....	....	....	67
68	...... -65	 64	62	....	....	....	....	....	....	....	....	....	....	....	....	....	....	68
69	65	63	....	...... -62	....	....	....	....	....	....	....	....	....	....	....	....	....	69
70	66	64	....	62	....	....	....	....	....	....	....	....	....	....	....	....	....	70
71	75. -74	....	....	....	...... -62	 63	...... -64	....	....	....	....	....	....	....	....	....	....	71
72	76. -73	....	....	....	62	 64	...... -63	....	....	....	....	....	....	....	....	....	....	72
73	...... -72	....	....	....	 63	...... -62	....	....	....	....	....	....	....	....	....	....	....	73
74	...... -71	....	....	....	 64	62	....	....	....	....	....	....	....	....	....	....	....	74
75	71	....	....	....	63	....	...... -62	....	....	....	....	....	....	....	....	....	....	75
76	72	....	....	....	64	....	62	....	....	....	....	....	....	....	....	....	....	76
77	....	65	 69	...... -68	....	....	....	...... -62	 63	...... -64	....	....	....	....	....	....	....	77
78	....	66	 70	...... -67	....	....	....	62	 64	...... -63	....	....	....	....	....	....	....	78
79	....	 67	...... -66	....	....	....	....	 63	...... -62	....	....	....	....	....	....	....	....	79
80	....	 68	...... -65	....	....	....	....	 64	62	....	....	....	....	....	....	....	....	80
81	....	69	....	65	....	....	....	63	....	...... -62	....	....	....	....	....	....	....	81
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	

	18	19	62	65	66	77	78	146	147	148	149	150	161	162	
62								146							62
				2	−2				148	−147	150	−149			
63				3							151				63
			1		4			147	−146			153			
64			1		3			148		146		152			64
				4							154				
65															65
			2					149	−152	153	−146				
66			2					150				146			66
									154	−151					
67															67
			3		1			151	−150			147			
68			3					152							68
				1						−149	148				
69				1					149		147				69
			4					153							
70			4		1			154		150		148			70
71															71
			5					155	−158	−159					
72			5					156							72
									159	−157					
73															73
			6					157	−156						
74			6					158							74
										−155					
75									155						75
			7					159							
76			7					160		156					76
77				2							149				77
			8					161	−164	165			−146		
78			8		2			162				150		146	78
									166	−163					
79															79
			9		3		1	163	−162			151		147	
80			9					164							80
				3		1				−161	152		148		
81				4		1			161		153		147		81
			10					165							
	18	19	62	65	66	77	78	146	147	148	149	150	161	162	

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
82	70		66				64		62								82
83	71			65													83
		75	−74							−62	63	−64					
84	72			66						62							84
		76	−73								64	−63					
85																	85
	73	−72		67						63	−62						
86											62						86
	74	−71		68						64							
87	75		71	69						63							87
												−62					
88	76		72	70						64		62					88
89																	89
	−72	73	−76	65									−62	63	−64		
90													62				90
	−71	74	−75	66										64	63		
91		71		67													91
	75												63	−62	−62		
92		72		68										62			92
	76												64				
93	73												63				93
			−72	69													
94	74												64		62		94
			−71	70													
95				71													95
																−62	
96				72												62	96
97																	97
				73												63	
98																	98
				74												64	
99				75												63	99
100				76												64	100
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	

	18	19	62	65	66	77	78	146	147	148	149	150	161	162	
82			10		4		1	166		162		154		148	82
83				5							155				83
			11					167	−170	171					
84			11		5			168				156			84
									172	−169					
85															85
			12		6			169	−168			157			
86			12					170							86
				6						−167	158				
87				7					167		159				87
			13					171							
88			13		7			172		168		160			88
89															89
			14	−5				173	−176	177	−156				
90			14					174							90
					−5				178	−175		−155			
91															91
			15		7			175	−174			159			
92			15					176							92
				7						−173	160				
93				6					173		157				93
			16					177							
94			16		6			178		174		158			94
95															95
	63	−64							−182	−183					
96															96
	64	−63							184	−181					
97															97
	−62								−180						
98	62														98
										−179					
99									179						99
		−62													
100		62								180					100
	18	19	62	65	66	77	78	146	147	148	149	150	161	162	

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
101{	116. −101	102 103	117. −102	118. −103	104 105															}101
102{	117. −103	101	−101	116																}102
103{	118. −102	101	116	−101																}103
104{	119. −105				101	−101	116													}104
105{	120. −104				101	116	−101													}105
106{		102	−103	117				101	−101	116										}106
107{		103	118	−102				101	116	−101										}107
108{		104	−105	119	102						101	−101	116							}108
109{		105	120	−104	103						101	116	−101							}109
110{		105	−104	120	102									101	−101	116				}110
111{		104	119	−105	103									101	116	−101				}111
112{					101												101	−101	116	}112
113{					105												101	116	−101	}113
114{																				}114
115{																				}115
116{	101	117 118	103	102	119 120															}116
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
117	102	116	101																	117
118	103	116		101																118
119	104				116	101														119
120	105				116		101													120
121		117	102					116	101											121
122		118		103				116		101										122
123		119	104		117						116	101								123
124		120		105	118						116		101							124
125		120	105		117									116	101					125
126		119		104	118									116		101				126
127					119												116	101		127
128					120												116		101	128
129																				129
130																				130
131	1	132 133	4	3	134 135	7	6		10	9		13			16	15		19	18	131
132	3	131	1	9		15	12		4			7			6					132
133	4	131	10	1		13	16			3			6			7				133
134	6		16	12	131	1	18					4				3		7		134
135	7		13	15	131	19	1						3		4				6	135
136	9	132	3	21				131	1			15			12					136
137	10	133		4				131		1			16			13				137
138	12	134	6		132	3			16		131	1			18	9		15		138
139	13	135		7	133		4			15	131		1		10	19			16	139
140	15	135	7		132		3		13			19	9	131	1				12	140
141	16	134		6	133					12		10	18	131		1		13		141
142	18				134	6						16				12	131	1		142
143	19				135		7						15		13		131		1	143
144																				144
145																				145
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
146	148. -147	150. -149	152. -151	154. -153	156. -155										146
147	…… -146	151 153	149 ……	…… -150	157 159										147
148	146 ……	152 154	150. ……	…… -149	158 160										148
149	153. -152	…… -146	…… 147	…… -148											149
150	154. -151	146 ……	148. ……	…… -147											150
151	…… -150	…… 147	…… -146												151
152	…… -149	…… 148	146 ……												152
153	149 ……	147 ……		…… -146											153
154	150 ……	148 ……		146 ……											154
155	159. -158				…… -146	…… 147	…… -148								155
156	160. -157				146 ……	…… 148	…… -147								156
157	…… -156				…… 147	…… -146									157
158	…… -155				…… 148	146 ……									158
159	155 ……				147 ……		…… -146								159
160	156 ……				148 ……		146 ……								160
161		149 ……	…… 153	…… -152				…… -146	…… 147	…… -148					161
162		150 ……	…… 154	…… -151				146 ……	…… 148	…… -147					162
163		…… 151	…… -150					…… 147	…… 146						163
164		…… 152	…… -149					147 148	146 ……						164
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	

	15	16	17	18	19	146	147	148	149	150	161	162	
146						 62	− 63. 1	 − 1		 − 2			146
147						63 1			3 	 4			147
148						1 64	62 131		 4	3 			148
149						 2	 − 3	 4					149
150						2 	 4	 − 3					150
151						 3	 − 2			 1			151
152						3 		 − 2	 1				152
153						 4	2 		1 				153
154						4 		− 2 		1 			154
155						 5	 − 6	 7					155
156						5 	 7	 − 6					156
157						 6	 − 5						157
158						6 		 − 5					158
159						 7	5 						159
160						7 		− 5 					160
161						 8	 − 9	 10	2 				161
162						8 	 10	 − 9	 2	2 			162
163						 9	 − 8			 3		 1	163
164						9 		 − 8	 3		 1		164
	15	16	17	18	19	146	147	148	149	150	161	162	

1	2	3	4	5	6	7	8	9	10	11	12	13	14	
165	153		149											165
									−146					
166	154		150				148		146					166
167	155			149										167
		159	−158							−146	147	148		
168	156			150						146				168
		160	−157								148	147		
169														169
	157	−156		151						147	−146			
170											146			170
	158	−155		152						148				
171	159			153						147				171
												146		
172	160		156	154						148		146		172
173														173
	−156	157	−160	149									−146	
174													146	174
	−155	158	−159	150										
175		155		151										175
	159												147	
176		156		152										176
	160												148	
177	157												147	177
			−156	153										
178	158												148	178
			−155	154										
179				155										179
180				156										180
181														181
				157										
182														182
				158										
183				159										183
184				160										184
1	2	3	4	5	6	7	8	9	10	11	12	13	14	

	15	16	17	18	19	146	147	148	149	150	161	162	
165							8		4		1		165
						10							
166						10		8		4		1	166
167									5				167
						11	− 12	13					
168						11				5			168
							13	− 12					
169									6				169
						12	− 11			6			
170						12							170
								− 11	6				
171							11		7				171
		146				13							
172						13		11		7			172
173													173
	−147	148				14	− 15	16	− 5				
174						14							174
	−148	147					16	− 15		− 5			
175													175
	146					15	− 14			7			
176	146					15							176
								− 14	7				
177							14						177
						16							
178		146				16		14		6			178
179													179
			−146	−147	148	17	− 18	19					
180			146			17							180
				−148	147		19	− 18					
181													181
			147	146		18	− 17						
182				146		18							182
			148					− 17					
183			147				17						183
					146	19							
184			148		146	19		17					184
	15	16	17	18	19	146	147	148	149	150	161	162	

London: Printed by Richard Taylor,
Red Lion Court, Fleet Street.

APPENDIX.

EXPRESSIONS OF M. PLANA

IN THE NOTATION AND WITH THE ARGUMENTS IN THE ORDER HITHERTO EMPLOYED; FROM THE "*THÉORIE DU MOUVEMENT DE LA LUNE.*"

◆

$$\frac{a}{r} =$$

$$1 + \frac{m^2}{6} - \frac{179}{288} m^4 - \frac{45}{16} m^2 e^2 + \frac{1}{4} m^2 E_{,}^2$$
$$- \frac{3}{4} m^2 (e_{,}^2 - E_{,}^2) - \frac{97}{48} m^5 + \frac{45}{16} m^3 e^2$$
$$[0]$$

$$+ \left\{ \begin{array}{l} m^2 + \frac{19}{6} m^3 + \frac{15}{4} m e^2 + \frac{131}{18} m^4 - \frac{1}{2} m^2 \gamma^2 - \frac{5}{2} m^2 e_{,}^2 \\ + \frac{189}{16} m^2 e^2 + \frac{383}{27} m^5 - \frac{239}{12} m^3 e_{,}^2 + \frac{11923}{256} m^3 e^2 - \frac{3}{128} m \gamma^4 \\ - \frac{29}{24} m^3 \gamma^2 - \frac{75}{8} m e^2 e_{,}^2 - \frac{15}{4} m e^2 \gamma^2 - \frac{15}{16} m e^4 \end{array} \right\} \cos 2\tau \quad [1]$$

$$+ \left\{ \begin{array}{l} 1 - \frac{e^2}{8} - \frac{\gamma^2}{4} + \frac{m^2}{6} - \frac{645}{128} m^3 - \frac{\gamma^4}{32} - \frac{47}{192} e^4 \\ + \frac{13}{32} e^2 \gamma^2 - \frac{152129}{4608} m^4 - \frac{265}{384} m^2 \gamma^2 - \frac{1067}{96} m^2 e^2 \\ - \frac{45}{8} m^2 e_{,}^2 + \frac{1}{4} m^2 E_{,}^2 - \frac{3}{4} m^2 (e_{,}^2 - E_{,}^2) \end{array} \right\} e \cos \xi \quad [2]$$

$$+ \left\{ \begin{array}{l} \frac{15}{8} m + \frac{187}{32} m^2 + \frac{31673}{1536} m^3 - \frac{45}{32} m \gamma^2 - \frac{75}{16} m e_{,}^2 \\ + \frac{1222495}{18432} m^4 + \frac{895}{128} m^2 e^2 - \frac{385}{32} m^2 e_{,}^2 - \frac{831}{128} m^2 \gamma^2 \end{array} \right\} e \cos (2\tau - \xi) \quad [3]$$

$$+ \left\{ \begin{array}{l} \frac{33}{16} m^2 + \frac{101}{16} m^3 + \frac{405}{64} m e^2 + \frac{5897}{384} m^4 - \frac{165}{32} m^2 e_{,}^2 \\ - \frac{99}{64} m^2 \gamma^2 + \frac{5829}{256} m^2 e^2 \end{array} \right\} e \cos (2\tau + \xi) \quad [4]$$

xpression of
I. Plana for
he reciprocal
f the radius
ector, vol. i.
, 664.

$$+\left\{-\frac{3}{2}m^2+\frac{449}{16}m^4-\frac{9}{4}m^2e^2-\frac{27}{16}m^2e_{,}^2+\frac{9}{4}m^2\gamma^2\right\}e_{,}\cos\xi_{,}$$

[5]

$$+\left\{\begin{array}{l}\frac{7}{2}m^2+\frac{157}{8}m^3+\frac{35}{4}me^2+\frac{799}{16}m^2e^2+\frac{3349}{48}m^4\\ -\frac{7}{4}m^2\gamma^2-\frac{123}{16}m^2e_{,}^2\end{array}\right\}e_{,}\cos(2\tau-\xi_{,})$$

[6]

$$+\left\{\begin{array}{l}-\frac{m^2}{2}-\frac{91}{24}m^3-\frac{15}{4}me^2--\frac{207}{16}m^2e^2-\frac{1265}{144}m^4\\ +\frac{1}{4}m^2\gamma^2+\frac{1}{16}m^2e_{,}^2\end{array}\right\}e_{,}\cos(2\tau+\xi_{,})$$

[7]

$$+\left\{1+\frac{2}{3}m^2-\frac{\gamma^2}{2}-\frac{1}{3}e^2-\frac{15}{2}m^3\right\}e^2\cos 2\xi$$

[8]

$$+\left\{-\frac{15}{4}m^2-\frac{225}{16}m^3+\frac{15}{8}m\gamma^2\right\}e^2\cos(2\tau-2\xi)$$

[9]

$$+\left\{\frac{7}{2}m^2+\frac{127}{12}m^3+10me^2\right\}e^2\cos(2\tau+2\xi)$$

[10]

$$+\left\{\begin{array}{l}-\frac{21}{8}m-\frac{837}{64}m^2-\frac{7063}{128}m^3-\frac{51}{64}me^2\\ -\frac{189}{64}me_{,}^2+\frac{147}{32}m\gamma^2\end{array}\right\}ee_{,}\cos(\xi+\xi_{,})$$

[11]

$$+\left\{\begin{array}{l}\frac{35}{8}m+\frac{1269}{64}m^2-\frac{615}{64}me_{,}^2+\frac{47255}{768}m^3\\ +\frac{35}{8}me^2-\frac{105}{32}m\gamma^2\end{array}\right\}ee_{,}\cos(2\tau-\xi-\xi_{,})$$

[12]

$$+\left\{-\frac{33}{32}m^2-\frac{5477}{384}m^3-\frac{45}{64}me^2\right\}ee_{,}\cos(2\tau+\xi+\xi_{,})$$

[13]

$$+\left\{\begin{array}{l}\frac{21}{8}m+\frac{1113}{64}m^2+\frac{833}{8}m^3+\frac{51}{64}me^2\\ +\frac{189}{64}me_{,}^2-\frac{147}{32}m\gamma^2\end{array}\right\}ee_{,}\cos(\xi-\xi_{,})$$

[14]

Expression of M. Plana for the reciprocal of the radius vector, vol. i. p. 664.

$$+\left\{\begin{array}{l} -\frac{15}{8}m-\frac{97}{64}m^2+\frac{16943}{256}m^3-\frac{15}{8}m\,e^2 \\ +\frac{15}{64}m\,e_{\prime}^2+\frac{45}{32}m\,\gamma^2 \end{array}\right\}e\,e_{\prime}\cos(2\,\tau-\xi+\xi_{\prime})$$

[15]

$$+\left\{\frac{231}{32}m^2+\frac{5727}{128}m^3+\frac{105}{64}m\,e^2\right\}e\,e_{\prime}\cos(2\,\tau+\xi-\xi_{\prime})-\frac{9}{4}m^2\,e_{\prime}^2\cos 2\,\xi_{\prime}$$

[16] [17]

$$+\left\{\frac{17}{2}m^2+\frac{799}{12}m^3+\frac{255}{16}m\,e^2\right\}e_{\prime}^2\cos(2\,\tau-2\,\xi_{\prime})$$

[18]

$$+\left\{-\frac{3}{4}m^3-\frac{45}{16}m\,e^2\right\}e_{\prime}^2\cos(2\,\tau+2\,\xi_{\prime})$$

[19]

$$+\left\{\frac{9}{8}+\frac{41}{32}m^2-\frac{27}{32}\gamma^2-\frac{81}{128}e^2\right\}e^3\cos 3\,\xi$$

[20]

$$+\left\{-\frac{105}{64}m-\frac{7703}{763}m^3\right\}e^3\cos(3\,\tau-3\,\xi)-\frac{1175}{384}e^3\cos(3\,\tau+3\,\xi)$$

[21] [22]

$$+\left\{-\frac{21}{4}m-\frac{789}{32}m^2\right\}\cos(2\,\xi+\xi_{\prime})-\frac{105}{8}m^2\,e^2\,e_{\prime}\cos(2\,\tau-2\,\xi-\xi_{\prime})$$

[23] [24]

$$-\frac{7}{2}m^2\,e^2\,e_{\prime}\cos(2\,\tau+2\,\xi+\xi_{\prime})$$

[25]

$$+\left\{\frac{21}{4}m+\frac{1161}{32}m^2\right\}e^2\,e_{\prime}\cos(2\,\xi-\xi_{\prime})$$

[26]

$$+\frac{15}{8}m^2\,e^2\,e_{\prime}\cos(2\,\tau-2\,\xi+\xi_{\prime})+\frac{49}{4}m^2\,e^2\,e_{\prime}\cos(2\,\tau+2\,\xi-\xi_{\prime})$$

[27] [28]

$$+\left\{-\frac{63}{32}m-\frac{1965}{256}m^2\right\}e\,e_{\prime}^2\cos(\xi+2\,\xi_{\prime})$$

[29]

$$+\left\{\frac{255}{32}m+\frac{12011}{256}m^2\right\}e\,e_{\prime}^2\cos(2\,\tau-\xi-2\,\xi_{\prime})$$

[30]

Expression of M. Plana for the reciprocal of the radius vector, vol. i. p. 664.

$$+\left\{\frac{63}{32}m+\frac{4635}{256}m^2\right\}e\,e_{\prime}^2\cos(\xi-2\,\xi_{\prime})$$

[32]

$$+\left\{-\frac{45}{32}-\frac{6219}{256}m^2\right\}e\,e_{\prime}^2\cos(2\,\tau-\xi+2\,\xi_{\prime})$$

[33]

$$+\frac{561}{32}m^2\,e\,e_{\prime}^2\cos(2\,\tau+\xi-2\,\xi_{\prime})$$

[34]

$$-\frac{53}{16}m^2\,e_{\prime}^3\cos 3\,\xi_{\prime}+\frac{845}{48}m^2\,e_{\prime}^3\cos(2\,\tau-3\,\xi_{\prime})$$

[35] [36]

$$+\frac{m^2}{48}e_{\prime}^3\cos(2\,\tau+3\,\xi_{\prime})+\frac{4}{3}m^2\cos 4\,\xi-\frac{567}{64}m\,e^3\,e_{\prime}\cos(3\,\xi+\xi_{\prime})$$

[37] [38] [41]

$$-\frac{285}{64}m\,e^3\,e_{\prime}\cos(2\,\tau-3\,\xi-\xi_{\prime})+\frac{567}{64}m\,e^3\,e_{\prime}\cos(3\,\xi-\xi_{\prime})$$

[42] [44]

$$+\frac{285}{64}m\,e^3\,e_{\prime}\cos(2\,\tau-3\,\xi+\xi_{\prime})-\frac{63}{16}m\,e^2\,e_{\prime}^2\cos(2\,\xi+2\,\xi_{\prime})$$

[45] [47]

$$+\frac{63}{16}m\,e^2\,e_{\prime}^2\cos(2\,\xi-2\,\xi_{\prime})-\frac{53}{64}m\,e\,e_{\prime}^3\cos(\xi+3\,\xi_{\prime})$$

[50] [53]

$$+\frac{845}{64}m\,e\,e_{\prime}^3\cos(2\,\tau-\xi-3\,\xi_{\prime})+\frac{53}{64}m\,e\,e_{\prime}^3\cos(\xi-3\,\xi_{\prime})$$

[54] [56]

$$-\frac{5}{64}m\,e\,e_{\prime}^3\cos(2\,\tau-\xi+3\,\xi_{\prime})-\frac{55}{16}m\,e^4\cos(2\,\tau-4\,\xi)$$

[57] [60]

$$+\left\{\frac{1}{2}m^2-\frac{5}{4}e^2+\frac{135}{32}m\,e^2-\frac{3}{4}m^3\right\}\gamma^2\cos 2\,\eta$$

[62]

$$+\left\{-\frac{3}{4}m^2+\frac{9}{16}m^3+\frac{3}{32}m\,\gamma^2\right\}\gamma^2\cos(2\,\tau-2\,\eta)$$

[63]

Expression of M. Plana for the reciprocal of the radius vector, vol. i. p. 664.

$$+\left\{-\frac{5}{8}+\frac{135}{64}m-\frac{481}{1536}m^2+\frac{75}{64}e^2\right\}e\gamma^2\cos(\xi-2\eta)$$

[65]

$$+\left\{\frac{33}{32}m^2-\frac{135}{64}e^2\right\}e\gamma^2\cos(\xi+2\eta)$$

[66]

$$+\left\{-\frac{21}{32}m-\frac{111}{128}m^2\right\}e\gamma^2\cos(2\tau-\xi-2\eta)$$

[67]

$$-\frac{165}{128}m^2e\gamma^2\cos(2\tau-\xi+2\eta)$$

[68]

$$+\left\{-\frac{33}{64}m+\frac{503}{512}m^2\right\}e\gamma^2\cos(2\tau+\xi-2\eta)$$

[69]

$$+\frac{3}{4}m^2e_{,}\gamma^2\cos(\xi_{,}-2\eta)+\frac{3}{4}m^2e_{,}\gamma^2\cos(\xi_{,}+2\eta)$$

[71] [72]

$$+\left\{\frac{7}{8}m-\frac{7}{8}m^2\right\}e_{,}\gamma^2\cos(2\tau-\xi_{,}-2\eta)$$

[73]

$$+\frac{3}{8}m^2e_{,}\gamma^2\cos(2\tau+\xi_{,}-2\eta)+\frac{e^2\gamma^2}{8}\cos(2\xi+2\eta)$$

[75] [78]

$$-\frac{21}{16}me^2\gamma^2\cos(2\tau-2\xi-2\eta)-\frac{75}{32}me^2\gamma^2\cos(2\tau-2\xi+2\eta)$$

[79] [80]

$$-\frac{33}{32}me^2\gamma^2\cos(2\tau+2\xi-2\eta)+\frac{75}{64}mee_{,}\gamma^2\cos(\xi+\xi_{,}-2\eta)$$

[81] [83]

$$-\frac{35}{32}mee_{,}\gamma^2\cos(2\tau-\xi-\xi_{,}-2\eta)$$

[85]

$$+\frac{33}{64}mee_{,}\gamma^2\cos(2\tau+\xi+\xi_{,}-2\eta)$$

[87]

$$-\frac{75}{64}mee_{,}\gamma^2\cos(\xi-\xi_{,}-2\eta)+\frac{21}{32}mee_{,}\gamma^2\cos(2\tau-\xi+\xi_{,}-2\eta)$$

[89] [91]

Expression of M. Plana for the reciprocal of the radius vector, vol. i. p. 664.

$$-\frac{49}{64} m\, e\, e_{,} \gamma^2 \cos(2\tau + \xi - \xi_{,} - 2\eta)$$

[93]

$$+\frac{51}{32} m\, e_{,}^2 \gamma^2 \cos(2\tau - 2\xi_{,} - 2\eta)$$

[97]

$$+\left\{-\frac{15}{16} m - \frac{81}{16} m^2 - \frac{5857}{256} m^3 + \frac{165}{64} m\gamma^2 - \frac{105}{32} m\, e^2 - \frac{15}{16} m\, e_{,}^2\right\} \frac{a}{a}$$

[10

$$+\frac{45}{16} m^2 \frac{a}{a_{,}} e \cos(\tau - \xi) + \left\{-\frac{15}{8} m - \frac{177}{16} m^2\right\} \frac{a}{a_{,}} e \cos(\tau + \xi)$$

[102] [103]

$$+\left\{\frac{15}{16} m - \frac{641}{32} m^2\right\} \frac{a}{a_{,}} e_{,} \cos(\tau - \xi_{,})$$

[104]

$$+\left\{\frac{5}{4} - \frac{45}{8} m + \frac{6673}{192} m^2 + \frac{25}{4} e^2 - \frac{5}{8} \gamma^2 + \frac{5}{4} e_{,}^2\right\} \frac{a}{a_{,}} e_{,} \cos(\tau + \xi_{,})$$

[105]

$$+\frac{435}{128} m \frac{a}{a_{,}} e^2 \cos(\tau - 2\xi) - \frac{405}{128} m \frac{a}{a_{,}} e^2 \cos(\tau + 2\xi)$$

[106] [107]

$$+\left\{\frac{5}{2} - \frac{45}{4} m\right\} \frac{a}{a_{,}} e\, e_{,} \cos(\tau + \xi + \xi_{,}) + \frac{15}{8} m \frac{a}{a_{,}} e\, e_{,} \cos(\tau + \xi - \xi_{,})$$

[109] [111]

$$+\frac{435}{128} m \frac{a}{a_{,}} e_{,}^2 \cos(\tau - 2\xi_{,}) - \frac{255}{128} m \frac{a}{a_{,}} e_{,}^2 \cos(\tau + 2\xi_{,})$$

[112] [113]

$$+\frac{45}{32} m \frac{a}{a_{,}} \gamma^2 \cos(\tau - 2\eta) + \left\{\frac{25}{64} m^2 - \frac{115}{128} m^3 {}^{*}\right\} \frac{a}{a_{,}} \cos 3\tau$$

[114] [116]

$$-\frac{475}{64} m^2 \frac{a}{a_{,}} e \cos(3\tau - \xi) + \frac{15}{16} m^2 \frac{a}{a_{,}} e \cos(3\tau + \xi)$$

[117] [118]

$$+\frac{125}{64} m^2 \frac{a}{a_{,}} e_{,} \cos(3\tau - \xi_{,}) + \frac{35}{16} m^2 \frac{a}{a_{,}} e_{,} \cos(3\tau + \xi_{,})$$

[119] [120]

---

[*] m^2 in the original, supposed to be an error of press.

Expression of M. Plana for the reciprocal of the radius vector, vol. i. p. 664.

$$-\frac{175}{64}m\frac{a}{a_{,}}e^2\cos(3\tau-2\xi)+\frac{75}{16}m\frac{a}{a_{,}}ee_{,}\cos(3\tau-\xi+\xi_{,})$$

[121] [125]

$$-\frac{25}{64}m\frac{a}{a_{,}}\gamma^2\cos(3\tau-2\eta)+\left\{\frac{7}{8}m^4+\frac{2737}{480}m^5-\frac{45}{64}m^3e^2\right\}\cos 4\tau$$

[129] [131]

$$+\left\{\frac{495}{128}m^3+\frac{13725}{512}m^4+\frac{6075}{512}m^2e^2\right\}e\cos(4\tau-\xi)$$

[132]

$$+\frac{805}{256}m^4e\cos(4\tau+\xi)+\frac{49}{8}m^4e_{,}\cos(4\tau-\xi_{,})$$

[133] [134]

$$-\frac{7}{8}m^4e_{,}\cos(4\tau+\xi_{,})+\left\{\frac{225}{64}m^2+\frac{3645}{128}m^3\right\}e^2\cos(4\tau-2\xi)$$

[135] [136]

$$+\frac{5775}{256}m^3ee_{,}\cos(4\tau-\xi-\xi_{,})-\frac{1485}{256}m^3ee_{,}\cos(4\tau-\xi+\xi_{,})$$

[138] [140]

$$+\frac{3}{8}m^3\gamma^2\cos(4\tau-2\eta)+\frac{625}{384}e^5\cos 5\xi$$

[144]

$$-\frac{5}{32}e^3\gamma^2\cos(3\xi-2\eta)-\frac{65}{32}e^2e_{,}\frac{a}{a_{,}}\cos(\tau-2\xi+\xi_{,})$$

$$+\frac{15}{32}e^2e_{,}\frac{a}{a_{,}}\cos(\tau+2\xi+\xi_{,})-\frac{5}{12}e_{,}\gamma^2\frac{a}{a_{,}}\cos(\tau+\xi_{,}-2\eta)$$

$$+\frac{675}{512}e^3m^3\cos(4\tau-3\xi)+\frac{45}{128}e\gamma^2m^2\cos(4\tau-\xi-2\eta)$$

$$-\frac{225}{32}e^2e_{,}m^2\cos(4\tau-2\xi+\xi_{,})+\frac{525}{32}e^2e_{,}m^2\cos(4\tau-2\xi-\xi_{,})$$

Expression of M. Plana for the reciprocal of the radius vector, vol. i. p. 664.

$$-\frac{49}{64} m\, e\, e_\prime \gamma^2 \cos(2\tau + \xi - \xi_\prime - 2\eta)$$

[93]

$$+\frac{51}{32} m\, e_\prime^2 \gamma^2 \cos(2\tau - 2\xi_\prime - 2\eta)$$

[97]

$$+\left\{-\frac{15}{16}m - \frac{81}{16}m^2 - \frac{5857}{256}m^3 + \frac{165}{64}m\gamma^2 - \frac{105}{32}m e^2 - \frac{15}{16}m e_\prime^2\right\}\frac{a}{a_\prime}$$

[101

$$+\frac{45}{16}m^2\frac{a}{a_\prime}e\cos(\tau - \xi) + \left\{-\frac{15}{8}m - \frac{177}{16}m^2\right\}\frac{a}{a_\prime}e\cos(\tau + \xi)$$

[102] [103]

$$+\left\{\frac{15}{16}m - \frac{641}{32}m^2\right\}\frac{a}{a_\prime}e_\prime\cos(\tau - \xi_\prime)$$

[104]

$$+\left\{\frac{5}{4} - \frac{45}{8}m + \frac{6673}{192}m^2 + \frac{25}{4}e^2 - \frac{5}{8}\gamma^2 + \frac{5}{4}e_\prime^2\right\}\frac{a}{a_\prime}e_\prime\cos(\tau + \xi_\prime)$$

[105]

$$+\frac{435}{128}m\frac{a}{a_\prime}e^2\cos(\tau - 2\xi) - \frac{405}{128}m\frac{a}{a_\prime}e^2\cos(\tau + 2\xi)$$

[106] [107]

$$+\left\{\frac{5}{2} - \frac{45}{4}m\right\}\frac{a}{a_\prime}e\,e_\prime\cos(\tau + \xi + \xi_\prime) + \frac{15}{8}m\frac{a}{a_\prime}e\,e_\prime\cos(\tau + \xi - \xi_\prime)$$

[109] [111]

$$+\frac{435}{128}m\frac{a}{a_\prime}e_\prime^2\cos(\tau - 2\xi_\prime) - \frac{255}{128}m\frac{a}{a_\prime}e_\prime^2\cos(\tau + 2\xi_\prime)$$

[112] [113]

$$+\frac{45}{32}m\frac{a}{a_\prime}\gamma^2\cos(\tau - 2\eta) + \left\{\frac{25}{64}m^2 - \frac{115}{128}m^3 {}^*\right\}\frac{a}{a_\prime}\cos 3\tau$$

[114] [116]

$$-\frac{475}{64}m^2\frac{a}{a_\prime}e\cos(3\tau - \xi) + \frac{15}{16}m^2\frac{a}{a_\prime}e\cos(3\tau + \xi)$$

[117] [118]

$$+\frac{125}{64}m^2\frac{a}{a_\prime}e_\prime\cos(3\tau - \xi_\prime) + \frac{35}{16}m^2\frac{a}{a_\prime}e_\prime\cos(3\tau + \xi_\prime)$$

[119] [120]

---

* m^2 in the original, supposed to be an error of press.

$$+\left\{\begin{array}{l}\left(2 + \frac{3}{2}m^2 - \frac{75}{64}m^3 - \frac{6559}{256}m^4 - \frac{48}{3}\right. \\ - e^2\left(\frac{1}{4} + 17\,m^2 + \frac{3195}{32}m^3 + \frac{4635}{76}\right. \\ - \gamma^2\left(\frac{1}{2} + \frac{63}{32}m^2 - \frac{1467}{256}m^3 - \frac{22857}{512}\right. \\ - e_,^2\left(\frac{45}{4}m^2 + \frac{6455}{64}m^3 + \frac{281095}{512}m^4\right) \\ \qquad\qquad - \frac{147}{256}m^4\Big) \\ - \gamma^4\left(\frac{3}{8} - \frac{135}{256}m - \frac{3749}{2048}m^2\right) + e \\ + e^2\gamma^2\left(\frac{23}{16} - \frac{405}{128}m + \frac{16029}{512}m^2\right) - \\ + \frac{5921}{161280}e^6 - \frac{75}{16}e_,^2\frac{a^2}{a_,^2} - \frac{135}{32}m^2\frac{a^2}{a_,^2} \\ + \frac{5171}{120}m^2\gamma^2 e_,^2 - \frac{458}{128}m^2\gamma^2 E_,^2 - \\ + \left\{\begin{array}{l}\frac{21}{4}m^2 + \frac{165}{32}m^3 - \frac{7167}{256}m^4 + \frac{7}{1} \\ + \frac{501}{128}m^2\gamma^2 + \frac{4107}{256}m^2 e^2 + \frac{1}{1}\end{array}\right.\end{array}\right.$$

Expression of M. Plana for the true longitude, vol. i. p. 574.

$$+\left\{\begin{array}{l}+\left\{\begin{array}{l}\frac{15}{4}m + \frac{263}{16}m^2 + \frac{50377}{768}m^3 + \frac{1}{} \\ + \frac{3496767841}{2654208}m^6 + \frac{918696205445}{1274019840}\end{array}\right. \\ - e^2\left(\frac{369}{64}m^2 + \frac{827639}{18432}m^3 + \frac{144286}{2457}\right. \\ - \gamma^2\left(\frac{39}{16}m + \frac{981}{64}m^2 + \frac{74095}{1024}m^3\right. \\ - e_,^2\left(\frac{75}{8}m + \frac{575}{16}m^2 - \frac{326681}{1536}m^3\right. \\ + \frac{3}{4}m\gamma^4 + \frac{105}{32}m\frac{a^2}{a_,^2} + \frac{195}{64}m\,e_,^4 \\ + \frac{1545}{256}m\,e_,^2\gamma^2 - \frac{99}{4}m^4 E_,^2 + \left(\frac{13}{8}\right.\end{array}\right.$$

C

Expression of M. Plana for the true longitude, vol. i. p. 574.

$$+\left\{\begin{array}{l}\left(\frac{17}{8}m\right. \\ + e^2\Big(\\ - \gamma^2 \\ - e_,^2\Big(\\ + \frac{451}{128}\end{array}\right\} e_,\sin(2\tau + \xi_,)$$

[7]

$$+\left\{\begin{array}{l}\Big(-3 \\ -e^2\Big(\\ + \gamma^2\,)\end{array}\right\} e\sin 2\xi$$

[8]

$$+\left\{\begin{array}{l}\left\{\frac{77}{16}\right. \\ \qquad -2\xi) \\ + e^2\Big\{ \\ -\gamma^2\Big(+2\xi) \\ - \frac{161}{16} \\ \qquad -\xi_,)\end{array}\right.$$

$$\lambda' = \mathrm{n}\,t + \varepsilon - \int \zeta\, \mathrm{d}\lambda' +$$

$$+\left\{\begin{array}{l}
\left\{\begin{array}{l}\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4 + \frac{2855}{108}m^5 + \frac{8304449}{165888}m^6 \\ + \frac{102859909}{1244160}m^7 + \frac{5490485501}{37324800}m^8\end{array}\right\} \\
+ e^2\left(\frac{75}{16}m + \frac{1101}{64}m^2 + \frac{71471}{1024}m^3 + \frac{2673553}{12288}m^4 + \frac{144641579}{294912}m^5\right) \\
- \gamma^2\left(\frac{3}{16}m + \frac{47}{64}m^2 + \frac{5149}{3072}m^3 + \frac{91745}{36864}m^4 + \frac{881273}{884736}m^5\right) \\
- e_{,}^2\left(\frac{55}{16}m^2 + \frac{697}{24}m^3 + \frac{16543}{144}m^4 + \frac{70043}{768}m^5\right) \\
- e^4\left(\frac{45}{32}m + \frac{199}{12}m^2 + \frac{904829}{9216}m^3\right) + \gamma^4\left(\frac{101}{128}m^2 + \frac{4529}{3072}m^3\right) \\
+ e_{,}^4\left(\frac{143}{128}m^2 - \frac{427}{48}m^3\right) + \frac{a^2}{a_{,}^2}\left(\frac{1925}{1024}m^2 + \frac{4415}{2048}m^3\right) \\
- e^2\gamma^2\left(\frac{69}{16}m + \frac{1431}{64}m^2 - \frac{85115}{2048}m^3\right) \\
- e^2 e_{,}^2\left(\frac{375}{32}m + \frac{11839}{192}m^2 + \frac{761227}{6144}m^3\right) \\
+ e_{,}^2\gamma^2\left(\frac{15}{32}m + \frac{269}{64}m^2 + \frac{76277}{2048}m^3\right) + \frac{45}{64}m\,e_{,}^2\gamma^4 + \frac{135}{16}m^3 e^2 E_{,}^2 \\
- \frac{39}{256}m\gamma^2 e_{,}^4 + \frac{975}{256}m\,e^2 e_{,}^4 + \frac{345}{32}m\,e^2 e_{,}^2\gamma^2 + \frac{2127}{1024}m\,e^2\gamma^4 \\
\qquad\qquad + \frac{9845}{1024}m\,e^4\gamma^2 \\
- \frac{45}{128}m\gamma^2\frac{a^2}{a_{,}^2} + \frac{225}{64}m\,e^4 e_{,}^2 + \frac{525}{128}m\,e^2\frac{a^2}{a_{,}^2} + \frac{375}{128}m\,e_{,}^2\frac{a^2}{a_{,}^2} \\
\qquad\qquad + \frac{5}{64}m\gamma^6 - \frac{19199}{768}m\,e^6 \\
+ \left\{\frac{33}{8}m^4 + \frac{175}{16}m^5 - \frac{27}{32}m^3\gamma^2 + \frac{1305}{52}m^3 e^2\right\}(e_{,}^2 - E_{,}^2)
\end{array}\right\}\sin 2 \quad [1]$$

Expression of M. Plana for the true longitude, vol. i. p. 574.

$$\left.\begin{array}{l}\left\{\begin{array}{l}-\frac{11}{16}m^2-\frac{257}{48}m^3-\frac{7337}{576}m^4+\frac{124223}{3456}m^5\\ +\frac{178759285}{331776}m^6+\frac{33403086163}{9953280}m^7\end{array}\right\}\\ -e^2\left\{\frac{75}{16}m+\frac{1113}{64}m^2-\frac{63385}{1024}m^3-\frac{14201441}{4096}m+\frac{2764481281}{294912}m^5\right\}\\ +\gamma^2\left\{\frac{3}{16}m+\frac{73}{64}m^2+\frac{24397}{3072}m^3\right\}+e_{,}^2\left(\frac{11}{128}m^2-\frac{1921}{384}m^3\right)\\ +\frac{69}{16}m\,e^2\gamma^2-\frac{3}{128}m\,e_{,}^2\gamma^2+\frac{75}{128}m\,e^2e_{,}^2+\frac{105}{32}m\,e^4-\frac{375}{128}m\frac{a^2}{a_{,}^2}\end{array}\right\}e_{,}\sin(2\tau+ \qquad [7]$$

$$\left.\begin{array}{l}+\left(\frac{5}{4}+\frac{23}{16}m^2-\frac{1485}{128}m^3-\frac{5597}{64}m^4-\frac{11972905}{24576}m^5\right.\\ \qquad\qquad\left.-\frac{15388364939}{8847360}m^6\right)\\ -e^2\left(\frac{11}{24}+\frac{6221}{192}m^2\right)-\gamma^2\left(\frac{15}{16}-\frac{135}{128}m-\frac{195}{256}m^2\right)-\frac{567}{16}m^2e_{,}^2\\ +\frac{141}{32}m^2E_{,}^2+\frac{17}{192}e^4-\frac{209}{192}\gamma^4+\frac{523}{192}e^2\gamma^2+\frac{21}{2}m^2(e_{,}^2-E_{,}^2)\end{array}\right\}e\sin 2\xi \qquad [8]$$

$$\left.\begin{array}{l}\left\{\begin{array}{l}\frac{45}{16}m+\frac{53}{4}m^2+\frac{276049}{3072}m^3+\frac{8005693}{18432}m^4\\ +\frac{2166605291}{884736}m^5+\frac{941693724529}{106168320}m^6\end{array}\right\}\\ +\frac{15}{64}m\,e^2-\frac{57}{32}m\,\gamma^2-\frac{225}{32}m\,e_{,}^2\end{array}\right\}+e^2\sin(2\tau-2\xi) \qquad [9]$$

$$\left\{\frac{95}{32}m^2+\frac{913}{96}m^3+\frac{132853}{5760}m^4+\frac{515}{64}m\,e^2-\frac{39}{64}m\,\gamma^2\right\}e^2\sin(2\tau+2\xi) \qquad [10]$$

$$\left.\begin{array}{l}\left\{\begin{array}{l}-\frac{21}{4}m-\frac{717}{32}m^2-\frac{3215}{32}m^3-\frac{4173895}{6144}m^4\\ -\frac{19962409}{9216}m^5-\frac{18012013679}{1769472}m^6\end{array}\right\}\\ -\frac{51}{32}m\,e^2+\frac{147}{16}m\,\gamma^2-\frac{189}{32}m\,e_{,}^2\end{array}\right\}e\,e_{,}\sin(\xi+\xi_{,}) \qquad [11]$$

ion of
ns for
e lon-
vol. i.

$$+\left\{\begin{array}{l}\left\{\begin{array}{l}\frac{35}{4}m+\frac{1801}{32}m^2+\frac{32429}{128}m^3+\frac{1830091}{2048}m^4\\+\frac{1211197}{8192}m^5+\frac{9159943057}{589824}m^6\end{array}\right\}\\-\frac{91}{16}m\gamma^2-\frac{615}{32}me_{,}^2\end{array}\right\}ee_{,}\sin(2\tau-\xi-\xi \qquad [12]$$

$$+\left\{\begin{array}{l}-\frac{17}{16}m^2-\frac{2633}{192}m^3-\frac{280873}{4608}m^4\\-\frac{9677359}{55296}m^5+\frac{3}{8}m\gamma^2-\frac{195}{32}me^2\end{array}\right\}ee_{,}\sin(2\tau+\xi+\xi_{,}) \qquad [13]$$

$$+\left\{\begin{array}{l}\left\{\begin{array}{l}\frac{21}{4}m+\frac{1233}{32}m^2+\frac{15165}{64}m^3+\frac{8548531}{6144}m^4\\+\frac{307373383}{36864}m^5+\frac{91915773791}{1769472}m^6\end{array}\right\}\\+\frac{51}{32}me^2-\frac{147}{16}m\gamma^2+\frac{189}{32}me_{,}^2\end{array}\right\}ee_{,}\sin(\xi-\xi_{,}) \qquad [14]$$

$$+\left\{\begin{array}{l}\left\{\begin{array}{l}-\frac{15}{4}m-\frac{173}{32}m^2+\frac{49045}{384}m^3+\frac{29319517}{18432}m^4\\+\frac{2827212637}{221184}m^5+\frac{224176571881}{2654208}m^6\end{array}\right.\\+\frac{39}{16}m\gamma^2+\frac{15}{32}me_{,}^2\end{array}\right\}ee_{,}\sin(2\tau-\xi+\xi_{,} \qquad [15]$$

$$+\left\{\begin{array}{l}\frac{119}{16}m^2+\frac{8131}{64}m^3+\frac{115757}{512}m^4+\frac{5266851}{6144}m^5\\-\frac{7}{8}m\gamma^2+\frac{455}{32}me^2\end{array}\right\}ee_{,}\sin(2\tau+\xi-\xi_{,}) \qquad [16]$$

$$+\left(-\frac{9}{4}m+\frac{5751}{128}m^3+\frac{13871}{32}m^4-\frac{81}{32}me^2+\frac{81}{32}m\gamma^2-\frac{7}{4}me_{,}^2\right)e_{,}^2\mathrm{s} \qquad [1$$

$$+\left\{\begin{array}{l}\frac{187}{16}m^2+\frac{9707}{96}m^3+\frac{78625}{144}m^4\\+\frac{60865501}{27648}m^5+\frac{1275}{64}me^2-\frac{51}{64}m\gamma^2\end{array}\right\}e_{,}^2\sin(2\tau-2\xi_{,}) \qquad [18]$$

Expression of M. Plana for the true longitude, vol. i. p. 574.

$$+\left\{-\frac{33}{32}m^3+\frac{9}{64}m\gamma^2-\frac{225}{64}me^2\right\}e_{\prime}^2\sin(2\tau+2\xi_{\prime})$$

[19]

$$+\left\{\frac{13}{12}+\frac{41}{24}m^2-\frac{23}{16}\gamma^2-\frac{43}{64}e^2\right\}e^3\sin 3\xi$$

[20]

$$+\left(\frac{105}{32}m+\frac{6011}{384}m^2\right)e^3\sin(2\tau-3\xi)+\frac{1093}{64}m^2e^3\sin(2\tau+3\xi)$$

[21] [22]

$$+\left\{-\frac{105}{16}m-\frac{3669}{128}m^2\right\}e^2e_{\prime}\sin(2\xi+\xi_{\prime})$$

[23]

$$+\left\{\frac{105}{16}m+\frac{577}{16}m^2\right\}e^2e_{\prime}\sin(2\tau-2\xi-\xi_{\prime})$$

[24]

$$-\frac{95}{64}m^2e^2e_{\prime}\sin(2\tau+2\xi+\xi_{\prime})$$

[25]

$$+\left\{\begin{matrix}\frac{105}{16}m+\frac{6081}{128}m^2+\frac{139475}{512}m^3+\frac{449}{160}me^2\\ -\frac{945}{64}m\gamma^2+\frac{945}{128}me_{\prime}^2\end{matrix}\right\}\sin(2\xi-\xi_{\prime})$$

[26]

$$+\left\{-\frac{45}{16}m+\frac{193}{32}m^2\right\}e^2e_{\prime}\sin(2\tau-2\xi+\xi_{\prime})$$

[27]

$$+\frac{665}{64}m^2e^2e_{\prime}\sin(2\tau+2\xi-\xi_{\prime})$$

[28]

$$+\left\{-\frac{63}{16}m-\frac{1245}{128}m^2-\frac{10519}{256}m^3\right\}e\,e_{\prime}^2\sin(\xi+2\xi_{\prime})$$

[29]

$$+\left\{\frac{255}{16}m+\frac{17179}{128}m^2+\frac{291049}{384}m^3+\frac{219639709}{73728}m^4\right\}e\,e_{\prime}^2\sin(2\tau-\xi-2\xi_{\prime})$$

[30]

xpression of M. Plana for he true lonitude, vol. i. . 574.

$$+\left\{\begin{matrix}\frac{63}{16}m+\frac{5355}{128}m^2+\frac{43557}{128}m^3-\frac{489}{128}me^2\\ -\frac{441}{64}m\gamma^2+\frac{49}{16}me_{,}^2\end{matrix}\right\}e\,e_{,}^2\sin(\xi-2\,\xi_{,})$$

[32]

$$+\left\{\begin{matrix}-\frac{45}{16}m-\frac{6219}{128}m^2-\frac{46669}{128}m^3+15me^2\\ +\frac{117}{64}m\gamma^2-\frac{35}{8}me_{,}^2+\frac{297}{256}e^4m^{-1}\end{matrix}\right\}e\,e_{,}^2\sin(2\tau-\xi+2\,\xi$$

[33]

$$+\frac{289}{16}m^2e\,e_{,}^2\sin(2\tau+\xi-2\,\xi_{,})-\frac{53}{24}m\,e_{,}^3\sin 3\,\xi_{,}$$

[34] [35]

$$+\frac{9295}{384}m^2e_{,}^3\sin(2\tau-3\,\xi_{,})+\frac{11}{384}m^2e_{,}^3\sin(2\tau+3\,\xi_{,})$$

[36] [37]

$$+\frac{103}{96}e^4\sin 4\,\xi+\frac{25}{4}m\,e^4\sin(2\tau-4\,\xi)-\frac{273}{32}m\,e^3e_{,}\sin(3\,\xi+\xi_{,}$$

[38] [39] [41]

$$+\frac{245}{32}m\,e^3e_{,}\sin(2\tau-3\,\xi-\xi_{,})+\frac{273}{32}m\,e^3e_{,}\sin(3\,\xi-\xi_{,})$$

[42] [44]

$$-\frac{105}{32}m\,e^3e_{,}\sin(2\tau-3\,\xi+\xi_{,})-\frac{315}{64}m\,e^2e_{,}^2\sin(2\,\xi+2\,\xi_{,})$$

[45] [47]

$$+\frac{765}{64}m\,e^2e_{,}^2\sin(2\tau-2\,\xi-2\,\xi_{,})$$

[48]

$$+\left\{\frac{885}{64}m+\frac{297}{256}e^2m^{-1}\right\}e^2e_{,}^2\sin(2\tau+2\,\xi+2\,\xi_{,})$$

[49]

$$+\frac{315}{64}m\,e^2e_{,}^2\sin(2\,\xi-2\,\xi_{,})-\frac{371}{96}m\,e\,e_{,}^3\sin(\xi+3\,\xi_{,})$$

[50] [53]

$$+\frac{845}{32}m\,e\,e_{,}^3\sin(2\tau-\xi-3\,\xi_{,})+\frac{371}{96}m\,e\,e_{,}^3\sin(\xi-3\,\xi_{,})$$

[54] [56]

Expression of M. Plana for the true longitude, vol. i. p. 574.

$$-\frac{5}{32}m\,e\,e_{\prime}^{3}\sin(2\tau-\xi+3\xi_{\prime})-\frac{77}{32}m\sin 4\xi_{\prime}$$

[57] [59]

$$+\left\{\begin{matrix}\left(-\frac{1}{4}+\frac{11}{16}m^2-\frac{33}{32}m^3-\frac{897}{512}m^4\right)+\frac{1}{8}\gamma^2\\ -e^2\left(\frac{9}{16}-\frac{675}{128}m\right)\end{matrix}\right\}\gamma^2\sin 2\eta$$

[62]

$$+\left\{\begin{matrix}\frac{9}{16}m-\frac{11}{8}m^2-\frac{2939}{3072}m^3-\frac{168653}{18432}m^4\\ -\frac{33}{64}m\gamma^2-\frac{75}{32}m\,e^2-\frac{45}{32}m\,e_{\prime}^{2}\end{matrix}\right\}\gamma^2\sin(2\tau-2\eta)$$

[63]

$$+\left\{-\frac{11}{32}m^2+\frac{3}{64}m\gamma^2-\frac{35}{64}m\,e^2\right\}\gamma^2\sin(2\tau+2\eta)$$

[64]

$$+\left\{\begin{matrix}\left(\frac{3}{4}-\frac{135}{32}m-\frac{1077}{256}m^2-\frac{21387}{2048}m^3\right)\\ +\gamma^2\left(\frac{3}{8}-\frac{405}{128}m\right)+e^2\left(-\frac{61}{32}+\frac{2025}{128}m\right)\end{matrix}\right\}e\gamma^2\sin(\xi-2\eta)$$

[65]

$$+\left\{-\frac{1}{2}+\frac{13}{16}m^2+\frac{3}{8}\gamma^2-\frac{1}{16}e^2\right\}e\gamma^2\sin(\xi+2\eta)$$

[66]

$$+\left\{\frac{3}{8}m-\frac{61}{16}m^2\right\}e\gamma^2\sin(2\tau-\xi-2\eta)$$

[67]

$$+\left\{-\frac{15}{16}m-\frac{19}{4}m^2\right\}e\gamma^2\sin(2\tau-\xi+2\eta)$$

[68]

$$+\left\{-\frac{33}{32}m+\frac{231}{256}m^2\right\}e\gamma^2\sin(2\tau+\xi-2\eta)$$

[69]

$$-\frac{39}{32}m^2e\gamma^2\sin(2\tau+\xi+2\eta)$$

[70]

ression of Plana for true lon- ide, vol. i. 74.

$$+\left\{\begin{array}{l}\frac{3}{16}m-\frac{123}{128}m^2-\frac{4481}{512}m^3+\frac{9}{64}m\,e^2\\ -\frac{21}{32}m\,\gamma^2+\frac{27}{128}m\,e_{,}^2\end{array}\right\}e_{,}\gamma^2\sin(\xi_{,}-2\eta)$$

[71]

$$+\left\{\frac{3}{16}m+\frac{153}{128}m^2\right\}e_{,}\gamma^2\sin(\xi_{,}+2\eta)$$

[72]

$$+\left\{\frac{21}{16}m-\frac{11}{4}m^2\right\}\gamma^2 e_{,}\sin(2\tau-\xi_{,}-2\eta)$$

[73]

$$-\frac{77}{64}m^2\gamma^2 e_{,}\sin(2\tau-\xi_{,}+2\eta)$$

[74]

$$+\left\{-\frac{9}{16}m-\frac{59}{32}m^2\right\}\gamma^2 e_{,}\sin(2\tau+\xi_{,}-2\eta)$$

[75]

$$+\frac{11}{64}m^2\gamma^2 e_{,}\sin(2\tau+\xi_{,}+2\eta)$$

[76]

$$+\left\{-\frac{1}{8}-\frac{135}{64}m\right\}e^2\gamma^2\sin(2\xi-2\eta)-\frac{13}{16}e^2\gamma^2\sin(2\xi+2\eta)$$

[77] [78]

$$-\frac{15}{32}m\,e^2\gamma^2\sin(2\tau-2\xi-2\eta)-\frac{5}{2}m\,e^2\gamma^2\sin(2\tau-2\xi+2\eta)$$

[79] [80]

$$-\frac{45}{32}m\,e^2\gamma^2\sin(2\tau+2\xi-2\eta)-\frac{45}{32}m\,e\,e_{,}\gamma^2\sin(\xi+\xi_{,}-2\eta)$$

[81] [83]

$$+\frac{27}{16}m\,e\,e_{,}\gamma^2\sin(\xi+\xi_{,}+2\eta)+\frac{7}{8}m\,e\,e_{,}\gamma^2\sin(2\tau-\xi-\xi_{,}-2\eta)$$

[84] [85]

$$-\frac{35}{16}m\,e\,e_{,}\gamma^2\sin(2\tau-\xi-\xi_{,}+2\eta)$$

[86]

Expression of M. Plana for the true longitude, vol. I. p. 574.

$$+\frac{33}{32}m\,e\,e_{,}\gamma^2\sin(2\tau+\xi+\xi_{,}-2\eta)-\frac{27}{16}m\,e\,e_{,}\gamma^2\sin(\xi-\xi_{,}+2\eta)$$

[87] [90]

$$-\frac{3}{8}m\,e\,e_{,}\gamma^2\sin(2\tau-\xi+\xi_{,}-2\eta)$$

[91]

$$+\frac{15}{16}m\,e\,e_{,}\gamma^2\sin(2\tau-\xi+\xi_{,}+2\eta)$$

[92]

$$-\frac{77}{32}m\,e\,e_{,}\gamma^2\sin(2\tau+\xi-\xi_{,}-2\eta)$$

[93]

$$+\frac{9}{64}m\,e_{,}^2\gamma^2\sin(2\xi_{,}-2\eta)+\frac{9}{64}m\,e_{,}^2\gamma^2\sin(2\xi_{,}+2\eta)$$

[95] [96]

$$+\frac{153}{64}m\,\gamma^2 e_{,}^2\sin(2\tau-2\xi_{,}-2\eta)$$

[97]

$$+\left\{-\frac{27}{64}m-\frac{135}{512}e^2m^{-1}\right\}\gamma^2e_{,}^2\sin(2\tau+2\xi_{,}-2\eta)$$

[99]

$$+\left\{\begin{array}{l}\left(-\frac{15}{8}m-\frac{93}{8}m^2-\frac{6927}{128}m^3-\frac{138189}{512}m^4-\frac{1553407}{1024}m^5\right.\\ \left.-\frac{223611751}{24576}m^5\right)-\frac{105}{16}m\,e^2+\frac{165}{32}m\,\gamma^2-\frac{15}{8}m\,e_{,}^2\end{array}\right\}\frac{a}{a_{,}}\sin\tau$$

[101]

$$+\left\{-\frac{165}{32}m-\frac{7317}{256}m^2-\frac{156147}{1024}m^3-\frac{14427729}{16384}m^4\right)e\frac{a}{a_{,}}\sin(\tau-\xi)$$

[102]

$$+\left\{-\frac{75}{32}m-\frac{117}{8}m^2-\frac{70415}{1024}m^3-\frac{1966573}{6144}m^4\right\}e\frac{a}{a_{,}}\sin(\tau+\xi)$$

[103]

$$+\left\{\begin{array}{l}\frac{15}{8}m-\frac{931}{32}m^2-\frac{37669}{768}m^3-\frac{265}{32}m\,e^2\\ \qquad+\frac{135}{64}m\,\gamma^2-\frac{75}{16}m\,e_{,}^2\end{array}\right\}e_{,}\frac{a}{a_{,}}\sin(\tau-\xi_{,})$$

[104]

$$+\left\{\begin{array}{l}\left(\frac{5}{2}-\frac{45}{4}m+\frac{2223}{32}m^2-\frac{124805}{768}m^3\right)+e^2\left(\frac{15}{2}+\frac{3335}{32}m\right)\\ -\gamma^2\left(\frac{15}{8}+\frac{325}{16}m\right)+e_{,}^2\left(\frac{5}{2}-\frac{375}{8}m\right)+\frac{15}{8}e^2\gamma^2m^{-1}\end{array}\right\}e_{,}\frac{a}{a_{,}}\sin(\tau+\xi_{,})$$

[105]

Expression of M. Plana for the true longitude, vol. i. p. 574.

$$+\left\{\begin{matrix}\frac{3}{16}m - \frac{123}{128}m^2 - \frac{4481}{512}m^3 + \frac{9}{64}m e^2 \\ - \frac{21}{32}m\gamma^2 + \frac{27}{128}m e_{\prime}^2\end{matrix}\right\} e_{\prime}\gamma^2 \sin(\xi_{\prime} - 2\eta)$$

[71]

$$+\left\{\frac{3}{16}m + \frac{153}{128}m^2\right\} e_{\prime}\gamma^2 \sin(\xi_{\prime} + 2\eta)$$

[72]

$$+\left\{\frac{21}{16}m - \frac{11}{4}m^2\right\} \gamma^2 e_{\prime} \sin(2\tau - \xi_{\prime} - 2\eta)$$

[73]

$$-\frac{77}{64}m^2\gamma^2 e_{\prime} \sin(2\tau - \xi_{\prime} + 2\eta)$$

[74]

$$+\left\{-\frac{9}{16}m - \frac{59}{32}m^2\right\} \gamma^2 e_{\prime} \sin(2\tau + \xi_{\prime} - 2\eta)$$

[75]

$$+\frac{11}{64}m^2\gamma^2 e_{\prime} \sin(2\tau + \xi_{\prime} + 2\eta)$$

[76]

$$+\left\{-\frac{1}{8} - \frac{135}{64}m\right\} e^2\gamma^2 \sin(2\xi - 2\eta) - \frac{13}{16}e^2\gamma^2 \sin(2\xi + 2\eta$$

[77] [78]

$$-\frac{15}{32}m e^2\gamma^2 \sin(2\tau - 2\xi - 2\eta) - \frac{5}{2}m e^2\gamma^2 \sin($$ [illegible] $$- 2\xi$$

[79]

$$-\frac{45}{32}m e^2\gamma^2 \sin(2\tau + 2\xi - 2\eta) - \frac{45}{}$$

[81]

$$+\frac{27}{16}m e e_{\prime}\gamma^2 \sin(\xi + \xi_{\prime} + 2$$

[84]

$$-\frac{35}{16}m e e_{\prime}\gamma^2 \sin(2$$

Expression of M. Plana for the true longitude, vol. i. p. 574.

$$+\frac{33}{32} m\, e\, e_{,} \gamma^2 \sin(2\tau + \xi + \xi_{,} - 2\eta) - \frac{27}{16} m\, e\, e_{,} \gamma^2 \sin(\xi - \xi_{,} + 2\eta)$$

[87] [90]

$$-\frac{3}{8} m\, e\, e_{,} \gamma^2 \sin(2\tau - \xi + \xi_{,} - 2\eta)$$

[91]

$$+\frac{15}{16} m\, e\, e_{,} \gamma^2 \sin(2\tau - \xi + \xi_{,} + 2\eta)$$

[92]

$$-\frac{77}{32} m\, e\, e_{,} \gamma^2 \sin(2\tau + \xi - \xi_{,} - 2\eta)$$

[93]

$$+\frac{9}{64} m\, e_{,}^2 \gamma^2 \sin(2\xi_{,} - 2\eta) + \frac{9}{64} m\, e_{,}^2 \gamma^2 \sin(2\xi_{,} + 2\eta)$$

[95] [96]

$$+\frac{153}{64} m\, \gamma^2 e_{,}^2 \sin(2\tau - 2\xi_{,} - 2\eta)$$

[97]

$$+\left\{-\frac{27}{64} m - \frac{135}{512} e^2 m^{-1}\right\} \gamma^2 e_{,}^2 \sin(2\tau + 2\xi_{,} - 2\eta)$$

[99]

$$+\left\{\begin{array}{l}\left(-\frac{15}{8} m - \frac{93}{8} m^2 - \frac{6927}{128} m^3 - \frac{138189}{512} m^4 - \frac{1553407}{1024} m^5\right. \\ \left. -\frac{223611751}{24576} m^5\right) - \text{[illegible]} - \frac{165}{32} m \gamma^2 - \text{[illegible]}\end{array}\right\} \frac{a}{a_{,}} \sin\tau$$

[101]

$$+\left\{-\frac{165}{32} m - \frac{731}{\text{[illegible]}} \text{[illegible]} m^3 - \frac{14}{\text{[illegible]}} \text{[illegible]}\right\} \sin(\tau - \xi)$$

[102]

$$+\left\{-\frac{75}{32} m - \text{[illegible]} \frac{15}{24} m^2 \text{[illegible]}\right\} \text{[illegible]} \xi)$$

$$\left\{\frac{15}{8} \text{[illegible]}\right.$$

$$\left.\begin{array}{r}\text{[illegible]} m) \\ \text{[illegible]} \gamma^2 m^{-1}\end{array}\right\} e_{,} \frac{a}{a_{,}} \sin(\tau + \xi_{,})$$

[105]

on of
a for
lon-
ol. i.

$$-\frac{435}{64} m e^2 \frac{a}{a_{\prime}} \sin(\tau - 2\xi) - \frac{195}{64} m e^2 \frac{a}{a_{\prime}} \sin(\tau + 2\xi)$$

[106] [107]

$$-\frac{45}{64} m e e_{\prime} \frac{a}{a_{\prime}} \sin(\tau - \xi - \xi_{\prime}) + \left\{\frac{25}{8} - \frac{225}{16} m\right\} e e_{\prime} \frac{a}{a_{\prime}} \sin(\tau + \xi + \xi_{\prime})$$

[108] [109]

$$+\left(\frac{25}{8} - \frac{1095}{32} m + \frac{15}{4} \gamma^2 m^{-1}\right) e e_{\prime} \frac{a}{a_{\prime}} \sin(\tau - \xi + \xi_{\prime})$$

[110]

$$+\frac{75}{32} m e e_{\prime} \frac{a}{a_{\prime}} \sin(\tau + \xi - \xi_{\prime}) + \frac{435}{64} m e_{\prime}^2 \frac{a}{a_{\prime}} \sin(\tau - 2\xi_{\prime})$$

[111] [112]

$$-\frac{255}{64} m e_{\prime}^2 \frac{a}{a_{\prime}} \sin(\tau + 2\xi_{\prime}) - \frac{75}{32} m \gamma^2 \frac{a}{a_{\prime}} \sin(\tau - 2\eta)$$

[113] [114]

$$+\frac{15}{32} m \gamma^2 \frac{a}{a_{\prime}} \sin(\tau + 2\eta) + \left\{\frac{15}{32} m^2 - \frac{5}{8} m^3 - \frac{1031}{64} m^4 - \frac{666889}{6144} m^5\right\}$$

[115]

$$+\left\{-\frac{2535}{256} m^2 - \frac{81865}{1024} m^3 - \frac{7611089}{16384} m^4\right\} e \frac{a}{a_{\prime}} \sin(3\tau - \xi)$$

[117]

$$+\frac{115}{128} m^2 e \frac{a}{a_{\prime}} \sin(3\tau + \xi) + \frac{75}{32} m^2 e_{\prime} \frac{a}{a_{\prime}} \sin(3\tau - \xi_{\prime})$$

[118] [119]

$$+\frac{35}{16} m^2 e_{\prime} \frac{a}{a_{\prime}} \sin(3\tau + \xi_{\prime}) - \frac{175}{32} m e_{\prime} \frac{a}{a_{\prime}} \sin(3\tau - 2\xi)$$

[120] [121]

$$+\frac{345}{64} m e e_{\prime} \frac{a}{a_{\prime}} \sin(3\tau - \xi + \xi_{\prime}) - \frac{25}{32} m \gamma^2 \frac{a}{a_{\prime}} \sin(3\tau - 2\eta)$$

[125] [129]

$$+\left\{\frac{201}{256} m^4 + \frac{649}{120} m^5 + \frac{647623}{28800} m^6 - \frac{33}{128} m^2 \gamma^2 + \frac{1425}{128} m^2 e^2\right\} \sin 4\tau$$

[131]

$$+\left\{\begin{array}{l} \frac{255}{64} m^3 + \frac{7701}{256} m^4 + \frac{631979}{4096} m^5 + \frac{413887919}{737280} m^6 \\ + \frac{2925}{256} m^2 e^2 - \frac{45}{64} m^2 \gamma^2 \end{array}\right\} e \sin(4\tau - \xi)$$

[132]

Expression of M. Plana for the true longitude, vol. i. p. 574.

$$+\frac{309}{128}m^4 e\sin(4\tau+\xi)$$

[133]

$$+\left\{\frac{1407}{256}m^4+\frac{19981}{384}m^5+\frac{12605}{256}m^3e^2-\frac{385}{128}m^3\gamma^2\right\}e_{,}\sin(4\tau-\xi_{,})$$

[134]

$$+\left\{-\frac{201}{256}m^4-\frac{5611}{640}m^5-\frac{3195}{256}m^3e^2+\frac{99}{128}m^3\gamma^2\right\}e_{,}\sin(4\tau+\xi_{,})$$

[135]

$$+\left\{\begin{matrix}\frac{1125}{256}m^2+\frac{29055}{512}m^3+\frac{1080043}{8192}m^4+\frac{73818043}{61440}m^5\,{*}\\ +\frac{466018749241}{235929600}m^6\end{matrix}\right\}e^2\sin(4\tau-2\xi)$$

[136]

$$+\left\{\frac{2975}{128}m^3+\frac{43949}{192}m^4+\frac{103321481}{73728}m^5\right\}e_{,}\sin(4\tau-\xi-\xi_{,})$$

[138]

$$+\left\{-\frac{765}{128}m^3-\frac{10841}{256}m^4-\frac{3547567}{8192}m^5\right\}e\,e_{,}\sin(4\tau-\xi+\xi_{,})$$

[140]

$$+\left\{-\frac{9}{256}m^2+\frac{255}{512}m^3+\frac{5425}{8192}m^4\right\}\gamma^2\sin(4\tau-2\eta)$$

[144]

$$+\frac{1097}{960}e^5\sin 5\xi+\gamma\frac{D^2}{a^2}\left\{\psi-K_{(2)}\right\}\left(\frac{19}{3}m^{-1}+\frac{13}{8}m^{-1}\right)\sin(\eta-fn\tau)$$

$$+\gamma\frac{D^2}{a^2}(\psi-K_{(2)})\left(-\frac{1}{3}m^{-1}\right)\sin 2\omega\sin(\eta+fnt)$$

$$+\frac{\gamma^4}{32}\sin 4\eta+\left\{-\frac{7}{24}+\frac{405}{256}m\right\}e^3\gamma^2\sin(3\xi-2\eta)$$

$$-\frac{59}{48}e^3\gamma^2\sin(3\xi+2\eta)-\frac{3}{16}e\gamma^4\sin(\xi-4\eta)$$

$$+\frac{e\gamma^4}{8}\sin(\xi+4\eta)-\frac{3}{32}m\gamma^4\sin(2\tau-4\eta)$$

$$+\frac{105}{16}e^2e_{,}\frac{a}{a_{,}}\sin(\tau-2\xi+\xi_{,})+\frac{65}{16}e^2e_{,}\frac{a}{a_{,}}\sin(\tau+2\xi+\xi_{,})$$

---

* m^4 in original, supposed error of press.

ression of Plana for true lon- ude, vol. i. 74.

$$+\frac{5}{24}\gamma^2 e_{,}\frac{a}{a_{,}}\sin(\tau+\xi_{,}-2\eta)-\frac{5}{8}\gamma^2 e_{,}\frac{a}{a_{,}}\sin(\tau+\xi_{,}+2\eta)$$

$$+\frac{675}{256}m^2 e^3\sin(4\tau-3\xi)+\frac{99}{128}m^2 e\gamma^2\sin(4\tau-\xi-2\eta)$$

$$-\frac{9}{128}m^2 e\gamma^2\sin(4\tau+\xi-2\eta)-\frac{1125}{128}m^2 e^2 e_{,}\sin(4\tau-2\xi+\xi_{,})$$

$$+\frac{2625}{128}m^2 e^2 e_{,}\sin(4\tau-2\xi-\xi_{,})+\frac{9}{128}m^2\gamma^2 e_{,}\sin(4\tau+\xi_{,}-2\eta)$$

$$-\frac{21}{128}m^2\gamma^2 e_{,}\sin(4\tau-\xi_{,}-2\eta)$$

ression for latitude, rding to Plana, i. p. 704.

$$\tan^{-1}s=$$

$$+\left\{\begin{array}{l}1-\frac{3}{8}\gamma^2-e^2+\frac{33}{128}m^3+\frac{241}{512}m^4+\frac{\gamma^4}{4}+\frac{5}{64}e^4\\ +\frac{7}{32}e^2\gamma^2+\frac{5}{128}m^2\gamma^2+\frac{27}{8}m^2 e_{,}^2-\frac{31}{512}m^2 e^2+\frac{69}{256}m^3\gamma^2\\ -\frac{82495}{24576}m^5-\frac{7977}{256}m^3 e^2+\frac{945}{512}m e^4-\frac{405}{256}m e^2\gamma^2-\frac{113}{128}m^3 e_{,}^2\end{array}\right\}\gamma\ \mathrm{si}$$

[14

$$+\left\{\begin{array}{l}\frac{3}{8}m+\frac{25}{32}m^2+\frac{2957}{1536}m^3-\frac{15}{16}m e_{,}^2+\frac{27}{32}m e^2+\frac{86485}{18432}m^4\\ -\frac{125}{128}m^2\gamma^2-\frac{199}{32}m^2 e_{,}^2+\frac{423}{128}m^2 e^2+\frac{4469761}{442368}m^5\\ +\frac{21363}{2048}m^3 e^2-\frac{7447}{6144}m^3\gamma^2-\frac{49555}{3072}m^3 e_{,}^2-\frac{15}{64}m e_{,}^2\gamma^2\\ -\frac{45}{8}m e^2 e_{,}^2-\frac{457}{128}m e^2\gamma^2+\frac{39}{128}m e_{,}^2-\frac{111}{512}m\gamma^4+\frac{45}{64}m\frac{a^2}{a_{,}^2}\\ -\frac{1545}{512}m e^4+\frac{9}{8}m^3(e_{,}^2-E_{,}^2)\end{array}\right\}+\sin(2\tau$$

[147]

Expression f
the latitude,
according to
M. Plana,
vol. i. p. 704.

$$+\left\{\begin{array}{l}\frac{11}{16}m^2+\frac{59}{24}m^3-\frac{3}{64}m\gamma^2+\frac{135}{32}me^2-\frac{91}{256}m^2\gamma^2+\frac{7063}{1152}m^4\\ +\frac{1929}{128}m^2e^2-\frac{55}{32}m^2e_{\prime}^2+\frac{356125}{27648}m^5-\frac{449}{24}m^3e_{\prime}^2\\ +\frac{134555}{2048}m^3e^2-\frac{22973}{12288}m^3\gamma^2+\frac{15}{128}me_{\prime}^2\gamma^2-\frac{165}{32}me^2\gamma^2\\ -\frac{225}{16}me^2e_{\prime}^2-\frac{1845}{256}m^4+\frac{3}{64}m\gamma^4\end{array}\right\}\gamma\sin(2\tau+\eta \qquad [148]$$

$$+\left\{\begin{array}{l}1-\frac{141}{64}m^2-\frac{5}{8}e^2-\frac{99}{16}m^3+\frac{135}{64}me^2-\frac{135}{64}m\gamma^2\\ -\frac{46579}{2048}m^4-\frac{79}{32}m^2e^2+\frac{9}{4}m^2E_{\prime}^2-\frac{515}{64}m^2e_{\prime}^2-\frac{649}{256}m^2\gamma^2\\ -\frac{5}{8}e^2\gamma^2+\frac{5}{32}e^4+\frac{1}{16}\gamma^4-\frac{779731}{12288}m^5+\frac{15}{4}m^2(e_{\prime}^2-E_{\prime}^2)\end{array}\right\}\gamma\sin(\xi-\eta) \qquad [149]$$

$$+\left\{\begin{array}{l}1+\frac{m^2}{2}-\frac{5}{8}\gamma^2-\frac{5}{4}e^2-\frac{105}{64}m^3-\frac{5479}{256}m^4\\ \qquad -\frac{5693}{256}m^2e^2+\frac{9}{4}m^2E_{\prime}^2\\ -\frac{255}{32}m^2e_{\prime}^2-\frac{593}{512}m^2\gamma^2+\frac{31}{32}e^2\gamma^2+\frac{7}{32}\gamma^4\\ \qquad +\frac{35}{24}e^4-\frac{1356205}{12288}m^5\\ +\frac{15}{4}m^2(e_{\prime}^2-E_{\prime}^2)\end{array}\right\}e\gamma\sin(\xi+\eta) \qquad [150]$$

$$+\left\{\begin{array}{l}\frac{3}{2}m+\frac{105}{16}m^2+\frac{3825}{128}m^3-\frac{3}{4}me^2-\frac{15}{4}me_{\prime}^2\\ \qquad -\frac{87}{64}m\gamma^2+\frac{102835}{1024}m^4\\ -\frac{111}{32}m^2e_{\prime}^2-\frac{543}{64}m^2e^2-\frac{2301}{256}m^2\gamma^2\end{array}\right\}e\gamma\sin(2\tau-\xi-\eta) \qquad [151]$$

$$+\left\{\begin{array}{l}\frac{15}{8}m+\frac{241}{32}m^2+\frac{45881}{1536}m^3-\frac{75}{16}me_{\prime}^2\\ \qquad -\frac{165}{64}me^2-\frac{123}{64}m\gamma^2\\ +\frac{1806823}{18432}m^4-\frac{7479}{512}m^2e^2-\frac{205}{8}m^2e_{\prime}^2-\frac{5285}{512}m^2\gamma^2\end{array}\right\}e\gamma\sin(2\tau-\xi+\eta) \qquad [152]$$

sion for
itude,
ing to
na,
p. 704.

$$+\left\{\begin{array}{l} \frac{3}{8}m + \frac{23}{32}m^2 + \frac{5587}{1536}m^3 - \frac{15}{16}m\,e_{\prime}^2 \\ \qquad + \frac{363}{64}m\,e^2 - \frac{39}{64}m\,\gamma^2 \\ -\frac{3547}{18432}m^4 - \frac{1919}{512}m^2e^2 - \frac{137}{512}m^2\gamma^2 - \frac{179}{32}m^2e_{\prime}^2 \end{array}\right\} e\,\gamma\sin(2\tau + \xi -$$

[153]

$$+\left\{\begin{array}{l} \frac{7}{4}m^2 + \frac{287}{48}m^3 + \frac{15}{2}m\,e^2 - \frac{27}{64}m\,\gamma^2 \\ \qquad + \frac{19907}{1152}m^4 + \frac{397}{16}m^2e^2 \\ -\frac{35}{8}m^2e_{\prime}^2 - \frac{451}{256}m^2\gamma^2 \end{array}\right\} e\,\gamma\sin(2\tau + \xi + \eta)$$

[154]

$$+\left\{\begin{array}{l} -\frac{3}{8}m - \frac{9}{64}m^2 + \frac{1107}{64}m^3 + \frac{15}{16}m\,e^2 + \frac{81}{64}m\,\gamma^2 - \frac{27}{64}m\,e_{\prime}^2 \\ + \frac{537375}{4096}m^4 \end{array}\right\} e_{\prime}\gamma\sin$$

[

$$+\left\{\begin{array}{l} -\frac{3}{8}m - \frac{69}{64}m^2 + \frac{2369}{128}m^3 + \frac{15}{16}m\,e^2 + \frac{81}{64}m\,\gamma^2 \\ -\frac{27}{64}m\,e_{\prime}^2 + \frac{1736297}{12288}m^4 \end{array}\right\} e_{\prime}\gamma\sin(\xi_{\prime} + \eta)$$

[156]

$$+\left\{\begin{array}{l} \frac{7}{8}m + \frac{255}{64}m^2 + \frac{3509}{256}m^3 + \frac{63}{32}m\,e^2 \\ -\frac{123}{64}m\,e_{\prime}^2 + \frac{25493}{512}m^4 + \frac{2031}{256}m^2e^2 - \frac{2325}{256}m^2e_{\prime}^2 \\ \qquad -\frac{1045}{256}m^2\gamma^2 \end{array}\right\} e_{\prime}\gamma\sin(2\tau -$$

[15

$$+\left\{\begin{array}{l} \frac{77}{32}m^2 + \frac{1949}{128}m^3 - \frac{7}{64}m\,\gamma^2 + \frac{315}{32}m\,e^2 + \frac{64091}{1024}m^4 \\ -\frac{1353}{256}m^2e_{\prime}^2 - \frac{1267}{512}m^2\gamma^2 + \frac{16383}{256}m^2e^2 \end{array}\right\} e_{\prime}\gamma\sin(2\tau -$$

[1

$$+\left\{\begin{array}{l} -\frac{3}{8}m - \frac{115}{64}m^2 - \frac{2083}{768}m^3 - \frac{27}{32}m\,e^2 + \frac{3}{64}m\,e_{\prime}^2 \\ + \frac{24337}{4608}m^4 + \frac{263}{256}m^2e_{\prime}^2 + \frac{119}{256}m^2\gamma^2 - \frac{823}{256}m^2e^2 \end{array}\right\} e_{\prime}\gamma\sin(2\tau +$$

[15

Expression for the latitude, according to M. Plana, vol. i. p. 704.

$$+\left\{\begin{array}{l} -\frac{11}{32}m^2-\frac{1127}{384}m^3+\frac{3}{64}m\gamma^2-\frac{135}{32}me^2 \\ \qquad -\frac{69271}{9216}m^4+\frac{11}{256}m^2e_{,}^2 \\ +\frac{463}{512}m^2\gamma^2-\frac{4359}{256}m^2e^2 \end{array}\right\}e_{,}\gamma\sin(2\tau+\xi_{,}+\eta)$$

[160]

$$+\left\{\frac{3}{4}-\frac{135}{64}m-\frac{1449}{512}m^2-\frac{77}{96}e^2+\frac{19}{32}\gamma^2\right\}e^2\gamma\sin(2\xi-\eta)$$

[161]

$$+\left\{\frac{9}{8}+\frac{37}{32}m^2-\frac{27}{16}e^2-\frac{17}{16}\gamma^2\right\}e^2\gamma\sin(2\xi+\eta)$$

[162]

$$+\left\{\begin{array}{l} \frac{147}{64}m+\frac{3173}{256}m^2+\frac{435649}{6144}m^3-\frac{825}{256}m\gamma^2 \\ \qquad -\frac{97}{256}me^2-\frac{1035}{128}me_{,}^2 \end{array}\right\}e^2\gamma\sin(2\tau-2\xi-\eta)$$

[163]

$$+\left\{\begin{array}{l} -\frac{15}{64}m-\frac{1555}{512}m^2+\frac{3473}{4096}m^3+\frac{19}{64}m\gamma^2 \\ \qquad +\frac{495}{256}me^2+\frac{75}{128}me_{,}^2 \end{array}\right\}e^2\gamma\sin(2\tau-2\xi+\eta)$$

[164]

$$+\left\{\begin{array}{l} \frac{27}{64}m+\frac{555}{256}m^2-\frac{8901}{1024}m^3-\frac{99}{64}m\gamma^2 \\ \qquad +\frac{513}{32}me^2-\frac{135}{128}me_{,}^2 \end{array}\right\}e^2\gamma\sin(2\tau+2\xi-\eta)$$

[165]

$$+\left\{\frac{425}{128}m^2-\frac{315}{16}m^3+\frac{12125}{256}me^2+\frac{641}{512}m\gamma^2\right\}e^2\gamma\sin(2\tau+2\xi+\eta)$$

[166]

$$+\left\{-\frac{9}{4}m-\frac{123}{8}m^2-\frac{18069}{250}m^3\right\}ee_{,}\gamma\sin(\xi+\xi_{,}-\eta)$$

[167]

$$+\left\{-3m-\frac{405}{32}m^2-\frac{1109}{32}m^3\right\}ee_{,}\gamma\sin(\xi+\xi_{,}+\eta)$$

[168]

$$+\left\{\begin{array}{l} \frac{7}{2}m+\frac{171}{8}m^2+\frac{19031}{128}m^3 \\ -\frac{123}{16}me_{,}^2-\frac{203}{64}m\gamma^2-\frac{63}{16}me^2 \end{array}\right\}ee_{,}\gamma\sin(2\tau-\xi-\xi_{,}-\eta)$$

[169]

xpression for
e latitude,
cording to
. Plana,
l. i. p. 704.

$$+\left\{\begin{array}{c}\frac{35}{8}m+\frac{423}{16}m^2+\frac{63571}{572}m^3-\frac{615}{64}m\,e_{\prime}^2\\ -\frac{273}{64}m\,\gamma^2-\frac{105}{16}m\,e^2\end{array}\right\}e\,e_{\prime}\gamma\sin(2\tau-\xi-\xi_{\prime}+\eta$$

[170]

$$+\left\{\frac{1}{2}m-\frac{11}{4}m^2\right\}e\,e_{\prime}\gamma\sin(2\tau+\xi+\xi_{\prime}-\eta)$$

[171]

$$-\frac{7}{8}m^2\,e\,e_{\prime}\gamma\sin(2\tau+\xi+\xi_{\prime}+\eta)$$

[172]

$$+\left\{\frac{9}{4}m+\frac{111}{8}m^2+\frac{34677}{256}m^3\right\}e\,e_{\prime}\gamma\sin(\xi-\xi_{\prime}-\eta)$$

[173]

$$+\left\{3\,m+\frac{609}{32}m^2+\frac{1545}{16}m^3\right\}e\,e_{\prime}\gamma\sin(\xi-\xi_{\prime}+\eta)$$

[174]

$$+\left\{-\frac{3}{2}m-\frac{15}{64}m^2\right\}e\,e_{\prime}\gamma\sin(2\tau-\xi+\xi_{\prime}-\eta)$$

[175]

$$+\left\{-\frac{15}{8}m-\frac{49}{16}m^2\right\}e\,e_{\prime}\gamma\sin(2\tau-\xi+\xi_{\prime}+\eta)$$

[176]

$$+\left\{\frac{7}{8}m+\frac{19}{4}m^2\right\}e\,e_{\prime}\gamma\sin(2\tau+\xi-\xi_{\prime}-\eta)$$

[177]

$$+\frac{49}{8}m^2\,e\,e_{\prime}\gamma\sin(2\tau+\xi-\xi_{\prime}+\eta)$$

[178]

$$+\left\{-\frac{9}{32}m+\frac{45}{256}m^2\right\}e_{\prime}^2\gamma\sin(2\,\xi_{\prime}-\eta)$$

[179]

$$+\left\{-\frac{9}{32}m-\frac{309}{256}m^2\right\}e_{\prime}^2\gamma\sin(2\,\xi_{\prime}+\eta)$$

[180]

$$+\left\{\frac{51}{32}m+\frac{2729}{256}m^2\right\}e_{\prime}^2\gamma\sin(2\tau-2\,\xi_{\prime}-\eta)$$

[181]

$$+\frac{187}{32}m^2\,e_{\prime}^2\gamma\sin(2\tau-2\,\xi_{\prime}+\eta)$$

[182]

Expression for the latitude, according to M. Plana, vol. i. p. 704.

$$+\left\{-\frac{9}{32}m-\frac{57}{256}m^2\right\}e_{,}^{2}\gamma\sin(2\tau+2\xi_{,}-\eta)$$
[183]

$$+\left\{\begin{array}{c}-\frac{15}{16}m-\frac{411}{128}m^2-\frac{13743}{512}m^3-\frac{15}{16}me^2\\ +\frac{195}{128}m\gamma^2-\frac{15}{16}me_{,}^{2}\end{array}\right\}\gamma\frac{a}{a_{,}}\sin(\tau-\eta)$$
[185]

$$+\left\{\begin{array}{c}-\frac{15}{16}m-\frac{83}{16}m^2-\frac{3471}{128}m^3-\frac{15}{32}me_{,}^{2}\\ -\frac{15}{4}me^2+\frac{375}{128}m\gamma^2\end{array}\right\}\gamma\frac{a}{a_{,}}\sin(\tau+\eta)$$
[186]

$$-\frac{45}{64}me\gamma\frac{a}{a_{,}}\sin(\tau-\xi-\eta)+\frac{45}{64}e\gamma\frac{a}{a_{,}}\sin(\tau-\xi+\eta)$$
[187] [188]

$$-\frac{195}{64}me\gamma\frac{a}{a_{,}}\sin(\tau+\xi-\eta)-\frac{135}{64}me\gamma\frac{a}{a_{,}}\sin(\tau+\xi+\eta)$$
[189] [190]

$$+\frac{15}{32}me_{,}\gamma\frac{a}{a_{,}}\sin(\tau-\xi_{,}-\eta)+\frac{15}{16}me_{,}\gamma\frac{a}{a_{,}}\sin(\tau-\xi_{,}+\eta)$$
[191] [192]

$$+\left\{\frac{5}{4}-\frac{45}{8}m\right\}e_{,}\gamma\frac{a}{a_{,}}\sin(\tau+\xi_{,}-\eta)$$
[193]

$$+\left\{\frac{5}{4}-\frac{45}{8}m\right\}e_{,}\gamma\frac{a}{a_{,}}\sin(\tau+\xi_{,}+\eta)$$
[194]

$$+\left\{\frac{17}{24}-\frac{135}{64}m\right\}e^3\gamma\sin(3\xi-\eta)+\frac{4}{3}e^3\gamma\sin(3\xi+\eta)$$
[195] [196]

$$+\frac{75}{16}me^3\gamma\sin(2\tau-3\xi-\eta)+\frac{15}{16}me^3\gamma\sin(2\tau-3\xi+\eta)$$
[197] [198]

$$+\frac{1}{2}me^3\gamma\sin(2\tau+3\xi-\eta)$$
[199]

Expression for the latitude, according to M. Plana, vol. i. p. 704.

$$-\frac{117}{32} m e^2 e_{,} \gamma \sin(2\xi + \xi_{,} - \eta) - \frac{405}{64} m \sin(2\xi + \xi_{,} + \eta)$$

[201] [202]

$$+\frac{343}{64} m e^2 e_{,} \gamma \sin(2\tau - 2\xi - \xi_{,} - \eta)$$

[203]

$$-\frac{35}{64} m e^2 e_{,} \gamma \sin(2\tau - 2\xi - \xi_{,} + \eta)$$

[204]

$$-\frac{27}{64} m e^2 e_{,} \gamma \sin(2\tau + 2\xi + \xi_{,} - \eta)$$

[205]

$$+\frac{117}{32} m e^2 e_{,} \gamma \sin(2\xi - \xi_{,} - \eta) + \frac{405}{64} m e^2 e_{,} \gamma \sin(2\xi - \xi_{,} + \eta$$

[207] [208]

$$-\frac{147}{64} m e^2 e_{,} \gamma \sin(2\tau - 2\xi + \xi_{,} - \eta)$$

[209]

$$+\frac{15}{64} m e^2 e_{,} \gamma \sin(2\tau - 2\xi + \xi_{,} + \eta)$$

[210]

$$+\frac{63}{64} m e^2 e_{,} \gamma \sin(2\tau + 2\xi - \xi_{,} - \eta)$$

[211]

$$-\frac{45}{32} m e e_{,}^2 \gamma \sin(2\tau - \xi + 2\xi_{,} + \eta) - \frac{27}{16} m e e_{,}^2 \gamma \sin(\xi + 2\xi_{,} -$$

[212] [213]

$$-\frac{9}{4} m e e_{,}^2 \gamma \sin(\xi + 2\xi_{,} + \eta) + \frac{51}{8} m e e_{,}^2 \gamma \sin(2\tau - \xi - 2\xi_{,} -$$

[214] [215]

$$-\frac{9}{32} m e e_{,}^2 \gamma \sin(2\tau + \xi + 2\xi_{,} - \eta) + \frac{27}{16} m \sin(\xi - 2\xi_{,} - \eta)$$

[217] [219]

$$+\frac{9}{4} m e e_{,}^2 \gamma \sin(\xi - 2\xi_{,} + \eta) - \frac{9}{8} m e e_{,}^2 \gamma \sin(2\tau - \xi + 2\xi_{,} +$$

[220] [221]

Expression for the latitude, according to M. Plana, vol. i. p. 704.

$$-\frac{45}{32}m\,e\,e_{,}^{2}\gamma\sin(2\tau-\xi+2\xi_{,}+\eta)+\frac{31}{32}m\sin(2\tau+\xi-2\xi_{,}-\eta)$$

[222] [223]

$$-\frac{53}{192}m\,e_{,}^{3}\gamma\sin(3\xi_{,}-\eta)-\frac{53}{192}m\,e_{,}^{3}\gamma\sin(3\xi_{,}+\eta)$$

[225] [226]

$$-\frac{95}{128}m^{2}\gamma\frac{a}{a_{,}}\sin(3\tau-\eta)+\frac{15}{64}m^{2}\gamma\frac{a}{a_{,}}\sin(3\tau+\eta)$$

[231] [232]

$$-\frac{25}{32}m\,e\,\gamma\frac{a}{a_{,}}\sin(3\tau-\xi-\eta)+\frac{15}{32}m\,e_{,}\gamma\sin(3\tau+\xi_{,}-\eta)$$

[233] [240]

$$+\left\{\begin{array}{l}\frac{621}{512}m^{4}-\frac{9}{512}m^{2}\gamma^{2}+\frac{405}{256}m^{2}e^{2}+\frac{127483}{122880}m^{5}\\ -\frac{165}{128}m^{3}e_{,}^{2}+\frac{177}{1024}m^{3}\gamma^{2}+\frac{3585}{512}m^{3}e^{2}\end{array}\right\}\gamma\sin(4\tau-\eta)$$

[241]

$$+\frac{161}{256}m^{4}\gamma\sin(4\tau+\eta)+\left\{\frac{45}{64}m^{2}+\frac{267}{64}m^{3}\right\}e\,\gamma\sin(4\tau-\xi-\eta)$$

[242] [243]

$$+\frac{105}{32}m^{3}e\,\gamma\sin(4\tau-\xi+\eta)+\frac{21}{32}m^{3}e\,\gamma\sin(4\tau+\xi-\eta)$$

[244] [245]

$$+\frac{385}{256}m^{3}e_{,}\gamma\sin(4\tau-\xi_{,}-\eta)-\frac{99}{256}m^{3}e_{,}\gamma\sin(4\tau+\xi_{,}-\eta)$$

[246] [247]

$$+\left\{-\frac{1}{24}+\frac{11}{32}m^{2}-\frac{33}{32}e^{2}+\frac{5}{128}\gamma^{2}\right\}\gamma^{3}\sin 3\eta$$

[251]

$$+\left\{\frac{1}{2}-\frac{155}{64}m+\frac{381}{256}m^{2}-\frac{29}{16}e^{2}-\frac{11}{128}\gamma^{2}\right\}e\,\gamma^{3}\sin(\xi-3\eta)$$

[252]

$$+\left\{-\frac{1}{8}+\frac{19}{16}m^{2}+\frac{19}{128}\gamma^{2}+\frac{9}{16}e^{2}\right\}e\,\gamma^{3}\sin(\xi+3\eta)$$

[253]

Expression for the latitude, according to M. Plana, vol. i. p. 704.

$$+ \frac{3}{64} m\, e_{,}\, \gamma^3 \sin (\xi_{,} - 3\,\eta) + \frac{3}{64} m\, e_{,}\, \gamma^3 (\xi_{,} + 3\,\eta)$$

[260] [261]

$$+ \frac{35}{64} m\, e_{,}\, \gamma^3 \sin (2\,\tau - \xi_{,} - 3\,\eta) - \frac{15}{64} m\, e_{,}\, \gamma^3 \sin (2\,\tau + \xi_{,} - 3\,\eta)$$

[262] [263]

$$+ \frac{3}{640} \gamma^5 \sin 5\,\eta - \frac{13}{64} e^2 \gamma^3 \sin (2\,\xi - 3\,\eta) - \frac{17}{64} e^2 \gamma^3 \sin (2\,\xi + 3\,\eta).$$

$$+ \frac{625}{384} e^4 \gamma \sin (4\,\xi - \eta) + \frac{99}{128} e^4 \gamma \sin (4\,\xi - \eta)$$

$$+ \frac{D^2 (K_2 - \psi)}{a^2\, m^2} \left\{ - \frac{2}{3} - \frac{1}{4} m - \frac{43}{18} m^2 + e_{,}^2 - \frac{1}{3} \gamma^2 \right\} \sin 2\,\varpi \sin f n t$$

$$- \frac{5}{16} e\, e_{,}\, \gamma \frac{a}{o_{,}} \sin (\tau - \xi + \xi_{,} + \eta) + \frac{25}{16} e\, e_{,}\, \dot{\gamma} \frac{a}{a_{,}} \sin (\tau - \xi + \xi_{,} - \eta)$$

$$+ \frac{45}{16} e\, e_{,}\, \gamma \frac{a}{a_{,}} \sin (\tau + \xi + \xi_{,} + \eta) + \frac{55}{48} e\, e_{,}\, \gamma \frac{a}{a_{,}} \sin (\tau + \xi + \xi_{,} - \eta)$$

$$+ \frac{45}{512} m^2 \gamma^3 \sin (4\,\tau - 3\,\eta) + \frac{585}{512} m^2 e^2 \gamma \sin (4\,\tau - 2\,\xi - \eta)$$

$$+ \frac{2025}{512} m^2 e^2 \gamma \sin (4\,\tau - 2\,\xi + \eta)$$

$$- \frac{45}{32} m^2\, e\, e_{,}\, \gamma \sin (4\,\tau - \xi + \xi_{,} - \eta)$$

$$+ \frac{105}{32} m^2\, e\, e_{,}\, \gamma \sin (4\,\tau - \xi - \xi_{,} - \eta)$$

Numerical values of the coefficients of the arguments in the expression for the equatorial parallax of the Moon, in terms of her mean longitude, according to MM. Damoiseau and Plana.

	Arg.	Damoiseau.	Plana.		Arg.	Damoiseau.	Plana.
0		+3420″·89	+3423″·1534	18	$2\tau-2\xi_{,}$	+″·08	+″·0522
1	2τ	+ 28·54	+ 27·5933	19	$2\tau+2\xi_{,}$		— ·0012
2	ξ	+ 186·48	+ 186·6545	20	3ξ	+ ·63	+ ·6340
3	$2\tau-\xi$	+ 34·43	+ 33·9214	21	$2\tau-3\xi$	— ·18	— ·1010
4	$2\tau+\xi$	+ 3·05	+ 3·0755	24	$2\tau-2\xi-\xi_{,}$	— ·03	+ ·0127
5	$\xi_{,}$	— ·32	— ·4284	33	$2\tau-\xi+2\xi_{,}$	— ·02	— ·0128
6	$2\tau-\xi_{,}$	+ 1·92	+ 2·1646	62	2η	+ ·03	— ·0159
7	$2\tau+\xi_{,}$	— ·26	— ·5447	63	$2\tau-2\eta$	— ·15	— ·1084
8	2ξ	+ 10·24	+ 10·2587	64	$2\tau+2\eta$	— ·03	
9	$2\tau-2\xi$	— ·41	+ ·1670	65	$\xi-2\eta$	— ·70	—1·1937
10	$2\tau+2\xi$	+ ·14	+ ·2702	67	$2\tau-\xi-2\eta$	— ·11	— ·0821
11	$\xi+\xi_{,}$	— ·92	— ·9192	69	$2\tau+\xi-2\eta$	— ·06	— ·0503
12	$2\tau-\xi-\xi_{,}$	+ 1·45	+ 1·4486	77	$2\xi-2\eta$	— ·06	
13	$2\tau+\xi+\xi_{,}$		— ·0014	101	τ	— ·98	— ·9247
14	$\xi-\xi_{,}$	+ 1·20	+ 1·0599	103	$\tau+\xi_{,}$	+ ·14	— ·1519
15	$2\tau-\xi+\xi_{,}$	— ·24	— ·3803	131	4τ	+ ·21	+ ·1365
16	$2\tau+\xi-\xi_{,}$	+ ·18	+ ·1877	132	$4\tau-\xi$	+ ·57	+ ·4985
17	$2\xi_{,}$		— ·0122	136	$4\tau-2\xi$	+ ·40	+ ·3250

M. Plana has many other minute equations, which I do not think it worth while to enumerate here.

	Arg.	Damoiseau.	Plana.		Arg.	Damoiseau.	Plana.
		″	″			″	″
93	$2\tau+\xi-\xi_{,}-2\eta$		− ·278	133	$4\tau+\xi$	− ·43	+ ·855
94	$2\tau+\xi-\xi_{,}+2\eta$			134	$4\tau-\xi_{,}$	+ ·82	+ 1·210
95	$2\xi_{,}-2\eta$		+ ·005	135	$4\tau+\xi_{,}$	− ·09	− ·201
96	$2\xi_{,}+2\eta$		+ ·005	136	$4\tau-2\xi$	+31·19	+34·518
97	$2\tau-2\xi_{,}-2\eta$		+ ·085	137	$4\tau+2\xi$	+ ·73	
98	$2\tau-2\xi_{,}+2\eta$			138	$4\tau-\xi-\xi_{,}$	+ 3·32	+ 3·837
99	$2\tau+2\xi_{,}-2\eta$		− ·020	139	$4\tau+\xi+\xi_{,}$		
100	$2\tau+2\xi_{,}+2\eta$			140	$4\tau-\xi+\xi_{,}$	− ·49	− ·922
101	τ	−122·48	−122·110	141	$4\tau+\xi-\xi_{,}$		
102	$\tau-\xi$	− 17·19	− 18·045	142	$4\tau-2\xi_{,}$		
103	$\tau+\xi$	− 8·40	− 8·237	143	$4\tau+2\xi_{,}$		
104	$\tau-\xi_{,}$	− ·48	− ·383	144	$4\tau-2\eta$	+ ·62	+ ·054
105	$\tau+\xi_{,}$	+ 17·56	+ 17·216	145	$4\tau+2\eta$		
106	$\tau-2\xi$	− 1·17	− ·796		5ξ	+ ·11	+ ·117
107	$\tau+2\xi$	− ·48	− ·357		$\eta-fnt$	− 6·8	− 6·577
108	$\tau-\xi-\xi_{,}$	− ·29	− ·025		$\eta+fnt$		+ ·343
109	$\tau+\xi+\xi_{,}$	+ 1·25	+ ·994		4η	+ ·38	+ ·424
110	$\tau-\xi+\xi_{,}$	+ 2·05	+ ·466		$3\xi-2\eta$	+ ·46	− ·048
111	$\tau+\xi-\xi_{,}$		+ ·084		$3\xi+2\eta$		− ·339
112	$\tau-2\xi_{,}$		+ ·075		$\xi-4\eta$		− ·140
113	$\tau+2\xi_{,}$		− ·044		$\xi+4\eta$		+ ·093
114	$\tau-2\eta$		− ·739		$2\tau-4\eta$		− ·095
115	$\tau+2\eta$		+ ·148		$\tau-2\xi+\xi_{,}$		+ ·172
116	3τ	+ ·01	+ ·887		$\tau+2\xi+\xi_{,}$		+ ·117
117	$3\tau-\xi$	− 3·04	− 2·950		$\tau+\xi_{,}-2\eta$		+ ·015
118	$3\tau+\xi$	+ ·33	+ ·143		$\tau+\xi_{,}+2\eta$		− ·046
119	$3\tau-\xi_{,}$	+ ·09	+ ·012		$4\tau-3\xi$	+ 1·39	+ ·502
120	$3\tau+\xi_{,}$	+ ·08	+ ·011		$4\tau-\xi-2\eta$	+ ·39	+ ·397
121	$3\tau-2\xi$		− ·640		$4\tau+\xi-2\eta$		− ·036
122	$3\tau+2\xi$				$4\tau-2\xi+\xi_{,}$	− ·50	− ·513
123	$3\tau-\xi-\xi_{,}$				$4\tau-2\xi-\xi_{,}$	+ 3·05	+ 1·197
124	$3\tau+\xi+\xi_{,}$				$4\tau+\xi_{,}-2\eta$		+ ·011
125	$3\tau-\xi+\xi_{,}$		+ ·194		$4\tau-\xi_{,}-2\eta$		− ·026
126	$8\tau+\xi-\xi_{,}$				$4\tau-4\xi$	+ ·03	
127	$3\tau-2\xi_{,}$				$6\tau-2\xi$	+ ·53	
128	$3\tau+2\xi_{,}$				♀ − ♅	− 1·10	
129	$3\tau-2\eta$		− ·246		2 (♀ − ♅)	− ·40	
130	$3\tau+2\eta$				♃ − ♅	− ·70	
131	4τ	+ 14·85	+ 14·409		2 (♃ − ♅)	+ ·20	
132	$4\tau-\xi$	+ 38·62	+ 38·001				

Numerical
of the coef
of the argu
in the exp
for the tru
gitude of th
in terms of
mean longi
according t
MM. Dam
and Plana.

Numerical values of the coefficients of the arguments in the expression for the latitude of the Moon, in terms of her mean longitude, according to M. Plana, vol. i. p. 717.

	Arg.			Arg.	
146	η	+ 18465″·538	193	$\tau+\xi_{,}-\eta$	+ ″·951
147	$2\tau-\eta$	+ 621·476	194	$\tau+\xi_{,}+\eta$	+ ·951
148	$2\tau+\eta$	+ 101·708	195	$3\xi-\eta$	+ 1·687
149	$\xi-\eta$	+ 1000·509	196	$3\xi+\eta$	+ 4·086
150	$\xi+\eta$	+ 1010·894	197	$2\tau-3\xi-\eta$	
151	$2\tau-\xi-\eta$	+ 166·615	198	$2\tau-3\xi+\eta$	
152	$2\tau-\xi+\eta$	+ 196·224	199	$2\tau+3\xi-\eta$	
153	$2\tau+\xi-\eta$	+ 35·039	200	$2\tau+3\xi+\eta$	
154	$2\tau+\xi+\eta$	+ 14·774	201	$2\xi+\xi_{,}-\eta$	− ·257
155	$\xi_{,}-\eta$	− 4·167	302	$2\xi+\xi_{,}+\eta$	− ·445
156	$\xi_{,}+\eta$	− 5·448	203	$2\tau-2\xi-\xi_{,}-\eta$	
157	$2\tau-\xi_{,}-\eta$	+ 29·535	204	$2\tau-2\xi-\xi_{,}+\eta$	
158	$2\tau-\xi_{,}+\eta$	+ 7·593	205	$2\tau+2\xi+\xi_{,}-\eta$	
159	$2\tau+\xi_{,}-\eta$	− 12·272	206	$2\tau+2\xi+\xi_{,}+\eta$	
160	$2\tau+\xi_{,}+\eta$	− 1·194	207	$2\xi-\xi_{,}-\eta$	+ ·257
161	$2\xi-\eta$	+ 32·343	208	$2\xi-\xi_{,}+\eta$	+ ·445
162	$2\xi+\eta$	+ 62·462	209	$2\tau-2\xi+\xi_{,}-\eta$	
163	$2\tau-2\xi-\eta$	+ 16·217	210	$2\tau-2\xi+\xi_{,}+\eta$	
164	$2\tau-2\xi+\eta$	− 1·883	211	$2\tau+2\xi+\xi_{,}-\eta$	
165	$2\tau+2\xi-\eta$	+ 2·377	212	$2\tau+2\xi-\xi_{,}+\eta$	
166	$2\tau+2\xi+\eta$	+ ·867	213	$\xi+2\xi_{,}-\eta$	− ·036
167	$\xi+\xi_{,}-\eta$	− 4·864	214	$\xi+2\xi_{,}+\eta$	− ·049
168	$\xi+\xi_{,}+\eta$	− 5·505	215	$2\tau-\xi-2\xi_{,}-\eta$	
169	$2\tau-\xi-\xi_{,}-\eta$	+ 8·241	216	$2\tau-\xi-2\xi_{,}+\eta$	
170	$2\tau-\xi-\xi_{,}+\eta$	+ 9·226	217	$2\tau+\xi+2\xi_{,}-\eta$	
171	$2\tau+\xi+\xi_{,}-\eta$	+ ·905	218	$2\tau+\xi+2\xi_{,}+\eta$	
172	$2\tau+\xi+\xi_{,}+\eta$	− ·084	219	$\xi-2\xi_{,}-\eta$	+ ·036
173	$\xi-\xi_{,}-\eta$	+ 5·183	220	$\xi-2\xi_{,}+\eta$	+ ·049
174	$\xi-\xi_{,}+\eta$	+ 6·360	221	$2\tau-\xi+2\xi_{,}-\eta$	
175	$2\tau-\xi+\xi_{,}-\eta$	− 1·945	222	$2\tau-\xi+2\xi_{,}+\eta$	
176	$2\tau-\xi+\xi_{,}+\eta$	− 2·692	223	$2\tau+\xi-2\xi_{,}-\eta$	
177	$2\tau+\xi-\xi_{,}-\eta$	+ 1·576	224	$2\tau+\xi-2\xi_{,}+\eta$	
178	$2\tau+\xi-\xi_{,}+\eta$	+ ·588	225	$3\xi_{,}-\eta$	− ·002
179	$2\xi_{,}-\eta$	+ ·106	226	$3\xi_{,}+\eta$	− ·002
180	$2\xi_{,}+\eta$	− ·146	227	$2\tau-3\xi_{,}-\eta$	
181	$2\tau-2\xi_{,}-\eta$	+ ·627	228	$2\tau-3\xi_{,}+\eta$	
182	$2\tau-2\xi_{,}+\eta$	+ ·172	229	$2\tau+3\xi_{,}-\eta$	
183	$2\tau+2\xi_{,}-\eta$	− ·062	230	$2\tau+3\xi_{,}+\eta$	
184	$2\tau+2\xi_{,}+\eta$		231	$3\tau-\eta$	− ·199
185	$\tau-\eta$	− 4·618	232	$3\tau+\eta$	+ ·060
186	$\tau+\eta$	− 5·016	233	$3\tau-\xi-\eta$	− ·150
187	$\tau-\xi-\eta$	− ·135	234	$3\tau-\xi+\eta$	
188	$\tau-\xi+\eta$	+ ·135	235	$3\tau+\xi-\eta$	
189	$\tau+\xi-\eta$	− ·585	236	$8\tau+\xi+\eta$	
190	$\tau+\xi+\eta$	− ·405	237	$3\tau-\xi_{,}-\eta$	
191	$\tau-\xi_{,}-\eta$	+ ·003	238	$3\tau-\xi_{,}+\eta$	
192	$\tau-\xi_{,}+\eta$	+ ·006			

ON

THE THEORY OF THE MOON,

AND ON

THE PERTURBATIONS OF THE PLANETS.

By J. W. LUBBOCK, Esq. F.R., R.A., and L.SS.

MEMBER OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES,
AND THE ACADEMY OF PALERMO.

PART II.

LONDON:
CHARLES KNIGHT, 22 LUDGATE STREET.
1836.

PRINTED BY RICHARD TAYLOR,
RED LION COURT, FLEET STREET.

PREFACE.

IN the following pages I have employed methods proposed in the former part of this work for the calculation of the Lunar inequalities, my object being partly to show by more numerous examples the facility with which (by means of equations in which the time is the independent variable,) expressions for the reciprocal of the radius vector, the longitude and the latitude of the moon may be obtained *directly*, and partly to verify some of the final results contained in the work of M. Plana, entitled *Théorie du Mouvement de la Lune.*

I employ throughout the well-known equation

$$\frac{d^2 . r^2}{2\, d t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2 \int d R + r \left(\frac{d R}{d r}\right) = 0$$

to determine the inequalities of the reciprocal of the radius vector in the following manner:

Making $\frac{\mathrm{a}}{r} = \mathrm{a} \left\{ \frac{1}{\mathrm{r}} + \delta \frac{1}{r} \right\}$

where r is the elliptic value of r or the same function of a and e as in the elliptic theory, when the perturbations are neglected, and a is connected with n the observed mean motion, by the equation $\mu = \mathrm{n}^2 \mathrm{a}^3$

$$\frac{r^2}{2} = \frac{\mathrm{r}^2}{2} - \mathrm{r}^3 \left(\delta \frac{1}{r}\right) + \frac{3}{2} \mathrm{r}^4 \left(\delta \frac{1}{r}\right)^2 - 2\, \mathrm{r}^5 \left(\delta \frac{1}{r}\right)^3 + \&c.$$

Since

$$\frac{\mathrm{r}^3}{\mathrm{a}^3} = 1 + 3 e^2 + \frac{3}{8} e^4 - 3 e \left(1 + \frac{3}{8} e^2\right) \cos \xi$$
$$+ \frac{5}{4} e^4 \cos 2 \xi + e^3 \cos 3 \xi + \frac{e^4}{8} \cos 4 \xi + \&c.$$

ξ being the *mean anomaly*,

if $\frac{r^2}{2 \mathrm{a}^2} = - \mathrm{a} \left(1 + 3 e^2 + \frac{3}{8} e^4\right) \delta \frac{1}{r} + \mathfrak{t}$

$$\frac{\mathrm{a}}{\mu}\left\{2\int \mathrm{d}R + r\frac{\mathrm{d}R}{\mathrm{d}r}\right\} = P$$

and if $\mathrm{a}\,\delta\frac{1}{r}$, $\mathfrak{r}$ and P are expressed in a series of cosines of angles of the form $i\,\mathrm{n}\,t + q$, so that Σ being the sign of summation,

$$\mathrm{a}\,\delta\frac{1}{r} = \Sigma\, E\, r_n \cos(i\,\mathrm{n}\,t + q)$$

$$\mathfrak{r} = \Sigma\, E\, \mathfrak{r}_n \cos(i\,\mathrm{n}\,t + q)$$

$$P = \Sigma\, E\, P_n \cos(i\,\mathrm{n}\,t + q)$$

E being the product of certain powers of the eccentricities and inclinations and n the *index of the argument*, (see p. 23,) then generally[*],

$$r_n\left\{i^2\left(1 + 3\,e^2 + \frac{3}{8}\,e^4\right) - 1\right\} = i^2\,\mathfrak{r}_n - P_n$$

which equation by continual approximation or methods of *false position* serves to determine all the inequalities of the parallax.

Thus, in the calculation of Arg. 3 (*Evection*), see p. 122,

$$i^2 = 1 - 4\,m + \frac{11}{2}\,m^2 + \frac{177}{16}\,m^3$$

$$\mathfrak{r}_3 = \frac{3}{2}\,m^2 + \frac{91}{16}\,m^3 + \frac{2657}{192}\,m^4$$

$$P_3 = \left\{4 + 4\,m + 8\,m^2\right\}\left\{\frac{9}{4}\,m^2 + \frac{15}{16}\,m^3 + \frac{235}{64}\,m^4\right\} - \frac{15}{4}\,m^4$$

By making continually $r_n =$ the sum of the terms previously found + an indeterminate quantity, and then equating coefficients, all combinations are avoided which do not concur in the formation of the term required, and the nature of this method is generally such that no operations are repeated which have once been performed. It is necessary, however, in some instances (as Arg. 33, p. 179, Arg. 65, p. 253, and Arg. 105, p. 284), also to leave r_n indeterminate in $\mathfrak{r}_n$, which always involves r_n implicitly.

---

[*] The exceptions are the Arguments 2, 8, 20, &c., which occur in the elliptic theory. See p. 136, p. 124, and p. 147, in which cases the equation is slightly modified by the terms which arise from the quantity

$$\frac{\mathrm{d}^2 .\, r^2}{2\,\mathrm{d}\,t^2} - \frac{\mu}{r}$$

In the former *Part*, I employed solely to determine inequalities of longitude the equation

$$d\lambda = \frac{h}{r^2}\left\{1 - \frac{1}{h}\int \frac{dR}{d\lambda'}\,dt\right\}$$

The coefficients relating to certain arguments are difficult to reach by means of this equation, when the divisor introduced by integration is a multiple of m, as in the instance of Arguments 5, 9, &c.; and this difficulty is greater when the divisor is a multiple of m^2 as in Arguments 51*, 77, and 110. Therefore, although the equation given above is generally preferable, the following is sometimes more advantageous. (See p. 218.)

$$\frac{d\lambda'}{dt} = \frac{h(1+s^2)}{2r^2} + \frac{\mu}{hr} - \frac{\mu}{2ha} - \frac{dr^2}{2h\,dt^2} - \frac{r^2}{2h(1+s^2)^2}\frac{ds^2}{dt^2}$$

$$- \frac{1}{h}\int dR - \frac{(1+s^2)}{2hr^2}\left\{\int \frac{dR}{d\lambda'}\,dt\right\}^2$$

where r denotes the distance of the body from the primary (not the curtate distance), and λ' the longitude reckoned upon the ecliptic or fixed plane.

This equation involves the double integral $\iint dR$, which is more easy to obtain than $\iint \frac{dR}{d\lambda}\,dt$ to a given power of m.

In the former *Part*, I abandoned the *direct* method of obtaining the inequalities of latitude, but upon further reflection I think that the equation

$$\frac{d^2 z}{dt^2} + \frac{\mu z}{r^3} + \frac{m_{,} z}{r_{,}^3} + \frac{3 m_{,} z r' \cos(\lambda' - \lambda)}{r_{,}^4} = 0$$

may be used advantageously to determine the inequalities of the coordinate z. The latitude of the moon is afterwards easily found from the expression

$$\text{Lat.} = \frac{z}{r} + \frac{z^3}{2 \,.\, 3\, r^3} + \frac{3 z^5}{2 \,.\, 4 \,.\, 5\, r^5} + \text{\&c.}$$

I have given numerous examples of this method of obtaining inequalities of latitude. (See p. 227.)

---

* In these cases it is useful to avail of the theory of the variation of the elliptic constants, as explained by M. Poisson.

In carrying the approximation to given powers of m and of the eccentricities, much trouble may be saved by avoiding to write combinations which give rise to quantities of a different order from those under consideration. I mention this circumstance because otherwise some of the expressions here given might appear incomplete. Thus, in p. 129, where quantities of the order m^3 only are sought resulting from the product of $\mathfrak{a} + \mathfrak{b}\,m + \mathfrak{c}\,m^2$ with $\mathfrak{A} + \mathfrak{B}\,m + \mathfrak{C}\,m^2$, the combinations $\mathfrak{a}\,\mathfrak{C}$, $\mathfrak{b}\,\mathfrak{B}$, $\mathfrak{c}\,\mathfrak{A}$ alone have to be retained, all others such as $\mathfrak{h}\,\mathfrak{C}$ multiplied by different powers of m being rejected. Great care, however, is requisite in order not to omit any of the combinations which concur in the formation of a given term, and the difficulty or rather impossibility of altogether avoiding numerical mistakes in such laborious calculations is the excuse I offer for such as may be discovered, "quilibet enim, qui omnes calculos hic expositos vel leviter perpendere voluerit, facile agnoscet, vix ullam adhuc quæstionem analyticam etiam nunc esse pertractatam, quæ tam intricatas calculi discussiones et tam prolixos calculos postulaverit."

By developing the inequalities in series after the manner of M. Plana, a difficulty, common to all methods of treating the problem, soon becomes manifest in the slow convergence of many of the expressions. In order, therefore, to obtain results which can safely be employed to form tables for finding the moon's place, it will be necessary to carry the developments of certain coefficients much beyond that limit which would otherwise suffice. Thus in the well-known instance of the quantity c − 1, the three first terms are numerically equal to ·004196, ·002942 and ·000996, the coefficients of m^2, m^3 and m^4 being $\frac{3}{2}$, $\frac{225}{32}$ and $\frac{4071}{128}$*. This divergence of the numerical coefficients appears to be chiefly confined to terms of the higher orders which form portions of the greater inequalities; thus

---

* "Le travail serait moindre, s'il était permis d'estimer la grandeur des inégalités d'apres la simple connaissance de l'ordre analytique du coefficient, alors on pourrait renfermer les règles à suivre pour cet objet dans un petit nombre de préceptes d'un application facile. Mais lorsqu'on pousse fort loin le calcul, les coefficiens numériques absolus nés des développemens sont d'une grandeur tellement différente entr'eux, qu'on pourrait être entrainé dans des erreurs graves, si l'on n'avait pas égard à leur influence."—*Plana, Théorie du Mouvement de la Lune*, vol. i. p. xiii.

$$\lambda_{12} = \frac{35}{4} m + \frac{1801}{32} m^2 + \frac{32989}{128} m^3 \qquad \text{(See p. 193.)}$$

(λ_{12} being the coefficient of $e\,e_{,}\sin(2\tau - \xi - \xi_{,})$ in the expression for the longitude); while the numerical coefficients belonging to the smaller inequalities are not generally divergent. The terms of a given order which constitute the smaller inequalities also result from fewer combinations and can be obtained with greater facility than those which belong to the greater inequalities. Without doubt, therefore, ultimately, the smaller inequalities will be determined accurately *à priori*, but it will be extremely difficult to obtain from theory the coefficients of the greater inequalities, such as those of Arguments 12, 13, 14, 15, &c.; it may prove necessary, and at all events it would be highly desirable, to obtain, empirically, with the utmost precision, the numerical values of the principal equations. It may be doubted whether any real advantage ensues from the form, different from that suggested by theory, adopted in Burchardt's Lunar Tables; and it is to be regretted that the expressions for the longitude and latitude of the moon, which have been used in these and other empirical Tables, are so extremely complicated in their nature as almost to preclude the possibility of deducing from them expressions similar to those furnished by the theory of gravitation.

The following are, I believe, the only terms of the fourth order where I have failed in arriving at results consistent with those of M. Plana.

In the reciprocal of the radius vector I find zero instead of

$$-\frac{45}{16} m^2 e^2. \quad \text{Arg. } 0.$$

In the latitude I find

$$-\frac{3}{8} m \text{ (Arg. 171) instead of } \frac{1}{2} m$$

and a term $+\frac{33}{128} m^3 \gamma \sin(4\tau - \eta)$ not given by M. Plana.

In terms of the fifth order I meet with more numerous discrepancies.

A distinct definition of the constants employed becomes indispensable in an extended view of the subject. In the elliptic theory, that is when the disturbing function is neglected,

$$\frac{a}{r} = 1 + e\left(1 - \frac{e^2}{8} + \&c.\right)\cos\xi + \&c.$$

$$\xi = \mathrm{c\,n}\,t + \varepsilon - \varpi \qquad \mathrm{c} = 1 \qquad a = \mathrm{a}$$

$$\frac{z}{\mathrm{a}} = \gamma\left(1 - \frac{e^2}{2} + \&c.\right)\sin\eta + \&c.$$

$$\eta = \mathrm{g\,n}\,t + \varepsilon - \beta \qquad \mathrm{g} = 1$$

ξ is the *mean anomaly*, and η the *mean argument of latitude*.

The quantities denoted by e and γ in this work are such that the coefficient of $\cos\xi$ in the expression for $\frac{a}{r}$ and of $\sin\eta$ in that for $\frac{z}{\mathrm{a}}$ have still the same form when the disturbing function is considered. This mode of definition is not identical with that employed by M. Plana, and reference to that circumstance is frequently necessary in order to compare our results and to ascertain whether or not they are consistent with each other. Wherever they are inconsistent, of course one or other must be inaccurate.

The nature of the problem is such that the value of any term influences many others, therefore I shall feel grateful for any assistance in ascertaining the cause of discrepancies which exist between my results and those of M. Plana. Already a few of these discrepancies have been removed by some numerical corrections with which I have been kindly favoured by M. de Pontécoulant.

When the analytical investigation shall have been concluded, and the form of the coefficients of all sensible inequalities shall have been determined, it will still be necessary to deduce from observation accurate values of the principal equations, from which to obtain with precision the numerical values of the constants m, e, and γ. The value of the quantity $e_,$ is not subject to so much uncertainty.

29 *Eaton Place*,
April, 1836.

CONTENTS.

TERMS in $\frac{\mathrm{a}\,\delta R}{\mu}$, in $\mathrm{a}^2\left(\delta \frac{1}{r}\right)^2$ and in $\frac{3\,\mathrm{r}^4}{2\,\mathrm{a}^2}\left(\delta \frac{1}{r}\right)^2$ multiplied by $m^3 e^2$ $m^3 e e_{,}$ and $m^3 e_{,}^2$.

Calculation of the following terms in $\mathrm{a}\,\delta \frac{1}{r}$,

$$\underset{[3]}{\frac{33113}{1536} m^3 e \cos(2\tau - \xi)} + \left\{\frac{12883}{256} m^3 e^2 - \frac{239}{12} m^3 e_{,}^2\right\} \underset{[1]}{\cos 2\tau}$$

$$+ \underset{[8]}{\frac{5}{3} m^2 e^2 \cos 2\xi} - \underset{[9]}{\frac{225}{16} m^3 e^2 \cos(2\tau - 2\xi)}$$

$$+ \underset{[10]}{\frac{127}{12} m^3 e^2 \cos(2\tau + 2\xi)} - \underset{[11]}{\frac{837}{64} m^2 e e_{,} \cos(\xi + \xi_{,})}$$

$$+ \underset{[12]}{\frac{1269}{64} m^3 e e_{,} \cos(2\tau - \xi - \xi_{,})} - \underset{[13]}{\frac{1687^{*}}{128} m^3 e e_{,} \cos(2\tau + \xi + \xi_{,})}$$

$$+ \underset{[14]}{\frac{1113}{64} m^3 e e_{,} \cos(\xi - \xi_{,})} - \underset{[15]}{\frac{97}{64} m^2 e e_{,} \cos(2\tau - \xi + \xi_{,})}$$

$$+ \underset{[16]}{\frac{5727}{128} m^3 e e_{,} \cos(2\tau + \xi - \xi_{,})} + \underset{[18]}{\frac{799}{12} m^3 e_{,}^2 \cos(2\tau - 2\xi_{,})}$$

$$- \underset{[19]}{\frac{3}{4} m^3 e_{,}^2 \cos(2\tau + 2\xi_{,})} + \left\{\frac{495}{128} m^3 + \frac{13725}{512} m^4\right\} \underset{[132]}{e \cos(4\tau - \xi)}$$

---

* The terms marked with an asterisk are not consistent with M. Plana's results.

$$+\frac{805}{256}m^4\,e\,\underset{[133]}{\cos(4\tau+\xi)}+\frac{105^*}{8}m^3\,e^2\,\underset{[131]}{\cos 4\tau}$$

$$+\frac{39795^*}{64}m^3\,e^2\,\underset{[136]}{\cos(4\tau-2\xi)}+\frac{5775}{256}m^3\,e\,e_{,}\,\underset{[138]}{\cos(4\tau-\xi-\xi_{,})}$$

$$-\frac{1485}{256}m^3\,e\,e_{,}\,\underset{[140]}{\cos(4\tau-\xi+\xi_{,})}$$ . . . . *page* 117 to 129.

Calculation of the following terms in the longitude,

$$\frac{1101}{64}m^2e^2\sin 2\tau+\frac{263}{16}m^2e\cos(2\tau-\xi)+\frac{43}{16}m^2e^2\sin 2\xi$$

$$-\frac{717}{32}m^2e\,e_{,}\sin(\xi+\xi_{,})+\frac{1801}{32}m^2e\,e_{,}\sin(2\tau-\xi-\xi_{,})$$

$$+\frac{1233}{32}m^2e\,e_{,}\sin(\xi-\xi_{,})-\frac{173}{32}m^2e\,e_{,}\sin(2\tau-\xi+\xi_{,})$$

Calculation of the term $-\frac{179}{288}m^4$ in a $\left(\delta\frac{1}{r}\right)$ (Arg. 0), and of the terms $-\frac{225}{32}m^3-\frac{4071}{128}m^4$ in the value of c . . . . *page* 129 to 136.

Calculation of the terms

$$\left\{\frac{383}{27}m^5+\frac{508315}{20736}m^6\right\}\cos 2\tau+\left\{\frac{2737}{480}m^5+\frac{594869}{7200}m^6\right\}\cos 4\tau$$

$+\frac{1535}{1792}m^6\cos 6\tau$ in a $\delta\frac{1}{r}$ and of the terms

$$\left\{\frac{2855}{108}m^5+\frac{6253189^*}{165888}m^6\right\}\sin 2\tau+\left\{\frac{649}{120}m^5+\frac{1179373^*}{28800}m^6\right\}\sin 4\tau$$

$+\frac{6627}{14336}m^6\sin 6\tau$ in the longitude . . . . . *page* 136 to 143.

* The terms marked with an asterisk are not consistent with M. Plana's results.

Calculation of the following terms in $a\,\delta\,\dfrac{1}{r}$

$$-\frac{75}{16}m e_{,}^{2} e \underset{[3]}{\cos(2\tau-\xi)}+\left\{\frac{405}{64}m^{2}+\frac{5037^{*}}{256}m^{2}e^{2}-\frac{165}{32}m^{2}e_{,}^{2}\right\}e\underset{[4]}{\cos(2\tau+\xi)}$$

$$+\left\{-\frac{9}{4}m^{2}e^{2}-\frac{27}{16}m^{2}e_{,}^{2}\right\}e_{,}\underset{[5]}{\cos\xi_{,}}$$

$$+\left\{\frac{799}{16}m^{2}e^{2}-\frac{123}{16}m^{2}e_{,}^{2}\right\}e_{,}\underset{[6]}{\cos(2\tau-\xi_{,})}$$

$$+\left\{-\frac{207}{16}m^{2}e^{2}+\frac{1}{16}m^{2}e_{,}^{2}\right\}e_{,}\underset{[7]}{\cos(2\tau+\xi_{,})}+\frac{95}{32}m^{2}e^{3}\underset{[20]}{\cos 3\xi}$$

$$-\frac{105}{64}m e^{3}\underset{[21]}{\cos(2\tau-3\xi)}+\frac{2125^{*}}{384}m^{2}e^{3}\underset{[22]}{\cos(2\tau+3\xi)}$$

$$+\left\{-\frac{21}{4}m-\frac{789}{32}m^{2}\right\}e^{2}e_{,}\underset{[23]}{\cos(2\xi+\xi_{,})}$$

$$-\frac{105}{8}m^{2}e^{2}e_{,}\underset{[24]}{\cos(2\tau-2\xi-\xi_{,})}-\frac{7^{*}}{4}m^{2}e^{2}e_{,}\underset{[25]}{\cos(2\tau+2\xi+\xi_{,})}$$

$$+\left\{\frac{21}{4}m+\frac{1161}{32}m^{2}\right\}e^{2}e_{,}\underset{[26]}{\cos(2\xi-\xi_{,})}$$

$$+\frac{15}{8}m^{2}e^{2}e_{,}\underset{[27]}{\cos(2\tau-2\xi+\xi_{,})}+\frac{49}{4}m^{2}e^{2}e_{,}\underset{[28]}{\cos(2\tau+2\xi-\xi_{,})}$$

$$-\frac{63}{32}m\,e\,e_{,}^{2}\underset{[29]}{\cos(\xi+2\xi_{,})}+\frac{255}{32}m\,e\,e_{,}^{2}\underset{[30]}{\cos(2\tau-\xi-2\xi_{,})}$$

---

* The terms marked with an asterisk are not consistent with M. Plana's results.

$$+\underset{[32]}{\frac{63}{32} m\, e\, e_{,}^{2} \cos(\xi - 2\,\xi_{,})} + \underset{[34]}{\frac{561}{32} m^{2} e\, e_{,}^{2} \cos(2\,\tau + \xi - 2\,\xi_{,})}$$

$$-\underset{[35]}{\frac{53}{16} m^{2} e_{,}^{3} \cos 3\,\xi_{,}} + \underset{[36]}{\frac{845}{48} m^{2} e_{,}^{3} \cos(2\,\tau - 3\,\xi_{,})} + \underset{[37]}{\frac{m^{2}}{48} e^{3} \cos(2\,\tau + 3\,\xi_{,})}$$

$$+\frac{6075}{512} m^{2} e^{3} \cos(4\,\tau - \xi) + \frac{525}{32} m^{2} e^{2} e_{,} \cos(4\,\tau - 2\,\xi - \xi_{,})$$

$$-\frac{225}{32} m^{2} e^{2} e_{,} \cos(4\,\tau - 2\,\xi - \xi_{,})$$

Calculation of the following terms in the longitude:

$$-\underset{[3]}{\frac{75}{8} m e^{2} \sin(2\tau - \xi)} + \underset{[4]}{\left\{\frac{195}{32} m e^{2} + \frac{3029^{*}}{128} m^{2} e^{2} - \frac{85}{16} m^{2} e_{,}^{2}\right\} e \sin(2\tau + \xi)}$$

$$+\underset{[6]}{\left\{\frac{175}{16} m\, e^{2} + \frac{4541}{64} m^{2} e^{2} - \frac{1353^{*}}{128} m^{2} e_{,}^{2}\right\} e_{,} \sin(2\,\tau - \xi_{,})}$$

$$+\underset{[7]}{\left\{-\frac{75}{16} m\, e^{2} - \frac{1113}{64} m^{2} e^{2} + \frac{11}{128} m^{2} e_{,}^{2}\right\} e_{,} \sin(2\,\tau + \xi_{,})}$$

$$+\underset{[21]}{\frac{105}{32} m\, e^{3} \sin(2\tau - 3\,\xi)} + \underset{[22]}{\frac{779^{*}}{192} m^{2} e^{3} \sin(2\,\tau + 3\,\xi)} - \underset{[23]}{\frac{105}{16} m\, e^{2} e_{,} \sin(2\,\xi + \xi_{,})}$$

$$-\underset{[25]}{\frac{5^{*}}{4} m^{2} e^{2} e_{,} \sin(2\,\tau + 2\,\xi + \xi_{,})} + \underset{[26]}{\frac{105}{16} m\, e^{2} e_{,} \sin(2\,\xi - \xi_{,})}$$

$$+\underset{[28]}{\frac{673^{*}}{64} m^{2} e^{2} e_{,} \sin(2\,\tau + 2\,\xi - \xi_{,})}$$

$$-\underset{[29]}{\frac{63}{16} m\, e\, e_{,}^{2} \sin(\xi + 2\,\xi_{,})} + \underset{[30]}{\frac{255}{16} m\, e\, e_{,}^{2} \sin(2\,\tau - \xi - 2\,\xi_{,})}$$

---

* The terms marked with an asterisk are not consistent with M. Plana's results.

$$+\underset{[32]}{\frac{63}{16} m\, e\, e_{,}^{2} \sin(\xi - 2\,\xi_{,})}$$

$$+\underset{[34]}{\frac{289}{16} m^{2} e\, e_{,}^{2} \sin(2\,\tau + \xi - 2\,\xi_{,})} - \underset{[35]}{\frac{53}{24} m\, e_{,}^{3} \sin 3\,\xi_{,}}$$

$$+\underset{[36]}{\frac{9295}{384} m^{2} e^{3} \sin(2\,\tau - 3\,\xi_{,})} + \underset{[37]}{\frac{11}{384} m^{2} e_{,}^{3} \sin(2\,\tau + 3\,\xi_{,})}$$ .. *page* 143 to 156.

Approximate expressions for $a\,\delta\frac{1}{r}$, $r\,\delta\frac{1}{r}$, and $\delta\lambda$ .. *page* 156 to 160.

Terms in the disturbing function of the order $m^{4} e^{2}$ and $m^{3} e^{3}$, &c. *p.* 160 to 167.

Terms in $\int dR$ of the order $m^{4} e^{2}$, in $a^{2}\left(\delta\frac{1}{r}\right)^{2}$, and $\frac{3\,r^{4}}{2\,a^{2}}\left(\delta\frac{1}{r}\right)^{2}$ of the order $m^{4} e^{2}$ and $m^{3} e^{3}$, &c. . . . . . . *page* 167 to 174.

Calculation of the following terms in $a\,\delta\frac{1}{r}$:

$$\underset{[3]}{\left\{-\frac{463^{*}}{128} m^{2} e^{2} - \frac{385}{32} m^{2} e_{,}^{2}\right\} e \cos(2\,\tau - \xi)} - \underset{[8]}{\frac{165}{64} m^{3} e^{2} \cos 2\,\xi}$$

$$-\underset{[11]}{\frac{7231}{128} m^{4} e\, e_{,} \cos(\xi + \xi_{,})} + \underset{[12]}{\frac{48935}{768} m^{4} e\, e_{,} \cos(2\,\tau - \xi - \xi_{,})}$$

$$+\underset{[14]}{\frac{1687}{16} m^{4} e\, e_{,} \cos(\xi - \xi_{,})} + \underset{[15]}{\frac{49277^{*}}{768} m^{4} e\, e_{,} \cos(2\,\tau - \xi + \xi_{,})}$$

$$-\underset{[21]}{\frac{7703}{384} m^{2} e^{2} \cos(2\,\tau - 3\,\xi)} - \underset{[29]}{\frac{1965^{*}}{256} m^{2} e\, e_{,}^{2} \cos(\xi + 2\,\xi_{,})}$$

$$+\underset{[30]}{\frac{12011}{256} m^{2} e\, e_{,}^{2} \cos(2\,\tau - \xi - 2\,\xi_{,})} + \underset{[32]}{\frac{4635}{256} m^{2} e\, e_{,}^{2} \cos(\xi - 2\,\xi_{,})}$$

$$-\underset{[33]}{\frac{45}{32} m\, e\, e_{,}^{2} \cos(2\,\tau - \xi - 2\,\xi_{,})} - \frac{227085^{*}}{2048} m^{2} e^{3} \cos(4\,\tau - 3\,\xi)$$ *p.* 174 to 180.

---

* The terms marked with an asterisk are not consistent with M. Plana's results.

Calculation of the following terms in the longitude:

$$\left\{\frac{74671}{1024} m^3 e^2 - \frac{691^*}{24} m^3 e_{,}^2\right\} \sin 2\tau$$

[1]

$$+ \left\{-\frac{369}{64} m^2 e^2 - \frac{575}{16} m^2 e_{,}^2\right\} e \sin(2\tau - \xi)$$

[3]

$$+ \frac{2765}{256} m^3 e^2 \sin 2\xi + \frac{913}{96} m^3 e^2 \sin(2\tau + 2\xi) - \frac{3379^*}{32} m^3 e e_{,} \sin(\xi + \xi_{,})$$

[8] [10] [11]

$$+ \frac{32989}{128} m^3 e e_{,} \sin(2\tau - \xi - \xi_{,}) - \frac{2633}{192} m^3 e e_{,} \sin(2\tau + \xi + \xi_{,})$$

[12] [13]

$$+ \frac{15333}{64} m^3 e e_{,} \sin(\xi - \xi_{,}) + \frac{48325}{384} m^3 e e_{,} \sin(2\tau - \xi + \xi_{,})$$

[14] [15]

$$+ \frac{3131}{64} m^3 e e_{,} \sin(2\tau + \xi - \xi_{,}) + \frac{9707}{96} m^3 e_{,}^2 \sin(2\tau - 2\xi_{,})$$

[16] [18]

$$- \frac{33}{32} m^3 e_{,}^2 \sin(2\tau + 2\xi_{,}) + \frac{10}{3} m^2 e^3 \sin 3\xi + \frac{6011}{384} m^2 e^3 \sin(2\tau - 3\xi)$$

[19] [20] [21]

$$- \frac{3669}{128} m^2 e^2 e_{,} \sin(2\xi + \xi_{,}) + \frac{6081}{128} m^2 e^2 e_{,} \sin(2\xi - \xi_{,})$$

[23] [26]

$$- \frac{1245}{128} m^2 e e_{,}^2 \sin(\xi + 2\xi_{,}) + \frac{17179}{128} m^2 e e_{,}^2 \sin(2\tau - \xi - 2\xi_{,})$$

[29] [30]

$$+ \frac{5355}{128} m^2 e e_{,}^2 \sin(\xi - 2\xi_{,}) + \frac{3075^*}{256} m^3 e^2 \sin 4\tau$$

[32] [131]

$$+ \left\{\frac{1125}{256} m^2 + \frac{321585^*}{512} m^3\right\} e^2 \sin(4\tau - 2\xi) + \frac{2975}{128} m^3 e e_{,} \sin(4\tau - \xi - \xi_{,})$$

[136] [138]

---

* The terms marked with an asterisk are not consistent with M. Plana's results.

$$-\frac{765}{128}m^3\,e\,e_{,}\sin(4\tau-\xi+\xi_{,})-\frac{227085^{*}}{1024}m^2e^3\sin(4\tau-3\xi)$$

[140]

$$-\frac{1125}{728}m^2e^2e_{,}\sin(4\tau-2\xi+\xi_{,})+\frac{2625}{128}m^2e^2e_{,}\sin(4\tau-2\xi-\xi_{,})$$

page 180 to 195.

Calculation of the following terms in the longitude,

$$\frac{285^{*}}{32}m^3e\sin\xi+\frac{51817}{768}m^3e\sin(2\tau-\xi)+\frac{169}{24}m^3e\sin(2\tau+\xi)$$

[2] [3] [4]

$$+\frac{255}{64}m^3e\sin(4\tau-\xi)$$ . . . . *page* 195 to 197.

[132]

Calculation of the following terms in $a\,\delta\frac{1}{r}$,

$$\frac{1337369}{18432}m^4e\cos(2\tau-\xi)+\frac{6293}{384}m^4e\cos(2\tau+\xi)$$ *page* 197 to 203.

[3] [4]

Calculation of the following terms in the longitude,

$$\frac{47519}{1152}m^4e\sin\xi+\frac{1912137}{9216}m^4e\sin(2\tau-\xi)$$

[2] [3]

$$+\frac{2005}{192}m^4e\sin(2\tau+\xi)+\frac{7701}{256}m^4e\sin(4\tau-\xi)$$

[4] [132]

$$+\frac{309}{128}m^4e\sin(4\tau+\xi)$$ . . . . *page* 203 to 205.

[133]

Calculation of the following terms in $a\,\delta\frac{1}{r}$,

$$\left\{-\frac{15}{16}m\,e^4-\frac{75}{8}m\,e^2e_{,}^2\right\}\cos 2\tau+\frac{75}{8}m^2e_{,}^2e^2\cos(2\tau-2\xi)$$

[1] [9]

---

* The terms marked with an asterisk are not consistent with M. Plana's results.

$+\left\{-\frac{51}{64}m e^2 - \frac{189}{64} m e_{,}^2\right\} e e_{,} \cos(\xi + \xi_{,})$
[11]

$+\left\{-\frac{315^*}{64} m e^2 - \frac{615}{64} m e_{,}^2\right\} e e_{,} \cos(2\tau - \xi - \xi_{,})$
[12]

$-\frac{405^*}{64} m e^3 e_{,} \cos(2\tau + \xi + \xi_{,}) + \left\{\frac{51}{64} m e^2 + \frac{189}{64} m e_{,}^2\right\} e e_{,} \cos(\xi - \xi_{,}$
[13] [14]

$+\left\{\frac{945^*}{128} m e^2 + \frac{15}{64} m e_{,}^2\right\} e e_{,} \cos(2\tau - \xi + \xi_{,})$
[15]

$+\frac{945^*}{64} m e^3 e_{,} \cos(2\tau + \xi - \xi_{,})$
[16]

$+\frac{255}{16} m e^2 e_{,}^2 \cos(2\tau - 2\xi_{,}) - \frac{45}{16} m e^2 e_{,}^2 \cos(2\tau + 2\xi_{,})$
[18] [19]

$-\frac{55^*}{16} m e^4 \cos(2\tau - 4\xi) - \frac{567}{64} m e^3 e_{,} \cos(3\xi + \xi_{,})$
[39] [41]

$-\frac{245^*}{64} m e^3 e_{,} \cos(2\tau - 3\xi - \xi_{,}) + \frac{567}{64} m e^3 e_{,} \cos(3\xi - \xi_{,})$
[42] [44]

$+\frac{625^*}{32} m e^3 e_{,} \cos(2\tau - 3\xi + \xi_{,}) - \frac{63}{16} m e^2 e_{,}^2 \cos(2\xi + 2\xi_{,})$
[45] [47]

$+\frac{63}{16} m e^2 e_{,}^2 \cos(2\xi - 2\xi_{,}) - \frac{371^*}{192} m e e_{,}^3 \cos(\xi + 3\xi_{,})$
[50] [53]

$+\frac{845}{64} m e e_{,}^3 \cos(2\tau - \xi - 3\xi_{,}) + \frac{371}{192} m e e_{,}^3 \cos(\xi - 3\xi_{,})$
[54] [56]

$-\frac{5}{64} m e e_{,}^3 \cos(2\tau - \xi + 3\xi_{,})$ . . . . *page* 205 to 212.
[57]

---

* The terms marked with an asterisk are not consistent with M. Plana's results.

Calculation of the following terms in the longitude,

$$\left\{\frac{15^*}{16} m e^4 - \frac{375}{32} m e^2 e_{,}^2\right\} \underset{[1]}{\sin 2\tau} + \frac{515}{64} m e^4 \underset{[10]}{\sin (2\tau + 2\xi)}$$

$$+ \left\{-\frac{51}{32} m e^2 - \frac{189}{32} m e_{,}^2\right\} e e_{,} \underset{[11]}{\sin (\xi + \xi_{,})}$$

$$+ \left\{-\frac{315^*}{32} m e^2 - \frac{615}{32} m e_{,}^2\right\} e e_{,} \underset{[12]}{\sin (2\tau - \xi - \xi_{,})}$$

$$- \frac{195}{32} m e^3 e_{,} \underset{[13]}{\sin (2\tau + \xi + \xi_{,})}$$

$$+ \left\{\frac{51}{32} m e^2 + \frac{189}{32} m e_{,}^2\right\} e e_{,} \underset{[14]}{\sin (\xi - \xi_{,})}$$

$$+ \left\{\frac{545^*}{64} m e^2 + \frac{15}{32} m e_{,}^2\right\} e e_{,} \underset{[15]}{\sin (2\tau - \xi + \xi_{,})}$$

$$+ \frac{455}{32} m e^3 e_{,} \underset{[16]}{\sin (2\tau + \xi - \xi_{,})}$$

$$+ \frac{1275}{64} m e^2 e_{,}^2 \underset{[18]}{\sin (2\tau - 2\xi_{,})} - \frac{225}{64} m e^2 e_{,}^2 \underset{[19]}{\sin (2\tau + 2\xi_{,})}$$

$$- \frac{45^*}{64} m e^4 \underset{[39]}{\sin (2\tau - 4\xi)} - \frac{273}{32} m e^3 e_{,} \underset{[41]}{\sin (3\xi + \xi_{,})}$$

$$+ \frac{245}{32} m e^2 e_{,} \underset{[42]}{\sin (2\tau - 3\xi - \xi_{,})} + \frac{273}{32} m e^3 e_{,} \underset{[44]}{\sin (3\xi - \xi_{,})}$$

$$- \frac{105}{32} m e^3 e_{,} \underset{[45]}{\sin (2\tau - 3\xi + \xi_{,})} - \frac{315}{64} m e^2 e_{,}^2 \underset{[47]}{\sin (2\xi + 2\xi_{,})}$$

---

* The terms marked with an asterisk are not consistent with M. Plana's results.

$+\frac{315}{64} m e^2 e_{,}^2 \sin(2\xi - 2\xi_{,}) - \frac{371}{96} m e e_{,}^3 \sin(\xi + 3\xi_{,})$
[50] [53]

$+\frac{371}{96} m e e_{,}^3 \sin(\xi - 3\xi_{,}) - \frac{5}{32} m e e_{,}^3 \sin(2\tau - \xi + 3\xi_{,})$ *page* 212 to 215.
[56] [57]

Calculation of the term $\frac{1261}{8} m^5 e_{,} \cos \xi_{,}$ in $a\,\delta \frac{1}{r}$ . . *page* 215 to 218.

Method of determining the inequalities of longitude when the divisor introduced by integration is a multiple of *m*. Example of this method in the calculation of the terms $\left\{-3m + \frac{735}{16} m^3 + \frac{4171^{*}}{16} m^4\right\} e_{,} \sin \xi_{,}$ in the longitude . . . . *page* 218 to 222.

Calculation of the terms $\left\{\frac{45}{16} m + \frac{53}{4} m^2 + \frac{48007^{*}}{384} m^3\right\} e^2 \sin(2\tau - 2\xi)$ in the longitude . . . . *page* 222 to 225.

Calculation of the terms $\left\{-\frac{9}{4} m + \frac{5751}{128} m^3\right\} e_{,}^2 \sin 2\xi_{,}$ in the longitude . . . . *page* 225 to 227.

Expression for $\frac{a^3}{r^3} + \frac{m_{,} a^3}{\mu r_{,}^3}$. Approximate expression for *z*. Approximate expression for the latitude of the moon . . . . *page* 227 to 248.

Calculation of terms in the reciprocal of the radius vector multiplied by γ^2 . . . . *page* 248 to 254.

Calculation of terms in the longitude multiplied by γ^2. Expression for s^2. Recapitulation of terms in $a\,\delta \frac{1}{r}$ multiplied by γ^2 obtained in this treatise. Expression for $\frac{d\lambda'}{n\,dt}$. Recapitulation of terms in λ' multiplied by γ^2 . . . . *page* 254 to 268.

Calculation of λ_{63}, λ_{73}, λ_{75}, and λ_{77}, by the method of page 219 *page* 268 to 273

[*] The terms marked with an asterisk are not consistent with M. Plana's results.

Calculation of terms in the reciprocal of the radius vector and of the longitude multiplied by $\frac{a}{a_{,}}$ . . . . *page* 273 to 287.

Calculation of terms in the parallactic inequality which depend upon the mass of the moon . . . . *page* 287 to 289.

Calculation of terms due to the non-sphericity of the earth. . *page* 289 to 292

CORRECTIONS.

In the calculation of $a^2 \left(\delta \frac{1}{r}\right)^2$, p. 55, the combination

$$\frac{19}{6} m^3 \cos 2\tau \left\{ \frac{15}{8} m e \cos (2\tau - \xi) \right\}$$

is omitted. This omission being supplied, gives the terms

$$\frac{4051}{288} m^4 e \cos \xi + \frac{377}{32} m^4 e \cos (4\tau - \xi), \quad \text{instead of}$$

$$\frac{2341}{288} m^4 e \cos \xi + \frac{187}{32} m^4 e \cos (4\tau - \xi)$$

I am indebted to M. de Pontécoulant for notice of this error.

I am also indebted to M. de Pontécoulant for the following corrections:

Page 126, line 8, read $A = -\frac{1687}{128}$; this correction introduced in p. 189 leads to $\lambda_{12} = -\frac{17}{16} m^2 - \frac{2633}{192} m^3$ p. 193, which agrees with the value of M. Plana.

$$\lambda_4 = \left\{ \left(1 - \frac{e^2}{2}\right) \left\{ 2 r_4 + \left(1 - \frac{e^2}{8}\right)(r_1 + e^2 r_{10}) + e^2 r_3 \right\} + \left(1 + \frac{e^2}{2}\right) \mathbf{R}_4 \right.$$

$$\left. + \left(1 + \frac{3}{8} e^2\right)(\mathbf{R}_1 + e^2 \mathbf{R}_{10}) + \frac{5}{4} e^2 \mathbf{R}_3 \right\} \frac{1}{(2 - 2m + c)}$$

$$= \frac{17}{8} m^2 + \left(\frac{405}{32} + \frac{15}{4} + \frac{15}{8}\right) \frac{m e^2}{3} \left(1 - \frac{2m}{3}\right)$$

$$+ \left(- \frac{41}{16} + \frac{5037}{128} - \frac{1}{8} + \frac{189}{16} + \frac{7}{2} + \frac{187}{32} + \frac{1}{4} - \frac{19}{16} + \frac{9}{32} - \frac{15}{8}\right.$$

$$\left. + \frac{3}{8} - \frac{45}{8}\right) \frac{m^2 e^2}{3} + \left(- \frac{165}{16} - \frac{5}{2} - \frac{5}{4} - \frac{15}{8}\right) \frac{m^2 e_{\prime}^2}{3}$$

$$= \frac{17}{8} m^2 + \frac{195}{32} m e^2 + \frac{2655}{128} m^2 e^2 - \frac{85}{16} m^2 e_{\prime}^2$$

which value agrees with that given by M. Plana.

Page 155, line 6, read $\frac{7}{4}$ for $\frac{9}{4}$ which gives $\lambda_{28} = \frac{665}{64} m^2$, agreeing with M. Plana.

Page 158, line 12, read $- \frac{135}{64} m e^3 \cos (2 \tau - 3 \xi)$ which gives

$$\frac{a R}{\mu} = \left\{ \frac{7}{32} m^2 - \frac{105}{128} m^3 \right\} e^3 \cos (2 \tau - 3 \xi) \text{ p. } 167;$$

this correction being introduced p. 178 gives $A = \frac{7703}{128}$, line 11, agreeing with M. Plana,

and p. 194 $\lambda_{21} = \frac{105}{32} m + \frac{6011}{384} m^2$, also agreeing with M. Plana.

Page 173, line 15, read $\frac{5355}{64} m^4 e e_{\prime} \cos (\xi - \xi_{\prime})$; this correction being introduced in r_{14}, p. 177, gives $A = \frac{1687}{16}$ agreeing with M. Plana, and

$$\lambda_{14} = \frac{21}{4} m + \frac{1233}{32} m^2 + \frac{15333}{64} m^2, \text{ p. } 193,$$

with my system of constants, which is consistent with M. Plana.

Page 183, line 9, read $+ \left(\frac{9}{4} m^2 + \frac{27}{16} m^3\right) (1 + m) e e_{\prime} \cos (2 \tau - \xi + \xi_{\prime})$, which gives $\left(\frac{9}{4} m^2 + \frac{63}{16} m^3\right) e e_{\prime} \cos (2 \tau - \xi + \xi_{\prime})$, p. 184, line 11, and $\lambda_{15} = - \frac{15}{4} m - \frac{173}{32} m^2 + \frac{48325}{384} m^3$, p. 193, which is consistent with the value of M. Plana.

The term $\frac{285}{32} m^3$ (p. 197) in λ_2, is consistent with the term $- \frac{75}{64} m^3$, given by M. Plana, for the equation given in p. 197 does obtain. Multiplying crosswise,

$$\left\{ 1 + \frac{2}{3} m^2 \right\} \left\{ 2 + \frac{3}{2} m^2 - \frac{75}{64} m^3 \right\} = \left\{ 2 + \frac{5}{2} m^2 + \frac{285}{32} m^3 \right\} \left\{ 1 + \frac{m^2}{6} - \frac{645}{128} m^3 \right\}$$

$$2 + \left\{ \frac{4}{3} + \frac{3}{2} \right\} m^2 - \frac{75}{64} m^3 = 2 + \left\{ \frac{5}{2} + \frac{1}{3} \right\} m^2 + \left\{ \frac{285}{32} - \frac{645}{64} \right\} m^3$$

ON

THE THEORY OF THE MOON,

AND ON

THE PERTURBATIONS OF THE PLANETS.

I PROPOSE here to carry the development of R, $\frac{dR}{d\lambda}$, &c., to the terms multiplied by $m^3 e^2$, $m^3 e e_{,}$, and $m^3 e_{,}^2$, and to find terms in the reciprocal of the radius vector and in the longitude by methods similar to those given page 52 and following of the former portion of this treatise.

Neglecting γ^2,

$$\delta R = -\mathrm{a}\left(\frac{dR}{d\mathrm{a}}\right) \mathrm{r}\,\delta\frac{1}{r} + \left(\frac{dR}{d\tau}\right)\delta\lambda$$

$$-\frac{\mathrm{a}^2}{\mu}\frac{dR}{d\mathrm{a}} = \frac{m^2}{2} + \frac{3}{2}m^2\cos 2\tau - m^2 e\cos\xi$$

$$-\frac{9}{2}m^2 e\cos(2\tau-\xi) + \frac{3}{2}m^2 e\cos(2\tau+\xi)$$

$$+\frac{3}{2}m^2 e_{,}\cos\xi_{,} + \frac{21}{4}m^2 e_{,}\cos(2\tau-\xi_{,})$$

$$-\frac{3}{4}m^2 e_{,}\cos(2\tau+\xi_{,})$$

$$\mathrm{r}\,\delta\frac{1}{r} = \frac{45}{16}m e^2\cos 2\tau + \frac{15}{8}m e\cos(2\tau-\xi) - \frac{15}{16}m e^2\cos(2\tau-2\xi)$$

$$-\frac{21}{8}m e e_{,}\cos(\xi+\xi_{,}) + \frac{35}{8}m e e_{,}\cos(2\tau-\xi-\xi_{,})$$

$$+\frac{21}{8}m e e_{,}\cos(\xi-\xi_{,}) - \frac{15}{8}m e e_{,}\cos(2\tau-\xi+\xi_{,})$$

$$\frac{\mathrm{a}}{\mu}\frac{dR}{d\tau} = \frac{3}{2}m^2\sin 2\tau - \frac{9}{2}m^2 e\sin(2\tau-\xi) + \frac{3}{2}m^2 e\sin(2\tau+\xi)$$

$$+\frac{21}{4}m^2 e_{,}\sin(2\tau-\xi_{,}) - \frac{3}{4}m^2 e_{,}\sin(2\tau+\xi_{,})$$

$$\delta\lambda = \frac{75}{16}m e^2\sin 2\tau + \frac{15}{4}m e\sin(2\tau-\xi) - 3 m e_{,}\sin\xi_{,}$$

$$+\frac{45}{16}\,m\,e^2\sin\left(2\,\tau-2\,\xi\right)-\frac{21}{4}\,m\,e\,c_{,}\sin\left(\xi+\xi_{,}\right)$$

$$+\frac{35}{4}\,m\,e\,e_{,}\sin\left(2\,\tau-\xi-\xi_{,}\right)+\frac{21}{4}\,m\,e\,e_{,}\sin\left(\xi-\xi_{,}\right)$$

$$-\frac{15}{4}\,m\,e\,e_{,}\sin\left(2\,\tau-\xi+\xi_{,}\right)-\frac{9}{4}\,m\,e_{,}^2\sin 2\,\xi_{,}$$

$$\frac{\mathrm{a}}{\mu}\,\delta R=\left\{\frac{135}{64}-\frac{135}{32}+\frac{225}{64}-\frac{135}{16}\right\}m^3e^2$$

$$+\left\{\left(\frac{45}{32}-\frac{15}{16}\right)e^2+\left(\frac{63}{8}+\frac{9}{8}\right)\,e_{,}^2\right\}m^3\cos 2\,\tau$$

$$+\left\{\frac{45}{32}-\frac{45}{64}+\frac{45}{16}+\frac{135}{64}\right\}m^3e^2\cos 2\,\xi+\left\{-\frac{15}{16}-\frac{15}{32}\right\}m^3e^2\cos(2\tau$$

$$+\left\{-\frac{45}{64}-\frac{21}{16}+\frac{105}{32}-\frac{45}{32}+\frac{105}{16}\right\}m^3e\,e_{,}\cos\left(\xi+\xi_{,}\right)$$

$$+\left\{\frac{45}{32}-\frac{63}{32}+\frac{35}{16}+\frac{27}{4}-\frac{63}{16}\right\}m^3e\,e_{,}\cos\left(2\,\tau-\xi-\xi_{,}\right)$$

$$+\left\{-\frac{63}{32}+\frac{9}{4}+\frac{63}{16}\right\}m^3e\,e_{,}\cos\left(2\,\tau+\xi+\xi_{,}\right)$$

$$+\left\{\frac{315}{64}+\frac{21}{16}-\frac{45}{32}+\frac{315}{32}-\frac{45}{16}\right\}m^3e\,e_{,}\cos\left(\xi-\xi_{,}\right)$$

$$+\left\{\frac{45}{32}+\frac{63}{32}-\frac{15}{16}-\frac{27}{4}+\frac{63}{16}\right\}m^3e\,e_{,}\cos\left(2\,\tau-\xi+\xi_{,}\right)$$

$$+\left\{\frac{63}{32}-\frac{9}{4}=\frac{63}{16}\right\}m^3e\,e_{,}\cos\left(2\,\tau+\xi-\xi_{,}\right)$$

$$+\left\{-\frac{63}{8}-\frac{27}{16}\right\}m^3e_{,}^2\cos\left(2\,\tau-2\,\xi_{,}\right)$$

$$+\left\{-\frac{9}{8}+\frac{27}{16}\right\}m^3e_{,}^2\cos\left(2\,\tau+2\,\xi_{,}\right)$$

$$+\left\{\frac{135}{64}+\frac{45}{32}-\frac{225}{64}-\frac{45}{16}\right\}m^3e^2\cos 4\,\tau$$

$$+\left\{-\frac{135}{32}-\frac{45}{64}+\frac{135}{16}-\frac{135}{64}\right\}m^3e^2\cos\left(4\,\tau-2\,\xi\right)$$

$$+\left\{\frac{315}{54}+\frac{105}{32}-\frac{315}{32}-\frac{105}{16}\right\}m^3e\,e_{,}\cos\left(4\,\tau-\xi-\xi_{,}\right)$$

$$+\left\{-\frac{45}{64}-\frac{45}{32}+\frac{45}{32}+\frac{45}{16}\right\}m^3e\,e_{,}\cos\left(4\,\tau-\xi+\xi_{,}\right)$$

$$\frac{\mathrm{a}}{\mu}\,\delta R=-\underset{[0]}{\frac{225}{32}\,m^3e^2}+\underset{[1]}{\left\{\frac{15}{32}e^2+9\,e_{,}^2\right\}m^3\cos 2\,\tau}+\frac{45}{8}\,m^3e^2\,\cos$$

[

$$-\underset{[9]}{\frac{45}{32}m^3e^2\cos(2\tau-2\xi)}+\underset{[11]}{\frac{411}{64}m^3ee_{\prime}\cos(\xi+\xi_{\prime})}$$

$$+\underset{[12]}{\frac{71}{16}m^3ee_{\prime}\cos(2\tau-\xi-\xi_{\prime})}+\underset{[13]}{\frac{135}{32}m^3ee_{\prime}\cos(2\tau+\xi+\xi_{\prime})}$$

$$+\underset{[14]}{\frac{759}{64}m^3ee_{\prime}\cos(\xi-\xi_{\prime})}-\underset{[15]}{\frac{3}{8}m^3ee_{\prime}\cos(2\tau-\xi+\xi_{\prime})}$$

$$-\underset{[16]}{\frac{135}{32}m^3ee_{\prime}\cos(2\tau+\xi-\xi_{\prime})}-\underset{[18]}{\frac{153}{16}m^3e_{\prime}^2\cos(2\tau-2\xi_{\prime})}$$

$$+\underset{[19]}{\frac{9}{16}m^3e_{\prime}^2\cos(2\tau+2\xi_{\prime})}-\underset{[131]}{\frac{45}{16}m^3e^2\cos 4\tau}$$

$$+\underset{[136]}{\frac{45}{32}m^3e^2\cos(4\tau-2\xi)}-\underset{[138]}{\frac{525}{64}m^3ee_{\prime}\cos(4\tau-\xi-\xi_{\prime})}$$

$$+\underset{[140]}{\frac{135}{64}m^3ee_{\prime}\cos(4\tau-\xi+\xi_{\prime})}$$

$\int d R$ contains no term of this order.

$$a^2\left(\delta\frac{1}{r}\right)^2=\left\{\frac{m^2}{3}+2m^2\cos 2\tau\right\}\left\{\frac{15}{4}me^2\cos 2\tau-\frac{21}{8}mee_{\prime}\cos(\xi+\xi_{\prime})\right.$$

$$+\frac{35}{8}mee_{\prime}\cos(2\tau-\xi-\xi_{\prime})+\frac{21}{8}mee_{\prime}\cos(\xi-\xi_{\prime})$$

$$\left.-\frac{15}{8}mee_{\prime}\cos(2\tau-\xi+\xi_{\prime})\right\}$$

$$+\frac{15}{4}me\cos(2\tau-\xi)\left\{\frac{2}{3}m^2e\cos\xi\right.$$

$$+\frac{187}{32}m^2e\cos(2\tau-\xi)+\frac{33}{16}m^2e\cos(2\tau+\xi)-\frac{3}{2}m^2e_{\prime}\cos\xi_{\prime}$$

$$\left.+\frac{7}{2}m^2e_{\prime}\cos(2\tau-\xi_{\prime})-\frac{m^2}{2}e_{\prime}\cos(2\tau+\xi_{\prime})\right\}$$

$$=\left(\frac{15}{4}+\frac{2805}{256}\right)m^3e^2+\left(\frac{5}{4}+\frac{5}{4}\right)m^3e^2\cos 2\tau+\frac{495}{128}m^3e^2\cos 2\xi$$

$$+\frac{5}{4}m^3e^2\cos(2\tau-2\xi)+\left(-\frac{7}{8}+\frac{35}{8}-\frac{15}{16}\right)m^3ee_{\prime}\cos(\xi+\xi_{\prime})$$

$$+\left(\frac{35}{24}\quad\frac{21}{8}-\frac{45}{16}\right)m^3 e\, e_{,} \cos(2\tau-\xi-\xi_{,})-\frac{21}{8}m^3 e\, e_{,}\cos(2\tau+\xi+\xi_{,})$$

$$+\left(\frac{7}{8}-\frac{15}{8}+\frac{105}{16}\right)m^3 e\, e_{,}\cos(\xi-\xi_{,})$$

$$+\left(-\frac{5}{8}+\frac{21}{8}-\frac{45}{16}\right)m^3 e\, e_{,}\cos(2\tau-\xi+\xi_{,})+\frac{21}{8}m^3 e\, e_{,}\cos(2\tau+\xi-\xi_{,})$$

$$+\left(\frac{15}{4}+\frac{495}{128}\right)m^3 e^2\cos 4\tau+\frac{2805}{256}m^3 e^2\cos(4\tau-2\xi)$$

$$+\left(\frac{35}{8}+\frac{105}{16}\right)m^3 e\, e_{,}\cos(4\tau-\xi-\xi_{,})$$

$$+\left(-\frac{15}{8}-\frac{15}{16}\right)m^3 e\, e_{,}\cos(4\tau-\xi+\xi_{,})$$

$$=\underset{[0]}{\frac{3765}{256}m^3 e^2}+\underset{[1]}{\frac{5}{2}m^3 e^2\cos 2\tau}+\underset{[8]}{\frac{495}{128}m^3 e^2\cos 2\xi}$$

$$+\underset{[9]}{\frac{5}{4}m^3 e^2\cos(2\tau-2\xi)}+\underset{[11]}{\frac{41}{16}m^3 e\, e_{,}\cos(\xi+\xi_{,})}$$

$$-\underset{[12]}{\frac{191}{48}m^3 e\, e_{,}\cos(2\tau-\xi-\xi_{,})}-\underset{[13]}{\frac{21}{8}m^3 e\, e_{,}\cos(2\tau+\xi+\xi_{,})}$$

$$+\underset{[14]}{\frac{89}{16}m^3 e\, e_{,}\cos(\xi-\xi_{,})}-\underset{[15]}{\frac{13}{16}m^3 e\, e_{,}\cos(2\tau-\xi+\xi_{,})}$$

$$+\underset{[16]}{\frac{21}{8}m^3 e\, e_{,}\cos(2\tau+\xi-\xi_{,})}+\underset{[131]}{\frac{975}{128}m^3 e^2\cos 4\tau}$$

$$+\underset{[136]}{\frac{2805}{256}m^3 e^2\cos(4\tau-2\xi)}+\underset{[138]}{\frac{175}{16}m^3 e\, e_{,}\cos(4\tau-\xi-\xi_{,})}$$

$$-\underset{[140]}{\frac{45}{16}m^3 e\, e_{,}\cos(4\tau-\xi+\xi_{,})}$$

$$\mathrm{r}^4=\mathrm{a}^4\{1-4e\cos\xi\}\qquad\qquad \frac{3}{2}\mathrm{r}^4=\mathrm{a}^4\left\{\frac{3}{2}-6e\cos\xi\right\}$$

$\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2$ contains in addition to the above, the terms

$$\frac{15}{8}m^3 e\cos\xi+\frac{5}{8}m^3 e\cos(2\tau-\xi)+\frac{15}{8}m^3 e\cos(4\tau-\xi)$$

(See p. 55.), which multiplied by $-6\,e\cos\xi$ give the terms

$$-\frac{45}{8}m^3e^2\left\{1+\cos 2\,\xi\right\}$$

$$-\frac{15}{8}m^3e^2\left\{\cos(2\,\tau-2\,\xi)+\cos 2\,\tau\right\}$$

$$-\frac{45}{8}m^3e^2\left\{\cos(4\,\tau-2\,\xi)+\cos 4\,\tau\right\}$$

Hence

$$\frac{3\,\mathrm{r}^4}{2\,\mathrm{a}^2}\left(\delta\frac{1}{r}\right)^2=\underset{[0]}{\frac{8415}{512}m^3e^2}+\underset{[1]}{\frac{15}{8}m^3e^2\cos 2\,\tau}+\underset{[2]}{\frac{45}{256}m^3e^2\cos 2\,\xi}$$

$$+\underset{[11]}{\frac{123}{32}m^3e\,e_{\prime}\cos(\xi+\xi_{\prime})}-\underset{[12]}{\frac{191}{32}m^3e\,e_{\prime}\cos(2\,\tau-\xi-\xi_{\prime})}$$

$$-\underset{[13]}{\frac{63}{16}m^3e\,e_{\prime}\cos(2\,\tau+\xi+\xi_{\prime})}+\underset{[14]}{\frac{267}{32}m^3e\,e_{\prime}\cos(\xi-\xi_{\prime})}$$

$$-\underset{[15]}{\frac{39}{32}m^3e\,e_{\prime}\cos(2\,\tau-\xi+\xi_{\prime})}$$

$$+\underset{[16]}{\frac{63}{16}m^3e\,e_{\prime}\cos(2\,\tau+\xi-\xi_{\prime})}+\underset{[131]}{\frac{1485}{256}m^3e^2\cos 4\,\tau}$$

$$+\underset{[136]}{\frac{5535}{512}m^3e^2\cos(4\tau-2\xi)}+\underset{[138]}{\frac{525}{32}m^3e\,e_{\prime}\cos(4\,\tau-\xi-\xi_{\prime})}$$

$$-\underset{[140]}{\frac{135}{32}m^3e\,e_{\prime}\cos(4\,\tau-\xi+\xi_{\prime})}$$

I shall now attempt to determine some of the terms in the reciprocal of the radius vector, integrating as before by the method of indeterminate coefficients the equation

$$\frac{\mathrm{d}^2.\,\mathrm{r}^2}{2\,\mathrm{d}\,t^2}-\frac{\mathrm{d}^2.\,\mathrm{r}^3\,\delta\frac{1}{r}}{\mathrm{d}\,t^2}+\frac{3\,\mathrm{d}^2.\,\mathrm{r}^4\left(\delta\frac{1}{r}\right)^2}{2\,\mathrm{d}\,t^2}-\frac{2\,\mathrm{d}^2.\,\mathrm{r}^5\left(\delta\frac{1}{r}\right)^3}{\mathrm{d}\,t^2}+\&\mathrm{c}.$$

$$-\frac{\mu}{r}+\frac{\mu}{a}+2\int\mathrm{d}R+r\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,r}\right)=0$$

$$r_3\left\{(2-2\,m-\mathrm{c})^2-1\right\}=(2-2\,m-\mathrm{c})^2\,\mathrm{r}_3$$

$$-2\left\{\left\{\frac{2-\mathrm{c}}{2-2m-\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_3+\frac{m_{,}\mathrm{a}}{(2-2m-\mathrm{c})\mu}R'_3\right\}$$

$$\mathrm{r}_3=\frac{3}{2}\left(m^2+\frac{19}{6}m^3+\frac{131}{18}m^4\right)+\frac{15}{16}m^3+\frac{187}{64}m^4 \qquad \text{See p. 56.}$$

$$=\frac{3}{2}m^2+\frac{91}{16}m^3+\frac{2657}{192}m^4$$

$$\mathrm{c}=1-\frac{3}{4}m^2-\frac{225}{32}m^3 \qquad \text{See p. 67.}$$

$$R_3=\frac{9}{4}m^2+\frac{15}{16}m^3+\frac{235}{64}m^4 \qquad R'_3=-\frac{15}{8}m^4 \qquad \text{See p. 53 and 55.}$$

$$\frac{2-\mathrm{c}}{2-2m-\mathrm{c}}+1=2+2m+4m^2$$

$$(2-2m-\mathrm{c})^2-1=-4m+\frac{11}{2}m^2+\frac{177}{16}m^3$$

Let $r_3=\frac{15}{8}m+\frac{187}{32}m^2+A\,m^3$, then

$$\left\{\frac{15}{8}m+\frac{187}{32}m^2+A\,m^3\right\}\left\{-4m+\frac{11}{2}m^2+\frac{177}{16}m^3\right\}$$

$$=\left\{1-4m+\frac{11}{2}m^2\right\}\left\{\frac{3}{2}m^2+\frac{91}{16}m^3+\frac{2657}{192}m^4\right\}$$

$$-\left\{4+4m+8m^2\right\}\left\{\frac{9}{4}m^2+\frac{15}{16}m^3+\frac{235}{64}m^4\right\}+\frac{15}{4}m^4$$

Equating the coefficients of m^4,

$$\frac{2655}{128}+\frac{2057}{64}-4A=\frac{2657}{192}-\frac{91}{4}+\frac{33}{4}-\frac{235}{16}-\frac{15}{4} \qquad 18+\frac{15}{4}$$

$$A=\frac{33113}{1536}$$

M. Plana has

$$\frac{31673}{1536}:$$

these figures may be arrived at by substituting

$$e\left(1-\frac{m^2}{2}\right) \text{ for } e$$

in $\left\{\frac{15}{8}m+\frac{33113}{1536}m^3\right\}e$. The discrepancy arises from coefficient of $\cos\xi$, Arg. 2, not agreeing with that of M. Pla See p. 50.

$$r_4\left\{(2-2m+\mathrm{c})^2-1\right\}=(2-2m+\mathrm{c})^2\mathrm{r}_4-2\left\{\frac{2+\mathrm{c}}{2-2m+\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}$$

$\mathfrak{r}_4 = \frac{3}{2} m^2 + \frac{19}{4} m^3 \qquad R_4 = - \frac{3}{4} m^2$

Let $r_4 = \frac{33}{16} m^2 + A\, m^3$

$$\left\{\frac{33}{16} m^2 + A\, m^3 \right\} \cdot \left\{(3 - 2\, m_{,}^2 - 1\right\} = (3 - 2\, m)^2 \left(\frac{3}{2}\, m^2 + \frac{19}{4} m^3\right)$$

$$- 2 \left(2 + \frac{2\, m}{3}\right) \left(- \frac{3}{4} m^2\right)$$

Equating the coefficients of m^3,

$$A = \frac{101}{16}$$

$$r_1 \left\{(1 + 3\, e^2) \left\{(2 - 2\, m)^2 - 1\right\} = (2 - 2\, m)^2\, \mathfrak{r}_1 - 2 \left\{\frac{2}{2 - 2\, m} + 1\right\} \frac{m \cdot a}{\mu} R_1\right.$$

$r_3 = \frac{15}{8} m + \frac{187}{32} m^2 + \frac{33113}{1536} m^3 \qquad r_4 = \frac{33}{16} m^2 + \frac{101}{16} m^3$

$\mathfrak{r}_1 = \frac{45}{16} m\, e^2 + \frac{759}{64} m^2\, e^2 + \frac{42809}{1024} m^3\, e^2$

$R_1 = \frac{15}{8} m^2\, e^2 + \frac{15}{32} m^3\, e^2 + 9\, m^3\, e_{,}^2$

Let $r_1 = m^2 + \frac{19}{6} m^3 + \frac{15}{4} m\, e^2 + \frac{189}{16} m^2\, e^2 - \frac{5}{2} m^2\, e_{,}^2 + A\, m^3\, e^2 + B\, m^3\, e_{,}^2$

$$\left\{m^2 + \frac{19}{6} m^3 + \frac{15}{4} m\, e^2 + \frac{189}{16} m^2 e^2 - \frac{5}{2}\, m^2\, e_{,}^2 + A\, m^3\, e^2 + B\, m^3\, e_{,}^2\right\}$$

$$\left\{3 - 8\, m + 4\, m^2 + 12\, e^2 - 24\, m\, e^2\right\}$$

$$= (4 - 8\, m + 4\, m^2) \left(\frac{45}{16} m\, e^2 + \frac{759}{64} m^2\, e^2 + \frac{42809}{1124} m^3\, e^2\right)$$

$$- 2\, (2 + m) \left(\frac{15}{8}\, m^2\, e^2 + \frac{15}{32} m^3\, e^2 + 9\, m^3\, e_{,}^2\right)$$

Equating the coefficients of $m^3\, e^2$ and $m^3\, e_{,}^2$,

$$- 24 + 38 + 15 - \frac{189}{2} + 3\, A = \frac{42809}{256} - \frac{759}{8} + \frac{45}{4} + \frac{15}{4} - \frac{15}{8}$$

$$A = \frac{12883}{256}$$

M. Plana has $\frac{11923}{256}$: these figures may be arrived at by substituting $e^2\, (1 - m^2)$ for e^2, in

$$\left\{\frac{15}{4} m + \frac{12883}{256} m^3\right\} e^2$$

$$20 + 3B = -36 - \frac{15}{4}$$

$$B = -\frac{239}{12}.$$

In the equation for determining the inequalities of the recipro of the radius vector,

$$\frac{d^2 . r^2}{2\,d\,t^2} - \frac{\mu}{r}$$

gives the term $c^2 - 1$; hence

$$c^2 - 1 + r_8\left\{4c^2 - 1\right\} = 4c^2 r_8 - 4\frac{m_{,}a}{\mu}R_8$$

$$r_8 = \frac{3}{2}r_2 = m^2 \qquad\qquad R_8{}^{*} = \frac{m^2}{8}$$

$$-\frac{3}{2}m^2 + 3r_8 = 4m^2 - \frac{m^2}{2}$$

$$r_8 = \frac{5}{3}m^2$$

M. Plana has $\frac{2}{3}m^2$: these figures may be arrived at, as in form instances, by substituting $e\left(1 - \frac{m^2}{2}\right)$ for e in my expression.

$$r_9\left\{(2 - 2m - 2c)^2 - 1\right\} = (2 - 2m - 2c)^2 r_9$$

$$-2\left\{\left\{\frac{2 - 2c}{2 - 2m - 2c} + 1\right\}\frac{m_{,}a}{\mu}R_9 + \frac{m_{,}a}{(2 - 2m - 2c)\mu}R'_9\right\}$$

$$r_9 = \frac{45}{16}m \qquad R_9 = -\frac{15}{8}m^2 - \frac{45}{32}m^3 \qquad R'_9 = -\frac{45}{16}m^4$$

Let $r_9 = -\frac{15}{4}m^2 + A\,m^3$

$$-\left\{-\frac{15}{4}m^2 + A\,m^3\right\} = -\frac{45}{4}m^3$$

$$-\left(-\frac{3}{2}m + 2\right)\left(-\frac{15}{8}m^2 - \frac{45}{32}m^3\right) - \frac{45}{16}m^3$$

$$A = -\frac{225}{16}$$

$$r_{10}\left\{(2-2m+2c)^2 - 1\right\}(2-2m+2c)^2 r_{10} - 2\left\{\frac{2+2c}{2-2m+2c} + 1\right\}\frac{m_{,}a}{\mu}R_{10}$$

$$r_{10} = \frac{99}{32}m^2 + \frac{303}{32}m^3 \qquad R_{10} = -\frac{3}{4}m^2$$

---

* The sign of this term in the development of R, p. 31, requires correction.

Let $r_{10} = \frac{7}{2}m^2 + A\,m^3$

$$\left(\frac{7}{2}m^2 + A\,m^3\right)(15-16\,m) = (16-16\,m)\left(\frac{99}{32}m^2 + \frac{303}{32}m_3\right)$$

$$-2\left(2+\frac{m}{2}\right)\left(-\frac{3}{4}m^2\right)$$

$$-56 + 15\,A = \frac{303}{2} - \frac{99}{2} + \frac{3}{4}$$

$$A = \frac{127}{12}$$

$$r_{11}\left\{(\mathrm{c}+m)^2 - 1\right\} = (\mathrm{c}+m)^2\,\mathrm{r}_{11} - 2\left\{\frac{\mathrm{c}}{\mathrm{c}+m} + 1\right\}\frac{m_{,}\,\mathrm{a}}{\mu}R_{11}$$

$$\mathrm{r}_{11} = -\frac{9}{4}m^2 + \frac{123}{32}m^3 \qquad R_{11} = \frac{3}{4}m^2 + \frac{411}{64}m^3$$

Let $r_{11} = -\frac{21}{8}m + A\,m^2$

$$\left\{-\frac{21}{8}m + A\,m^2\right\}\ \left\{2\,m - \frac{1}{2}m^2\right\} = (1+2\,m)\left(-\frac{9}{4}m^2 + \frac{123}{32}m^3\right)$$

$$-2\,(2-m)\left(\frac{3}{4}m^2 + \frac{411}{64}m^3\right)$$

$$\frac{21}{16} + 2\,A = \frac{123}{32} - \frac{9}{2} - \frac{411}{16} + \frac{3}{2}$$

$$A = -\frac{837}{64}$$

$$r_{12}\left\{(2-3\,m-\mathrm{c})^2 - 1\right\} = (2-3\,m-\mathrm{c})^2\,\mathrm{r}_{12} - 2\left\{\frac{2-\mathrm{c}}{2-3\,m-\mathrm{c}} + 1\right\}\frac{m_{,}\,\mathrm{a}}{\mu}R_{12}$$

$$\mathrm{r}_{12} = \frac{21}{4}m^2 + \frac{471}{16}m^3 - \frac{191}{32}m^3 = \frac{21}{4}m^2 + \frac{751}{32}m^3$$

$$R_{12} = \frac{63}{8}m^2 + \frac{71}{16}m^3$$

Let $r_{12} = \frac{35}{8}m + A\,m^2$

$$\left(\frac{35}{8}m + A\,m^2\right)\left(-6\,m + \frac{21}{2}m^2\right) = (1-6\,m)\left(\frac{21}{4}m^2 + \frac{751}{32}m^3\right)$$

$$-2\,(2+3\,m)\left(\frac{63}{8}m^2 + \frac{71}{16}m^3\right)$$

$$\frac{735}{16} - 6A = \frac{751}{32} - \frac{126}{4} - \frac{71}{4} - \frac{189}{4}$$

$$A = \frac{1269}{64}$$

$$r_{13}\left\{(2-m+\mathrm{c})^2-1\right\} = (2-m+\mathrm{c})^2\,\mathfrak{r}_{13} - 2\left\{\frac{2+\mathrm{c}}{2-m+\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{13}$$

$$\mathfrak{r}_{13} = -\frac{3}{4}m^2 - \frac{91}{16}m^3 - \frac{63}{16}m^3 = -\frac{3}{4}m^2 - \frac{154}{16}m^3$$

$$R_{13} = \frac{3}{8}m^2 + \frac{135}{32}m^3$$

Let $r_{13} = -\frac{33}{32}m^2 + A\,m^3$

$$\left(-\frac{33}{32}m^2 + A\,m^3\right)(8-6\,m) = (9-6\,m)\left(-\frac{3}{4}m^2 - \frac{154}{16}m^3\right)$$

$$-2\left(2+\frac{m}{3}\right)\left(\frac{3}{8}m^2 + \frac{135}{32}m^3\right)$$

$$\frac{99}{16} + 8\,A = -\frac{1386}{16} + \frac{18}{4} - \frac{135}{8} - \frac{1}{4}$$

$A = -\frac{1727}{128}$. M. Plana has $-\frac{5477}{384}$.

$$r_{14}\left\{(\mathrm{c}-m)^2-1\right\} = (\mathrm{c}-m)^2\,\mathfrak{r}_{14} - 2\left\{\frac{\mathrm{c}}{\mathrm{c}-m}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{14}$$

$\mathfrak{r}_{14} = -\frac{9}{4}m^2 + \frac{267}{32}m^3$ $\quad R_{14} = \frac{3}{4}m^2 + \frac{759}{64}m^3$ $\quad r_{14} = \frac{21}{8}\,m + A$

$$\left(\frac{21}{8}m + A\,m^2\right)\left(-2\,m - \frac{1}{2}m^2\right) = (1-2\,m)\left(-\frac{9}{4}m^2 + \frac{267}{32}m^3\right)$$

$$-2\,(2+m)\left(\frac{3}{4}m^2 + \frac{759}{64}m^3\right)$$

$$-\frac{21}{16} - 2\,A = \frac{267}{32} + \frac{9}{2} - \frac{759}{16} - \frac{3}{2}$$

$$A = \frac{1113}{64}$$

$$r_{15}\left\{(2-m-\mathrm{c})^2-1\right\} = (2-2\,m-\mathrm{c})^2\,\mathfrak{r}_{15} - 2\left\{\frac{2-\mathrm{c}}{2-m-\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{15}$$

$$\mathfrak{r}_{15} = -\frac{3}{4}m^2 - \frac{91}{16}m^3 - \frac{39}{32}m^3 = -\frac{3}{4}m^2 - \frac{221}{32}m^3$$

$R_{15} = -\frac{9}{8}m^2 - \frac{3}{8}m^3$ $\quad r_{15} = -\frac{15}{8}m + A\,m^2$

$$\left(-\frac{15}{8}\,m + A\,m^2\right)\left(-2\,m + \frac{5}{2}m^2\right) = (1-2\,m)\left(-\frac{3}{4}m^2 - \frac{221}{32}m^3\right)$$

$$-2\,(2+m)\left(-\frac{9}{8}m^2 - \frac{3}{8}m^3\right)$$

$$= \frac{75}{16} - 2\,A = -\frac{221}{32} + \frac{3}{2} + \frac{3}{2} + \frac{9}{4}$$

$$A = -\frac{97}{64}$$

$$r_{16}\left\{(2-3m+\mathrm{c})^2-1\right\} = (2-3m+\mathrm{c})^2\,\mathrm{r}_{16} - 2\left\{\frac{2+\mathrm{c}}{2-3m+\mathrm{c}}+1\right\}\frac{m_{,}\,\mathrm{a}}{\mu}R_{16}$$

$$\mathrm{r}_{16} = \frac{21}{4}m^2 + \frac{417}{16}m^3 + \frac{63}{16}m^3 = \frac{21}{4}m^2 + \frac{534}{16}m^3$$

$$R_{16} = -\frac{21}{8}m^2 - \frac{135}{32}m^3 \qquad\qquad r_{16} = \frac{231}{32}\,m^2 + A\,m^3$$

$$\left(\frac{231}{32}\,m^2 + A\,m^3\right)(8-18\,m) = (9-18\,m)\left(\frac{21}{4}m^2 + \frac{534}{16}\,m^3\right)$$

$$-\,2\,(2+m)\left(-\frac{21}{8}m^2 - \frac{135}{32}\,m^3\right)$$

$$-\frac{4158}{32} + 8\,A = \frac{4806}{16} - \frac{378}{4} + \frac{135}{8} + \frac{21}{4}$$

$$A = \frac{5727}{128}.$$

$$r_{18}\left\{(2-4\,m)^2-1\right\} = (2-4\,m)^2\,\mathrm{r}_{18} - 2\left\{\frac{2}{2-4\,m}+1\right\}\frac{m_{,}\,\mathrm{a}}{\mu}R_{18}$$

$$\mathrm{r}_{18} = 0 \qquad\qquad R_{18} = -\frac{51}{8}m^2 - \frac{153}{16}\,m^3 \qquad\qquad r_{18} = \frac{17}{2}m^2 + A\,m^3$$

$$\left(\frac{17}{2}m^2 + A\,m^3\right)(3-16\,m) = -\,2\,(2+m)\left(-\frac{51}{8}\,m^2 - \frac{153}{16}m^3\right)$$

$$-\frac{272}{2} + 3\,A = \frac{153}{4} + \frac{51}{2}$$

$$A = \frac{799}{12}$$

$$3\,r_{19} = 4\,\mathrm{r}_{19} - 4\,\frac{m_{,}\,\mathrm{a}}{\mu}R_{19} \qquad\qquad R_{19} = \frac{9}{16}m^3$$

$$r_{19} = -\,\frac{3}{4}m^3$$

$$r_{132}\left\{(4-4\,m-\mathrm{c})^2-1\right\} = (4-4\,m-\mathrm{c})^2\,\mathrm{r}_{132}$$

$$-\,2\left\{\left\{\frac{4-\mathrm{c}}{4-4\,m-\mathrm{c}}+1\right\}\frac{m_{,}\,\mathrm{a}}{\mu}R_{132} + \frac{m_{,}\,\mathrm{a}}{(4-4\,m-\mathrm{c})\,\mu}R'_{132}\right\}$$

$$\mathrm{r}_{132} = \frac{45}{16}\,m^3 \qquad\qquad R_{132} = -\,\frac{45}{32}m^3$$

$$8\,r_{132} = \frac{9\,.\,45}{16}\,m^3 + \frac{4\,.\,45}{32}\,m^3 \qquad\qquad r_{132} = \frac{495}{128}m^3.$$

Now, let

$$r_{132} = \frac{495}{128}\,m^3 + A\,m^4$$

$$R_{132} = -\,\frac{45}{32}m^3 - \frac{957}{128}m^4 \qquad\qquad R'_{132} = \frac{45}{16}m^4 \qquad \text{(See p. 54 and 55.)}$$

$$\mathfrak{r}_{132} = \frac{45}{16} m^3 + \frac{21}{16} m^4 + \frac{465}{64} m^4 = \frac{45}{16} m^3 + \frac{549}{64} m^4$$

$$\left(\frac{495}{128} m^3 + A m^4\right)(8 - 24\, m) = (9 - 24\, m)\left(\frac{45}{16} m^3 + \frac{549}{64} m^4\right)$$

$$- 2\left(2 + \frac{4}{3} m\right)\left(-\frac{45}{32} m^3 - \frac{957}{128} m^4\right) - \frac{2\,.\,45}{3\,.\,16} m^4$$

$$-\frac{495\,.\,24}{128} + 8\, A = \frac{9\,.\,549}{64} - \frac{24\,.\,45}{16} + \frac{8\,.\,15}{32} + \frac{2\,.\,957}{64} - \frac{30}{16}$$

$A = \frac{8595}{512}$ M. Plana has $\frac{13725}{512}$.

$$r_{133}\left\{(4 - 4\, m + \mathrm{c})^2 - 1\right\} = (4 - 4\, m + \mathrm{c})^2\, \mathfrak{r}_{133} - 2\left\{\frac{4 + \mathrm{c}}{4 - 4\, m + \mathrm{c}} + 1\right\}\frac{m_{,}\mathrm{a}}{\mu} R_{133}$$

$R_{133} = -\frac{45}{64} m^4$ $\mathfrak{r}_{133} = \frac{21}{16} m^4 + \frac{51}{32} m^4 = \frac{93}{32} m^4$

$24\, r_{133} = \frac{25\,.\,93}{32} m^4 + \frac{4\,.\,45}{64} m^4$ $r_{133} = \frac{805}{256} m^4$

$$r_{131}\left\{(4 - 4\, m)^2 - 1\right\} = (4 - 4\, m)^2\, \mathfrak{r}_{131} - 2\left\{\frac{4}{4 - 4\, m} + 1\right\}\frac{m_{,}\mathrm{a}}{\mu} R_{131}$$

$\mathfrak{r}_{131} = \frac{1485}{256} m^3 e^2 + \frac{1485}{256} m^3 e^2 = \frac{1485}{128} m^3 e^2$ $R_{131} = -\frac{45}{16} m^3 e^2$

$$15\, r_{131} = \frac{16\,.\,1485}{128} m^3 e^2 + \frac{2\,.\,2\,.\,45}{16} m^3 e^2$$

$r_{131} = \frac{105}{8} m^3 e^2$ M. Plana has $-\frac{45}{64} m^3 e^2$.

$$r_{136}\left\{(4 - 4\, m - 2\, \mathrm{c})^2 - 1\right\} = (4 - 4\, m - 2\, \mathrm{c})^2\, \mathfrak{r}_{136} - 2\left\{\frac{4 - 2\, \mathrm{c}}{4 - 4\, m - 2\, \mathrm{c}} + 1\right\}\frac{m_{,}\mathrm{a}}{\mu} R_{136}$$

$$\mathfrak{r}_{136} = \frac{675}{256} m^2 + \frac{1485}{256} m^3 + \frac{5535}{12} m^3 = \frac{675}{256} m^2 + \frac{358695}{768} m^3$$

$$R_{136} = \frac{45}{32} m^3$$

$$\left\{\frac{225}{64} m^2 + A m^3\right\}\left\{3 - 16\, m\right\} = (4 - 16\, m)\left\{\frac{675}{256} m^2 + \frac{358695}{768} m^3\right\} - \frac{2\,.\,2\,.\,45}{32} m^3$$

$$-\frac{225}{4} + 3\, A = \frac{358695}{192} - \frac{675}{16} - \frac{45}{8}$$

$A = \frac{39795}{64}$ M. Plana has $\frac{3645}{128}$.

$$r_{138}\left\{(4 - 5\, m - \mathrm{c})^2 - 1\right\} = (4 - 5\, m - \mathrm{c})^2\, \mathfrak{r}_{138} - 2\left\{\frac{4 - \mathrm{c}}{4 - 5\, m - \mathrm{c}} + 1\right\}\frac{m_{,}\mathrm{a}}{\mu} R_{138}$$

$\mathfrak{r}_{138} = \frac{525}{32} m^3$ $R_{138} = -\frac{525}{64} m^3$

$$8\, r_{138} = \frac{9\,.\,525}{32}\, m^3 + \frac{4\,.\,525}{64}\, m^3 \qquad\qquad r_{138} = \frac{5775}{256}\, m^3$$

$$r_{140}\left\{(4 - 3\,m - \mathrm{c})^2 - 1\right\} = (4 - 3\,m - \mathrm{c})^2\, \mathfrak{r}_{140} - 2\left\{\frac{4 - \mathrm{c}}{4 - 3\,m - \mathrm{c}} + 1\right\}\frac{m_{,}\,\mathrm{a}}{\mu}\, R_{140}$$

$$\mathfrak{r}_{140} = -\frac{135}{32}\, m^3 \qquad\qquad R_{140} = \frac{135}{64}\, m^3$$

$$8\, r_{140} = -\frac{9\,.\,135}{32}\, m^3 - \frac{2\,.\,2\,.\,135}{64}\, m^3 \qquad\qquad r_{140} = -\frac{1485}{256}\, m^3$$

The expression for the longitude is to be obtained from the equation

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,t} = \frac{h}{r^2} - \frac{1}{r^2}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t.$$

Let

$$\frac{\mathrm{a}^2}{r^2} = \mathfrak{a} + \mathfrak{b}\, m + \mathfrak{c}\, m^2 = \frac{\mathrm{a}^2}{\mathrm{r}^2} + \frac{2\,\mathrm{a}}{\mathrm{r}}\,\delta\,\frac{1}{r} + \mathrm{a}^2\left(\delta\,\frac{1}{r}\right)^2$$

and let

$$h - \int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t = \sqrt{\mathrm{a}\,\mu}\,\{\mathfrak{A} + \mathfrak{B}\, m + \mathfrak{C}\, m^2\}$$

$$h = \sqrt{\mathrm{a}\,\mu}\left\{1 - \frac{e^2}{2} - \frac{m^2}{3} - \frac{803}{384}\, m^2 e^2 - \frac{m^2\, e_{,}^2}{2}\right\}.$$ See Phil. Trans. 1834, p. 139.

$$\frac{\mathrm{d}\,\lambda}{\mathrm{n}\,\mathrm{d}\,t} = \{\mathfrak{a}\;\mathfrak{C} + \mathfrak{b}\;\mathfrak{B} + \mathfrak{c}\;\mathfrak{A}\}\, m^2 + \&\mathrm{c}.$$

$$\mathfrak{a} = \frac{\mathrm{a}^2}{\mathrm{r}^2} = 1 + \frac{e^2}{2} + 2\,e\cos\xi + \frac{5}{2}\,e^2\cos 2\,\xi$$

$$\begin{aligned}\mathfrak{C} = {} & -\frac{1}{3} - \frac{803}{384}\,e^2 - \frac{e_{,}^2}{2} + \left\{\frac{3}{4} - \frac{15}{8}\,e^2 - \frac{15}{8}\,e_{,}^2\right\}\cos 2\,\tau - \frac{9}{2}\,e\cos(2\,\tau - \xi) \\ & + \frac{e}{2}\cos(2\,\tau + \xi) + \frac{21}{8}\,e_{,}\cos(2\,\tau - \xi_{,}) - \frac{3}{8}\,e_{,}\cos(2\,\tau + \xi_{,}) \\ & - \frac{45}{32}\,e^2\cos(2\,\tau - 2\,\xi) + \frac{3}{8}\,e^2\cos(2\,\tau + 2\,\xi) - \frac{63}{4}\,e\,e_{,}\cos(2\,\tau - \xi - \xi_{,}) \\ & - \frac{e\,e_{,}}{4}\cos(2\,\tau + \xi + \xi_{,}) + \frac{9}{4}\,e\,e_{,}\cos(2\,\tau - \xi + \xi_{,}) \\ & + \frac{7}{4}\,e\,e_{,}\cos(2\,\tau + \xi - \xi_{,}) + \frac{51}{8}\,e_{,}^2\cos(2\,\tau - 2\,\xi_{,}).\end{aligned}$$

The term of which the argument is $2\,\tau - 2\,\xi$ in the preceding value of $\mathfrak{C}$ arises thus. $\frac{\mathrm{a}\,m_{,}}{\mu}\,R$ contains the term

$$+\frac{15}{4}m^2 e^2 \cos(2\tau - 2\xi)$$

This term gives in $\frac{\mathrm{a}\, m_{\prime}}{\mu}\int \frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t$ the term

$$+\left\{-\frac{15}{8}m - \frac{45}{32}m^2\right\} e^2 \cos(2\tau - 2\xi)$$

and this value is so far complete, because $\delta\frac{\mathrm{d}R}{\mathrm{d}\lambda}$ contains no term of which the argument is $2\tau - 2\xi$ multiplied by $m^3 e^2$.

In order to have the terms multiplied by e^2, $e\,e_{\prime}$ and $e_{\prime}^2$ only, $\mathfrak{b}$ may be taken equal to zero, because

$$\mathfrak{B} = -\frac{15}{8}e^2 \cos(2\tau - 2\xi),$$

and because there is no term in $\mathfrak{b}$ independent of the eccentricities.

$$\frac{\mathrm{a}}{\mathrm{r}} = 1 + e\cos\xi + e^2 \cos 2\xi$$

$$\mathfrak{C} = \frac{1}{3} + \frac{225}{128}e^2 + \frac{e_{\prime}^2}{2} + \left(2 - 5e_{\prime}^2 + \frac{189}{8}e^2\right)\cos 2\tau + \frac{4}{3}e\cos\xi$$

$$+\frac{187}{16}e\cos(2\tau - \xi) + \frac{33}{8}e\cos(2\tau + \xi) - 3e_{\prime}\cos\xi_{\prime} + 7e_{\prime}\cos(2\tau - \xi_{\prime})$$

$$-e_{\prime}\cos(2\tau + \xi_{\prime}) + \frac{10}{3}e^2\cos 2\xi - \frac{15}{2}e^2\cos(2\tau - 2\xi) + 7e^2\cos(2\tau + 2\xi)$$

$$-\frac{837}{32}e\,e_{\prime}\cos(\xi + \xi_{\prime}) + \frac{1269}{32}e\,e_{\prime}\cos(2\tau - \xi - \xi_{\prime}) - \frac{33}{16}e\,e_{\prime}\cos(2\tau + \xi + \xi_{\prime})$$

$$+\frac{1113}{32}e\,e_{\prime}\cos(\xi + \xi_{\prime}) - \frac{97}{32}e\,e_{\prime}\cos(2\tau - \xi + \xi_{\prime}) + \frac{231}{16}e\,e_{\prime}\cos(2\tau + \xi - \xi_{\prime})$$

$$-\frac{9}{2}e_{\prime}^2\cos 2\xi_{\prime} + 17e_{\prime}^2\cos(2\tau - 2\xi_{\prime}) + \frac{1125}{128}e^2\cos(4\tau - 2\xi)$$

$$+2e\cos\xi\left\{\cos 2\tau + \frac{2e}{3}\cos\xi + \frac{187}{32}e\cos(2\tau - \xi) + \frac{33}{16}e\cos(2\tau + \xi)\right.$$

$$\left.-\frac{3}{2}e_{\prime}\cos\xi_{\prime} + \frac{7}{2}e_{\prime}\cos(2\tau - \xi_{\prime}) - \frac{e_{\prime}}{2}\cos(2\tau + \xi_{\prime})\right\}$$

$$+2e^2\cos 2\xi\left\{\frac{1}{6} + \cos 2\tau\right\}$$

$\mathfrak{C}$ contains the terms $\frac{225}{125}e^2$ and $\frac{225}{125}e^2\cos(4\tau - 2\xi)$ which are the only terms arising from $\left(\delta\frac{1}{r}\right)^2$.

$$\mathfrak{a} = 1 - \frac{e^2}{2}$$

$$\frac{d\lambda}{n\,dt} = m^2 \Bigg\{ -\frac{1}{3} + \frac{1}{3} + \Big\{ -\frac{803}{384} - \frac{1}{6} + \frac{225}{128} - \frac{1}{6} + \frac{2}{3} \Big\} e^2 + \Big\{ -\frac{1}{2} + \frac{1}{2} \Big\} e_{,}^2$$

$$+ \Bigg\{ \frac{3}{4} + 2 + \Big\{ -\frac{15}{8} + \frac{3}{8} - \frac{9}{2} + \frac{1}{2} + \frac{189}{8} - 1 + \frac{187}{32} + \frac{33}{16} \Big\} e^2$$

$$+ \Big\{ -\frac{15}{8} - \frac{5}{2} \Big\} e_{,}^2 \Bigg\} \cos 2\tau$$

$$+ \Big\{ -\frac{2}{3} + \frac{4}{3} \Big\} e \cos \xi + \Big\{ \frac{9}{2} + \frac{3}{4} + \frac{187}{16} + 1 \Big\} e \cos (2\tau - \xi)$$

$$+ \Big\{ \frac{1}{2} + \frac{3}{4} + \frac{33}{8} + 1 \Big\} e \cos (2\tau + \xi) - 3 e_{,} \cos \xi_{,} + \Big(7 + \frac{21}{8} \Big) e_{,} \cos (2\tau - \xi_{,})$$

$$+ \Big\{ -1 - \frac{3}{8} \Big\} e_{,} \cos (2\tau + \xi_{,}) + \Big\{ -\frac{5}{6} + \frac{10}{3} + \frac{1}{3} + \frac{2}{3} \Big\} e^2 \cos 2\xi$$

$$+ \Big\{ -\frac{45}{32} - \frac{9}{2} + \frac{15}{16} - \frac{15}{2} + 1 + \frac{187}{32} \Big\} e^2 \cos (2\tau - 2\xi)$$

$$+ \Big\{ \frac{3}{8} + \frac{1}{2} + \frac{15}{16} + 7 + 1 + \frac{33}{16} \Big\} e^2 \cos (2\tau + 2\xi)$$

$$+ \Big\{ -\frac{837}{32} - \frac{3}{2} \Big\} e\, e_{,} \cos (\xi + \xi_{,})$$

$$+ \Big\{ -\frac{63}{4} + \frac{21}{8} + \frac{1269}{32} + \frac{7}{2} \Big\} e\, e_{,} \cos (2\tau - \xi - \xi_{,})$$

$$+ \Big\{ -\frac{1}{4} - \frac{3}{8} - \frac{33}{16} - \frac{1}{2} \Big\} e\, e_{,} \cos (2\tau + \xi + \xi_{,})$$

$$+ \Big\{ \frac{1113}{32} - \frac{3}{2} \Big\} e\, e_{,} \cos (\xi - \xi_{,})$$

$$+ \Big\{ \frac{9}{4} - \frac{3}{8} - \frac{97}{32} - \frac{1}{2} \Big\} e\, e_{,} \cos (2\tau - \xi + \xi_{,})$$

$$+ \Big\{ \frac{7}{4} + \frac{21}{8} + \frac{231}{16} + \frac{7}{2} \Big\} e\, e_{,} \cos (2\tau + \xi - \xi_{,})$$

$$- \frac{9}{2} e_{,}^2 \cos 2\xi_{,} + \Big\{ \frac{51}{8} + 17 \Big\} e_{,}^2 \cos (2\tau - 2\xi_{,}) + \frac{1125}{128} e^2 \cos (4\tau - 2\xi) \Bigg\}$$

$$= m^2 \Bigg\{ \Big\{ \frac{11}{4} + \frac{801}{32} e^2 - \frac{55}{16} e_{,}^2 \Big\} \cos 2\tau + \frac{2}{3} e \cos \xi + \frac{143}{16} e \cos (2\tau - \xi)$$

$$+ \frac{51}{8} e \cos (2\tau + \xi) - 3 e_{,} \cos \xi_{,} + \frac{77}{8} e_{,} \cos (2\tau - \xi_{,}) - \frac{11}{8} e_{,} \cos (2\tau + \xi_{,})$$

$$+ \frac{21}{6} e^2 \cos 2\xi - \frac{45}{8} e^2 \cos (2\tau - 2\xi) + \frac{95}{8} e^2 \cos (2\tau + 2\xi) - \frac{805}{32} e\, e_{,} \cos (\xi + \xi_{,})$$

$$+\frac{961}{32}e\,e_{,}\cos(2\tau-\xi-\xi_{,})-\frac{51}{16}e\,e_{,}\cos(2\tau+\xi+\xi_{,})+\frac{1065}{32}e\,e_{,}\cos(\xi-\xi_{,})$$

$$-\frac{53}{32}e\,e_{,}\cos(2\tau-\xi+\xi_{,})+\frac{357}{16}e\,e_{,}\cos(2\tau+\xi-\xi_{,})-\frac{9}{2}e_{,}^{2}\cos 2\,\xi_{,}$$

$$\left.+\frac{187}{8}e_{,}^{2}\cos(2\tau-2\,\xi_{,})+\frac{1125}{128}e^{2}\cos(4\tau-2\,\xi)\right\}$$

The following terms occur in the value of λ, (see p. 42.)

$$\frac{75\,m\,e^{2}}{16}\sin 2\tau+\frac{15}{4}m\,e\sin(2\tau-\xi)-\frac{21}{4}m\,e\,e_{,}\sin(\xi+\xi_{,})$$

$$+\frac{35}{4}m\,e\,e_{,}\sin(2\tau-\xi-\xi_{,})+\frac{21}{4}m\,e\,e_{,}\sin(\xi-\xi_{,})$$

$$-\frac{15}{4}m\,e\,e_{,}\sin(2\tau-\xi+\xi_{,})$$

By differentiating these, the terms in $\frac{d\lambda}{n\,dt}$ multiplied by m ma be obtained, and considering only these arguments,

$$\frac{d\lambda}{n\,dt}=\left\{\frac{75}{8}m+\frac{801}{32}m^{2}\right\}e^{2}\cos 2\tau$$

$$+\left\{\frac{15}{4}m+\frac{143}{16}m^{2}\right\}e\cos(2\tau-\xi)$$

$$+\left\{-\frac{21}{4}m-\frac{805}{32}m^{2}\right\}e\,e_{,}\cos(\xi+\xi_{,})$$

$$+\left\{\frac{35}{4}m+\frac{961}{32}m^{2}\right\}e\,e_{,}\cos(2\tau-\xi-\xi_{,})$$

$$+\left\{\frac{21}{4}m+\frac{1065}{32}m^{2}\right\}e\,e_{,}\cos(\xi-\xi_{,})$$

$$+\left\{-\frac{15}{4}m-\frac{53}{32}m^{2}\right\}e\,e_{,}\cos(2\tau-\xi+\xi)$$

Integrating

$$\lambda=\left\{\frac{75}{16}m+\frac{1101}{64}m^{2}\right\}e^{2}\sin 2\tau$$

$$+\left\{\frac{15}{4}m+\frac{263}{16}m^{2}\right\}e\cos(2\tau-\xi)$$

$$+\left\{-\frac{21}{4}m-\frac{717}{32}m^{2}\right\}e\,e_{,}\sin(\xi+\xi_{,})$$

$$+ \left\{\frac{35}{4} m + \frac{1801}{32} m^2\right\} e\, e_{,} \sin(2\tau - \xi - \xi_{,})$$

$$+ \left\{\frac{21}{4} m + \frac{1233}{32} m^2\right\} e\, e_{,} \cos(\xi - \xi_{,})$$

$$+ \left\{-\frac{15}{4} m - \frac{173}{32} m^2\right\} e\, e_{,} \sin(2\tau - \xi + \xi_{,})$$

These terms are identical with those given by M. Plana.

$\frac{\mathrm{d}\lambda}{\mathrm{n\,d}t}$ contains the term

$$\left\{\frac{5}{2} + \frac{21}{6} m^2\right\} e^2 \cos 2\xi,$$

which being integrated, gives

$$\left\{\frac{5}{4} + \frac{43}{16} m^2\right\} e^2 \sin 2\xi.$$

M. Plana has $\left\{\frac{5}{4} + \frac{23}{16} m^2\right\} e^2 \sin 2\xi$: these figures may be arrived at by substituting, as before, $e^2(1 - m^2)$, for e^2 in my expression.

The relation between the constants h and a is to be obtained from the equations

$$\frac{r^2\,\mathrm{d}\lambda}{\mathrm{d}t} = h - \int \frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t \qquad \text{and}$$

$$\frac{r^2\,\mathrm{d}\lambda^2 + \mathrm{d}r^2}{\mathrm{d}t^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}R = 0.$$

Neglecting the eccentricities,

$$\frac{r^2}{a^2} = 1 + m^2 - 2m^2 \cos 2\tau$$

$$\cos(2\lambda - 2\lambda_{,}) = \cos 2\tau - \frac{11}{4} m^2 \sin^2 2\tau$$

$$= \cos 2\tau - \frac{11}{8} m^2 + \frac{11}{8} m^2 \cos 4\tau$$

$$m = \frac{n_{,}}{\mathrm{n}} \qquad \mathrm{n}^2 = n^2(1 - 2m^2) \qquad n_{,}^2 = n^2 m^2 (1 - 2m^2)$$

$$R = -\frac{m_{,} r^2}{r_{,}^3}\{1 + 3\cos(2\lambda - 2\lambda_{,})\}$$

$$= -\frac{n_{\prime}^2 a^2}{2}\left\{1 + m^2 - 2 m^2 \cos 2\tau\right\}$$

$$\left\{1 + 3\cos 2\tau - \frac{33}{8} m^2 + \frac{33}{8} m^2 \cos 4\tau\right\}$$

$$= -\frac{m^2 (1 - 2m^2)^*}{2}\left\{1 + m^2 - 2 m^2 \cos 2\tau\right\}$$

$$\left\{1 + 3\cos 2\tau - \frac{33}{8} m^2 + \frac{33}{8} m^2 \cos 4\tau\right\}$$

$$= -\frac{m^2}{2} + \frac{m^4}{2} + \frac{3}{2} m^4 + \frac{33}{16} m^4 +, \&c.$$

$$= -\frac{m^2}{2} + \frac{65}{16} m^4$$

Hence, if $\frac{a}{r} = 1 + r_0 +$ periodic terms

$$r_0 = -\frac{m^2}{2} + \frac{65}{16} m^4$$

$$h^2 = a\mu\left\{1 - \frac{145}{32} m^4\right\}. \text{ (See p. 65.)}$$

Considering only the constant term in the equation

$$\frac{d\lambda}{dt} = \frac{h}{r^2} - \frac{1}{r^2}\int \frac{dR}{d\lambda} dt$$

$$\frac{d\lambda}{dt} = \sqrt{\frac{\mu}{a^3}}\left\{1 - \frac{145}{64} m^4\right\}\left\{1 + 2r_0 + r_0^2 + \frac{r_1^2}{2} + \frac{3}{4} m^4\right\}$$

$$= \sqrt{\frac{\mu}{a^3}}\left\{1 - \frac{145}{64} m^4 - m^2 + \frac{65}{8} m^4 + \frac{m^4}{4} + \frac{m^4}{2} + \frac{3}{4} m^4\right\}$$

$$= \sqrt{\frac{\mu}{a^3}}\left\{1 - m^2 + \frac{471}{64} m^4\right\}$$

If this quantity be called n and if $\mathrm{n} = \sqrt{\frac{\mu}{\mathrm{a}^3}}$

$$\mathrm{a} = a\left\{1 - m^2 + \frac{471}{64} m^4\right\}^{-\frac{2}{3}} = a\left\{1 + \frac{2}{3} m^2 - \frac{1253}{288} m^4\right\}$$

$$\frac{\mathrm{a}}{r} = \left\{1 - \frac{m^2}{2} + \frac{65}{16} m^4\right\}\left\{1 + \frac{2}{3} m^2 - \frac{1253}{288} m^4\right\}$$

---

* This investigation is intended to correct a mistake in p. 65, which arose from neglecting the factor $1 - 2m^2$.

$= 1 + \frac{m^2}{6} - \frac{179}{288} m^4 +$ periodic terms, and terms dependent upon the eccentricities.

$$h = \sqrt{a\mu}\left\{1 - \frac{145}{64} m^4\right\}\left\{1 - m^2 + \frac{471}{64} m^4\right\}^{\frac{1}{2}}$$

$$= \sqrt{a\mu}\left\{1 - \frac{m^2}{3} + \frac{11}{144} m^4\right\}$$

I shall now endeavour to obtain the terms in c, proportional to m^3 and m^4, by extending to these terms the method employed in p. 49, for finding the value of c.

$$a\,\delta\frac{1}{r} = \frac{m^2}{6} - \frac{179}{288} m^4 + \left\{\frac{2}{3} m^2 - \frac{1253}{288} m^4\right\} e \cos \xi$$

$$r^3 = a^3\left\{1 - 3\,e \cos \xi\right\} \qquad r^4 = a^4\left\{1 - 4\,e \cos \xi\right\}$$

$$r^3\,\delta\frac{1}{r} = a^2\left\{-\frac{m^2}{2} + \frac{179}{96} m^4 + \frac{2}{3} m^2 - \frac{1253}{288} m^4\right\}$$

$$= a^2\left\{-\frac{m^2}{6} - \frac{179}{72} m^4\right\} e \cos \xi$$

$$\frac{3}{2} r^4\,\delta\left(\frac{1}{r}\right)^2 = a^2\left\{\frac{45}{16} m^3 + \frac{3443}{192} m^4\right\} e \cos \xi$$ See p. 55 and the Errata.

$$\frac{\delta r^2}{2} = \frac{r^2}{2} - r^3\,\delta\frac{1}{r} + \frac{3}{2} r^4\left(\delta\frac{1}{r}\right)^2$$

$$= a^2\left\{-1 - \frac{m^2}{6} + \frac{179}{72} m^4 + \frac{45}{16} m^3 + \frac{3443}{192} m^4\right\} e \cos \xi$$

$$= a^2\left\{-1 - \frac{m^2}{6} + \frac{45}{16} m^3 + \frac{11761}{576} m^4\right\} e \cos \xi$$

Hence,

$$c^2\left\{1 + \frac{m^2}{6} - \frac{45}{16} m^3 - \frac{11761}{576} m^4\right\}$$

$$= 1 + \frac{2}{3} m^2 - \frac{1253}{288} m^4 - 4\left\{\frac{m^2}{2} + \frac{135}{32} m^3 + \frac{5999}{384} m^4 + \frac{135}{32} m^4\right\}$$

$$c^2 = \left\{1 - \frac{4}{3} m^2 - \frac{135}{8} m^3 - \frac{24110}{288} m^4\right\}\left\{1 - \frac{m^2}{6} + \frac{45}{16} m^3 + \frac{11761}{576} m^4 + \frac{m^4}{36}\right\}$$

$$= 1 - \frac{3}{2} m^2 - \frac{225}{16} m^3 - \frac{4035}{64} m^4$$

$$c = 1 - \frac{3}{4} m^2 - \frac{225}{32} m^3 - \frac{4071}{128} m^4$$

The preceding method is that employed p. 67, where, however, a different value is obtained, partly in consequence of the mistake already noticed in p. 134, and partly on account of a mistake in p. 55*, for the discovery of which I am indebted to M. de Pontécoulant. The following method, which leads to the same result, is more simple.

In the equation for determining the inequalities of the reciprocal of the radius vector,

$$\frac{d^2 . r^2}{2 d t^2} - \frac{\mu}{r}$$

gives the term $c^2 - 1$; hence

$$c^2 - 1 + r_2 \left\{ c^2 - 1 \right\} = c^2 r_2 - 4 \frac{m_{,} a}{\mu} R_2 - 2 \frac{m_{,} a}{\mu} R'_2$$

$$r_2 = \frac{m^2}{2} - \frac{179}{96} m^4 + \frac{45}{16} m^3 + \frac{3443}{192} m^4 = \frac{m^2}{2} + \frac{45}{16} m^3 + \frac{3085}{192} m^4$$

Let $c^2 = 1 - \frac{3}{2} m^2 + A m^3 + B m^4$, then

$$\left\{ -\frac{3}{2} m^2 + A m^3 + B m^4 \right\} \left\{ 1 + \frac{2}{3} m^2 \right\} = \left(1 - \frac{3}{2} m^2\right) \left(\frac{m^2}{2} + \frac{45}{16} m^3 + \frac{3085}{192} m^4\right)$$

$$- 4 \left\{ \frac{m^2}{2} + \frac{135}{32} m^3 + \frac{5999}{384} m^4 + \frac{135}{32} m^4 \right\}$$

$$A = \frac{45}{16} - \frac{135}{8} = - \frac{225}{16}$$

$$1 + B = - \frac{3}{4} + \frac{3085}{192} - \frac{5999}{96} - \frac{135}{8}$$

$$B = - \frac{4035}{64}$$

$$c^2 = 1 - \frac{3}{2} m^2 - \frac{225}{16} m^3 - \frac{4035}{64} m^4 \text{ as before.}$$

I shall now endeavour to obtain the terms in $\frac{a}{r}$ and in λ, of which the arguments are 2τ, 4τ, and 6τ, independent of the eccentricities, and multiplied by m^5 and m^6. The following are the terms already obtained. (See p. 55 and p. 64.)

$$\frac{a}{r} = 1 + \frac{m^2}{6} - \frac{179}{288} m^4 + \left\{ m^2 + \frac{19}{6} m^3 + \frac{131}{18} m^4 \right\} \cos 2\tau$$

$$+ \frac{7}{8} m^4 \cos 4\tau$$

$$\lambda = n t + \left\{ \frac{11}{8} m^2 + \frac{59}{12} m^3 + \frac{893}{72} m^4 \right\} \sin 2\tau + \frac{201}{256} m^4 \sin 4\tau$$

$$a \delta \frac{1}{r} - \frac{m^2}{6} - \frac{179}{128} m^4 + \left\{ m^2 + \frac{19}{6} m^3 + \frac{131}{18} m^4 \right\} \cos 2\tau + \frac{7}{8} m^4 \cos 4\tau$$

---

* See the Errata.

$$\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2 = \left\{\frac{1}{2}+\frac{1}{36}\right\}m^4 + \frac{19}{6}m^5 + \left\{\frac{131}{18}+\frac{361}{72}-\frac{179}{864}\right\}m^6$$

$$+\left\{\frac{m^4}{3}+\frac{19}{18}m^5+\left\{\frac{131}{54}-\frac{179}{144}+\frac{7}{8}\right\}m^6\right\}\cos 2\tau$$

$$+\left\{\frac{m^4}{2}+\frac{19}{6}m^5+\left\{\frac{131}{18}+\frac{361}{72}+\frac{7}{24}\right\}m^6\right\}\cos 4\tau + \frac{7}{8}m^6\cos 6\tau$$

$$=\frac{19}{36}m^4+\frac{19}{6}m^5+\frac{1159}{96}m^6+\left\{\frac{m^4}{3}+\frac{19}{18}m^5+\frac{889}{432}m^6\right\}\cos 2\tau$$

$$+\left\{\frac{m^4}{2}+\frac{19}{6}m^5+\frac{151}{12}m^6\right\}\cos 4\tau+\frac{7}{8}m^6\cos 6\tau$$

$$\mathrm{a}^3\left(\delta\frac{1}{r}\right)^3 = \left\{\frac{1}{18}+\frac{19}{36}+\frac{1}{4}\right\}m^6\cos 2\tau + \left\{\frac{1}{12}+\frac{1}{6}\right\}m^6\cos 4\tau + \frac{m^6}{4}\cos 6\tau$$

$$=\frac{5}{6}m^6\cos 2\tau+\frac{m^6}{4}\cos 4\tau+\frac{m^6}{4}\cos 6\tau$$

When the eccentricity is neglected, $\frac{\mathrm{r}}{\mathrm{a}} = 1$,

$$\frac{r^2}{\mathrm{a}^2} = 1 - 2\,\mathrm{a}\,\delta\frac{1}{r} + 3\,\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2 - 4\,\mathrm{a}^3\left(\delta\frac{1}{r}\right)^3 + \&c.$$

$$R = -\frac{m_{,}r^2}{4\,r_{,}^3}\left\{1 + 3\cos(2\lambda - 2\lambda_{,})\right\} + \&c.$$

$$\frac{r^2}{\mathrm{a}^2} = 1 - \frac{m^2}{3} + \frac{407}{144}m^4$$

$$-\left\{2m^2+\frac{19}{3}m^3+\frac{122}{9}m^4\right\}\cos 2\tau - \frac{m^4}{4}\cos 4\tau$$

$$\cos(2\lambda - 2\lambda_{,}) = \cos(2\tau + 2\delta\lambda) = \cos(2\tau\cos 2\delta\lambda - \sin 2\tau\sin 2\delta\lambda)$$

$$=\cos 2\tau\left\{1 - 2\delta\lambda^2\right\} - 2\sin 2\tau\,\delta\lambda + \&c.$$

$$=\cos 2\tau\left\{1 - \frac{121}{32}m^4\sin^2 2\tau\right\}$$

$$-2\sin 2\tau\left\{\left\{\frac{11}{8}m^2+\frac{59}{12}m^3+\frac{893}{72}\right\}m^4\sin 2\tau + \frac{201}{256}m^4\sin 4\tau\right\}$$

$$=-\frac{11}{8}m^2-\frac{59}{12}m^3-\frac{893}{72}m^4+\left\{1-\frac{443}{256}m^4\right\}\cos 2\tau$$

$$+\left\{\frac{11}{8}m^2+\frac{59}{12}m^3+\frac{893}{72}m^4\right\}\cos 4\tau + \frac{443}{256}m^4\cos 6\tau$$

$$\frac{\mathrm{a}R}{\mu} = -\frac{m^2}{4}\left\{1-\frac{m^2}{3}+\frac{407}{144}m^4-\left\{2m^2+\frac{19}{3}m^3+\frac{122}{9}m^4\right\}\cos 2\tau-\frac{m^4}{4}\cos 4\tau\right\}$$

$$\left\{1 - \frac{33}{8} m^2 - \frac{59}{4} m^3 - \frac{893}{24} m^4 + \left(3 - \frac{1329}{256} m^4\right) \cos 2\tau \right.$$

$$\left. + \left(\frac{33}{8} m^2 + \frac{59}{4} m^3 + \frac{893}{24} m^4\right) \cos 4\tau + \frac{1329}{256} m^4 \cos 6\tau \right\}$$

Considering only the terms multiplied by m^5 and m^6,

$$\frac{\mathrm{a}R}{\mu} = -\frac{m^2}{4} \left[\left\{ -\frac{19}{2} - \frac{59}{4} \right\} m^3 + \left\{ \frac{407}{144} - \frac{61}{3} + \frac{11}{8} - \frac{893}{24} \right\} m^4 \right.$$

$$+ \left\{ -\frac{19}{3} m^3 + \left(-\frac{122}{9} + \frac{407}{48} - \frac{3}{8} + \frac{33}{4} - \frac{33}{8} - \frac{1329}{256} \right) m^4 \right\} \cos 2\tau$$

$$+ \left\{ \left(-\frac{19}{2} + \frac{59}{4} \right) m^3 + \left(-\frac{1}{4} - \frac{61}{3} - \frac{11}{8} + \frac{893}{24} \right) m^4 \cos 4\tau \right.$$

$$\left. + \left(-\frac{3}{8} - \frac{33}{8} + \frac{1329}{256} \right) m^4 \cos 6\tau \right\}$$

$$= \frac{97}{16} m^5 + \frac{7681}{576} m^6 + \left\{ \frac{19}{12} m^5 + \frac{15767}{9216} m^6 \right\} \cos 2\tau$$

$$+ \left\{ -\frac{21}{16} m^5 - \frac{61}{16} m^6 \right\} \cos 4\tau - \frac{177}{1024} m^6 \cos 6\tau$$

Uniting to these, terms found before (see p. 53),

$$\frac{\mathrm{a}R}{\mu} = \left\{ -\frac{m^2}{4} + \frac{179}{96} m^4 + \frac{97}{16} m^5 + \frac{7681}{576} m^6 \right.$$

$$+ \left\{ -\frac{3}{4} m^2 + \frac{3}{4} m^4 + \frac{19}{12} m^5 + \frac{15767}{9216} m^6 \right\} \cos 2\tau$$

$$+ \left\{ -\frac{9}{32} m^4 - \frac{21}{16} m^5 - \frac{61}{16} m^6 \right\} \cos 4\tau - \frac{177}{1024} m^6 \cos 6\tau$$

In the equation which serves to determine the expression for the radius vector, d R signifies the differential of R only as regards the coordinates of the moon. Hence d R = the differential of R, supposing only n t variable + the differential of R with regard to $\mathrm{n}_{,} t$, only in as much as it is contained in the terms in r, λ and s due to the perturbations.

In order to have the terms, now under consideration, $\left(\delta \frac{1}{r}\right)^2$, $(\delta \lambda)^2$, and $\left(\delta \frac{1}{r}\right) \delta \lambda$ may be neglected in the second portion of dR, hence dR = the differential of R, supposing only $n\,t$ variable.

$$+ \left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) \mathrm{d}\,\delta r + \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right) \mathrm{d}\,.\,\delta \lambda$$

$\mathrm{d}\,.\,\delta r$ and $\mathrm{d}\,\delta\lambda$ being restrained to mean the differentials of those quantities with regard to $\mathrm{n}_{\prime}t$ only.

$$\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}\,\delta r+\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\mathrm{d}\,\delta\lambda=-\mathrm{a}\left(\frac{\mathrm{d}R}{\mathrm{d}\mathrm{a}}\right)\mathrm{d}\,.\,\mathrm{r}\,\delta\frac{1}{r}+\frac{\mathrm{d}R}{\mathrm{d}\tau}\mathrm{d}\,\delta\lambda$$

$$\frac{\mathrm{a}}{\mu}\mathrm{d}\,R=\left\{\frac{m^2}{2}+\frac{3}{2}m^2\cos 2\,\tau\right\}\left\{2\,m^3+\frac{19}{3}m^4\right\}\sin 2\,\tau\,\mathrm{d}\,t$$

$$+\frac{3}{2}m^2\sin 2\,\tau\left\{-\frac{11}{4}m^3-\frac{59}{6}m^4\right\}\cos 2\,\tau\,\mathrm{d}\,t$$

$$=\left\{m^5+\frac{19}{6}m^6\right\}\sin 2\,\tau\,\mathrm{d}\,t+\left\{-\frac{9}{16}m^5-\frac{21}{8}m^6\right\}\sin 4\,\tau\,\mathrm{d}\,t$$

Integrating, the following terms arise in $\int\mathrm{d}\,R$,

$$\left\{-\frac{m^5}{2}-\frac{25}{12}m^6\right\}\cos 2\,\tau+\left\{\frac{9}{64}m^5+\frac{51}{64}m^6\right\}\cos 4\,\tau$$

$$\mathrm{r}_1=\frac{m^4}{2}+\frac{19}{12}m^5+\frac{889}{288}m^6-\frac{5}{3}m^6=\frac{m^4}{2}+\frac{19}{12}m^5+\frac{409}{288}m^6$$

Hence, if $r_1=\left\{m^2+\frac{19}{6}m^3+\frac{131}{18}m^4+A\,m^5+B\,m^6\right\}$

$$\left\{m^2+\frac{19}{6}m^3+\frac{131}{18}m^4+A\,m^5+B\,m^6\right\}\left\{3-8\,m+4\,m^2\right\}$$

$$=\left\{4-8\,m+4\,m^2\right\}\left\{\frac{m^4}{2}+\frac{19}{12}m^5+\frac{409}{288}m^6\right\}$$

$$-2\left\{\left\{\frac{2}{2-2\,m}+1\right\}\left\{-\frac{3}{4}m^2+\frac{3}{4}m^4+\frac{19}{12}m^5+\frac{15767}{9216}m^6\right\}-\frac{m^5}{2}-\frac{25}{12}m^6\right\}$$

$$=\left\{4-8\,m+4\,m^2\right\}\left\{\frac{m^4}{2}+\frac{19}{12}m^5+\frac{409}{288}m^6\right\}$$

$$-2\left\{\left\{2+m+m^2+m^3+m^4\right\}\left\{-\frac{3}{4}m^2+\frac{3}{4}m^4+\frac{19}{12}m^5+\frac{15767}{9216}m^6\right\}\right.$$

$$\left.-\frac{m^5}{2}-\frac{25}{12}m^6\right\}$$

Equating coefficients,

$$\frac{38}{3}-\frac{524}{9}+3\,A=\frac{19}{3}-4-\frac{19}{3}-\frac{3}{2}+\frac{3}{2}+1$$

$$A=\frac{383}{27}$$

$$\frac{262}{9}-\frac{3064}{27}+3\,B=\frac{409}{72}-\frac{76}{6}+2-\frac{15767}{2304}-\frac{19}{6}-\frac{3}{2}+\frac{3}{2}+\frac{25}{6}$$

$$B=\frac{508315}{20736}$$

$$r_{131} = \frac{3}{4}m^4 + \frac{19}{4}m^5 + \frac{151}{8}m^6 - \frac{m^6}{2} = \frac{3}{4}m^4 + \frac{19}{4}m^5 + \frac{147}{8}m^6$$

If $r_{131} = \frac{7}{8}m^4 + A m^5 + B m^6$

$$\left\{\frac{7}{8}m^4 + A m^5 + B m^6\right\} \left\{15 - 32m + 16m^2\right\}$$

$$= \left\{16 - 32m + 16m^2\right\} \left\{\frac{3}{4}m^4 + \frac{19}{4}m^5 + \frac{147}{8}m^6\right\}$$

$$- 2\left\{\left\{2 + m + m^2\right\} \left\{-\frac{9}{32}m^4 - \frac{21}{16}m^5 - \frac{61}{16}m^6\right\} + \frac{9}{64}m^5 + \frac{51}{64}m^6\right\}$$

$$-28 + 15A = 76 - 24 + \frac{9}{16} + \frac{21}{4} - \frac{9}{32}$$

$$A = \frac{2737}{480}$$

$$14 - \frac{2737}{15} + 15B = 294 - 152 + 12 + \frac{61}{4} + \frac{21}{8} + \frac{9}{16} - \frac{51}{32}$$

$$B = \frac{594869}{7200}.$$

If $a\,\delta\,\frac{1}{r} = A m^6 \cos 6\tau$

$$35A = \frac{36.13}{16} + \frac{2.2.177}{1024} \qquad A = \frac{1535}{1792}$$

$$\frac{dR}{d\lambda} = \frac{3m_{,}r^2}{2r_{,}^3}\sin(2\lambda - 2\lambda_{,})$$

$$\sin(2\lambda - 2\lambda_{,}) = \sin 2\tau\left\{1 - 2\left(\frac{11}{8}m^2\sin 2\tau\right)^2\right\}$$

$$+ 2\cos 2\tau\left\{\left\{\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4\right\}\sin 2\tau + \frac{201}{256}m^4\sin 4\tau\right\}$$

$$= \left\{1 + \left(-\frac{121}{64} - \frac{121}{128} + \frac{201}{256}\right)m^4\right\}\sin 2\tau$$

$$+ \left\{\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4\right\}\sin 4\tau + \left\{\frac{121}{128} + \frac{201}{256}\right\}m^4\cos 6\tau$$

$$= \left\{1 - \frac{525}{256}m^4\right\}\sin 2\tau + \left\{\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4\right\}\sin 4\tau$$

$$+ \frac{443}{256}m^4\cos 6\tau$$

$$\frac{a}{\mu}\frac{dR}{d\lambda} = \frac{3}{2}m^2\left\{1 - \frac{m^2}{3} + \frac{407}{144}m^4\right.$$

$$- \left\{2\,m^2 + \frac{19}{3}m^3 + \frac{122}{9}m^4\right\}\cos 2\tau - \frac{m^4}{4}\cos 4\tau \Bigg\}$$

$$\Bigg\{ \left\{1 - \frac{525}{256}m^4\right\}\sin 2\tau$$

$$+ \left\{\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4\right\}\sin 4\tau + \frac{443}{256}m^4 \sin 6\tau \Bigg\}$$

Considering only the terms multiplied by m^5 and m^6,

$$\frac{\mathrm{a}}{\mu}\frac{\mathrm{d}R}{\mathrm{d}\lambda} = \frac{3}{2}m^2 \Bigg\{ \left\{-\frac{525}{256} - \frac{11}{8} + \frac{407}{144} + \frac{1}{8}\right\} m^4 \sin 2\tau$$

$$+ \left\{ \left\{-\frac{19}{6} + \frac{59}{12}\right\} m^3 + \left\{\frac{893}{72} - \frac{11}{24} - \frac{61}{9}\right\} m^4 \right\} \sin 4\tau$$

$$+ \left\{\frac{443}{256} - \frac{11}{8} - \frac{1}{8}\right\} m^4 \sin 6\tau \Bigg\}$$

Uniting to these, terms found before (see p. 59),

$$\frac{\mathrm{a}}{\mu}\frac{\mathrm{d}R}{\mathrm{d}\lambda} = \left\{\frac{3}{2}m^2 - \frac{m^4}{2} - \frac{1093}{1536}m^6\right\}\sin 2\tau + \left\{\frac{9}{16}m^4 + \frac{21}{8}m^5 + \frac{31}{4}m^6\right\}\sin 4\tau$$

$$+ \frac{177}{512}m^6 \sin 6\tau.$$

Integrating,

$$- \frac{\mathrm{a}}{\mathrm{n}\,\mu}\int \frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t = \left\{\frac{3}{4}m^2 + \frac{3}{4}m^3 + \frac{m^4}{2} + \frac{m^5}{2} + \frac{443}{3072}m^6\right\}\cos 2\tau$$

$$+ \left\{\frac{9}{64}m^4 + \frac{51}{64}m^5 + \frac{175}{64}m^6\right\}\cos 4\tau + \frac{177}{3072}m^6 \cos 6\tau$$

$$\frac{\mathrm{a}^2}{r^2} = 1 + \frac{m^2}{3} + \left\{\frac{19}{36} - \frac{179}{144}\right\} m^4$$

$$+ \Bigg\{ 2\,m^2 + \frac{19}{3}m^3 + \left(\frac{131}{9} + \frac{1}{3}\right)m^4 + \left(\frac{766}{27} + \frac{19}{18}\right)m^5$$

$$+ \left\{\frac{508315}{10368} + \frac{889}{432}\right\} m^6 \Bigg\} \cos 2\tau$$

$$+ \left\{ \left(\frac{7}{4} + \frac{1}{2}\right)m^4 + \left(\frac{2737}{240} + \frac{19}{6}\right)m^5 + \left(\frac{594869}{3600} + \frac{151}{12}\right)m^6 \right\} \cos 4\tau$$

$$+ \left\{\frac{1535}{1792} + \frac{7}{8}\right\} m^6 \cos 6\tau$$

$$= 1 + \frac{m^2}{3} - \frac{103}{144}m^4 + \left\{2\,m^2 + \frac{19}{3}m^3 + \frac{134}{9}m^4 + \frac{1589}{54}m^5 + \frac{4766859}{93312}m^6\right\}\cos 2\tau$$

$$+ \left\{\frac{9}{4}m^4 + \frac{3497}{240}m^5 + \frac{640169}{3600}m^6\right\}\cos 4\tau + \frac{3103}{1792}m^6 \cos 6\tau$$

$$h \doteq \sqrt{a\mu}\left\{1 - \frac{m^2}{3} + \frac{11}{144}m^4\right\}$$ See p. 135.

$$\frac{d\lambda}{dt} = \frac{1}{r^2}\left\{h - \int \frac{dR}{d\lambda}dt\right\}$$

$$\frac{d\lambda}{n\,dt} = \left\{1 + \frac{m^2}{3} - \frac{103}{144}m^4\right\}\left\{1 - \frac{m^2}{3} + \frac{11}{144}m^4\right\} + \frac{3}{4}m^4$$

$$+ \left\{\left(1 - \frac{m^2}{3} + \frac{11}{144}m^4\right)\left(2m^2 + \frac{19}{3}m^3 + \frac{134}{9}m^4 + \frac{1589}{54}m^5 + \frac{4766859}{93312}m^6\right)\right.$$

$$+ \left(1 + \frac{m^2}{3} - \frac{103}{144}m^4\right)\left(\frac{3}{4}m^2 + \frac{3}{4}m^3 + \frac{m^4}{2} + \frac{m^5}{2} + \frac{443}{3072}m^6\right)$$

$$\left. + \frac{9.3}{2.4.4}m^6 + \frac{2.9}{2.64}m^6\right\}\cos 2\tau$$

$$+ \left\{\left(1 - \frac{m^2}{3}\right)\left(\frac{9}{4}m^4 + \frac{3497}{240}m^5 + \frac{640169}{3600}m^6\right)\right.$$

$$+ \left(m^2 + \frac{19}{6}m^3 + \frac{134}{18}m^4\right)\left(\frac{3}{4}m^2 + \frac{3}{4}m^3 + \frac{m^4}{2}\right)$$

$$\left. + \left(1 + \frac{m^2}{3}\right)\left(\frac{9}{64}m^4 + \frac{51}{64}m^5 + \frac{175}{64}m^6\right)\right\}\cos 4\tau$$

$$\frac{d\lambda}{n\,dt} = 1$$

$$+ \left\{\frac{11}{4}m^2 + \frac{85}{12}m^3 + \frac{539}{36}m^4 + \left(\frac{1589}{54} - \frac{19}{9} + \frac{1}{2} + \frac{1}{4}\right)m^5\right.$$

$$\left. + \left(\frac{4766859}{93312} - \frac{134}{27} + \frac{11}{72} + \frac{443}{3072} + \frac{1}{6} - \frac{309}{576} + \frac{27}{32} + \frac{9}{64}\right)m^6\right\}\cos 2\tau$$

$$+ \left\{\frac{201}{64}m^4 + \left(\frac{3497}{240} + \frac{19}{8} + \frac{3}{4} + \frac{51}{64}\right)m^5\right.$$

$$\left. + \left(\frac{640169}{3600} - \frac{3}{4} + \frac{19}{8} + \frac{134}{24} + \frac{1}{2} + \frac{175}{64} + \frac{3}{64}\right)m^6\right\}\cos 4\tau$$

$$+ \left\{\frac{177}{3072} + \frac{3.9}{8.4} + \frac{9}{64} + \frac{3103}{1792}\right\}m^6\cos 6\tau$$

$$\frac{d\lambda}{n\,dt} = 1 + \left\{\frac{11}{4}m^2 + \frac{85}{12}m^3 + \frac{539}{36}m^4 + \frac{3031}{108}m^5\right.$$

$$\left. + \frac{1867909}{82944}m^6\right\}\cos 2\tau$$

$$+ \left\{\frac{201}{64}m^4 + \frac{17753}{960}m^5 + \frac{1023613}{7200}m^6\right\}\cos 4\tau$$

$$+ \frac{19881}{7168}m^6\cos 6\tau$$

Integrating,

$$\lambda = \mathrm{n}t + \left\{\frac{11}{4}m^2 + \frac{85}{12}m^3 + \frac{539}{36}m^4 + \frac{3031}{108}m^5 \right.$$

$$\left. + \frac{1867909}{82944}m^6\right\} \frac{1}{2}\left\{1 + m + m^2 + m^3 + m^4\right\} \sin 2\tau$$

$$+ \left\{\frac{201}{64}m^4 + \frac{17753}{960}m^5 + \frac{1023613}{7200}m^6\right\} \frac{1}{4}\left\{1 + m + m^2\right\} \sin 4\tau$$

$$+ \frac{19881}{7168}m^6 \sin 6\tau$$

Finally,

$$\lambda = \mathrm{n}t + \left\{\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4 \right.$$

$$\left. + \frac{2855}{108}m^5 + \frac{6253189}{165888}m^6\right\} \sin 2\tau$$

[1]

$$+ \left\{\frac{201}{256}m^4 + \frac{649}{120}m^5 + \frac{1179373}{28800}m^6\right\} \sin 4\tau$$

[131]

$$+ \frac{6627}{14336}m^6 \sin 6\tau$$

M. Plana has

$$\frac{8304449}{165888}m^6 \sin 2\tau + \frac{647623}{28800}m^6 \sin 4\tau$$

The calculations which lead to these terms are so complicated, that I place little confidence in my results; but the details which I have furnished will, I trust, lead to the discovery of the errors if there be any.

I now propose to obtain some of the terms in the reciprocal of the radius vector depending upon the cubes of the eccentricities, by methods precisely the same as those employed in page 37.

The development of R required is to be found at p. 30.

$$a^3\left(\delta\,\frac{1}{r}\right)^2 = \frac{15}{4}m\,e\cos(2\tau - \xi)\left\{\frac{15}{4}m\,e^2\cos 2\tau - \frac{21}{8}m\,e\,e_{\prime}\cos(\xi + \xi_{\prime}) \right.$$

$$+ \frac{35}{8}m\,e\,e_{\prime}\cos(2\tau - \xi - \xi_{\prime}) + \frac{21}{8}m\,e\,e_{\prime}\cos(\xi - \xi_{\prime})$$

$$\left. - \frac{15}{8}m\,e\,e_{\prime}\cos(2\tau - \xi + \xi_{\prime})\right\}$$

$$= \frac{225}{32} m^2 e^3 \cos \xi + \frac{75}{16} m^2 e^2 e_{,} \cos \xi_{,}$$

[2] [5]

$$+ \frac{315}{64} m^2 e^2 e_{,} \cos (2\tau - \xi_{,}) - \frac{315}{64} m^2 e^2 e_{,} \cos (2\tau + \xi_{,})$$

[6] [7]

$$- \frac{315}{64} m^2 e^2 e_{,} \cos (2\tau - 2\xi - \xi_{,})$$

[24]

$$+ \frac{315}{64} m^2 e^2 e_{,} \cos (2\tau - 2\xi + \xi_{,})$$

[27]

$$+ \frac{525}{64} m^2 e^2 e_{,} \cos (4\tau - 2\xi - \xi_{,})$$

$$- \frac{225}{64} m^2 e^2 e_{,} \cos (4\tau - 2\xi + \xi_{,})$$

$\mathfrak{a}^2 \left(\delta \frac{1}{r}\right)^2$ contains the terms

$$\frac{225}{128} m^2 e^2 \left\{1 + \cos (4\tau - 2\xi)\right\}$$

Hence $\frac{3\,\mathfrak{r}^4}{2\,\mathfrak{a}^2} \left(\delta \frac{1}{r}\right)^2$ contains the terms

$$- \frac{225}{128} m^2 e^2 \left\{1 + \cos (4\tau - 2\xi)\right\} 6\, e \cos \xi$$

$$= - \frac{675}{64} m^2 e^3 \cos \xi - \frac{675}{128} m^2 e^3 \cos (4\tau - \xi)$$

$$- \frac{675}{128} m^2 e^3 \cos (4\tau - 3\xi)$$

$$\mathfrak{r}_2 = \frac{3}{2}\left(1 + \frac{3e^2}{8}\right)(r_0 + r_8)$$

$$\mathfrak{r}_3 = \frac{3}{2}\left(1 + \frac{3e^2}{8}\right)(r_9 + r_1) = \frac{45}{8} m e^2 + \frac{405}{32} m^2 e^2 - \frac{15}{4} m^2 e_{,}^2$$

$$\mathfrak{r}_4 = \frac{3}{2}\left(1 + \frac{3e^2}{8}\right)(r_1 + r_{10}) = \frac{45}{8} m e^2 + \frac{753}{32} m^2 e^2 - \frac{15}{4} m^2 e_{,}^2$$

$$\mathfrak{r}_5 = \frac{3}{2}(r_{14} + r_{11}) + \frac{225}{32} m^2 e^2 = \frac{27}{\text{[illegible]}} m^2 e^2$$

$$= \frac{3}{2}(r_{12} + r_{16}) + \frac{945}{128} m^2 e^2 = \frac{105}{16} m e^2 + \frac{3069}{64} m^2 e^2$$

$$= \frac{3}{2}(r_{15} + r_{13}) - \frac{945}{128} m^2 e^2 = -\frac{45}{16} m e^2 - \frac{717}{64} m^2 e^2$$

$= \frac{3}{2} r_8 = 3 m^2$ $\qquad \mathfrak{r}_{21} = \frac{3}{2} r_9 - \frac{r_1}{16} = -\frac{91}{16} m^2$

$= \frac{3}{2} r_{10} - \frac{r_1}{16} = \frac{83}{16} m^2$ $\qquad \mathfrak{r}_{23} = \frac{3}{2} r_{11} = -\frac{63}{16} m - \frac{2511}{128} m^2$

$= \frac{3}{2} r_{13} = -\frac{99}{64} m^2$ $\qquad \mathfrak{r}_{26} = \frac{3}{2} r_{14} = \frac{63}{16} m + \frac{3339}{128} m^2$

$= \frac{3}{2} r_{16} = \frac{693}{64} m^2$ $\qquad \mathfrak{r}_{29} = \frac{3}{2} r_{17} = -\frac{27}{8} m^2$

$= \frac{3}{2} r_{18} = \frac{51}{4} m^2$ $\qquad \mathfrak{r}_{31} = \frac{3}{2} r_{19} = 0$

$= \frac{3}{2} r_{17} = -\frac{27}{8} m^2$ $\qquad \mathfrak{r}_{33} = \frac{3}{2} r_{19} = 0$

$= \frac{3}{2} r_{18} = \frac{51}{4} m^2$ $\qquad \mathfrak{r}_{35} = 0$

$= 0$ $\qquad \mathfrak{r}_{37} = 0$

$$\{(1 + 3e^2)(2 - 2m - c)^2 - 1\} = (2 - 2m - c)^2 \mathfrak{r}_3$$

$$- 2\left\{\frac{2 - c}{2 - 2m - c} + 1\right\} \frac{m_{/} a}{\mu} R_3$$

Let $r_3 = \frac{15}{8} m + \frac{187}{32} m^2 + A m e^2 + B m e_{/}^2$, then

$$\frac{15}{8} m + \frac{187}{32} m^2 + A m e^2 + B m e_{/}^2 \Big\} \{-4m + 3e^2 - 12 m e^2\}$$

$$= (1 - 4m)\left\{\frac{45}{8} m e^2 + \frac{405}{32} m^2 e^2 - \frac{15}{4} m^2 e_{/}^2\right\}$$

$$- 4\left\{-\frac{39}{32} m^2 e^2 - \frac{45}{8} m^2 e_{/}^2\right\}$$

efficients,

$$A = -\frac{45}{2} + \frac{405}{32} + \frac{39}{8}$$

$$= \frac{225}{32}\, m^2 e^3 \cos\xi + \frac{75}{16} m^2 e^2 e_{\prime} \cos\xi_{\prime}$$

$$\qquad [2] \qquad\qquad [5]$$

$$+ \frac{315}{64} m^2 e^2 e_{\prime} \cos(2\tau - \xi_{\prime}) - \frac{315}{64} m^2 e^2 e_{\prime} \cos(2\tau + \xi_{\prime})$$

$$\qquad [6] \qquad\qquad [7]$$

$$- \frac{315}{64} m^2 e^2 e_{\prime} \cos(2\tau - 2\xi - \xi_{\prime})$$

$$[24]$$

$$+ \frac{315}{64} m^2 e^2 e_{\prime} \cos(2\tau - 2\xi + \xi_{\prime})$$

$$[27]$$

$$+ \frac{525}{64} m^2 e^2 e_{\prime} \cos(4\tau - 2\xi - \xi_{\prime})$$

$$- \frac{225}{64} m^2 e^2 e_{\prime} \cos(4\tau - 2\xi + \xi_{\prime})$$

$a^2 \left(\delta \frac{1}{r}\right)^2$ contains the terms

$$\frac{225}{128} m^2 e^2 \left\{ 1 + \cos(4\tau - 2\xi) \right\}$$

Hence $\frac{3\,r^4}{2\,a^2} \left(\delta \frac{1}{r}\right)^2$ contains the terms

$$- \frac{225}{128} m^2 e^3 \left\{ 1 + \cos(4\tau - 2\xi) \right\} 6\, e \cos\xi$$

$$= - \frac{675}{64} m^2 e^3 \cos\xi - \frac{675}{128} m^2 e^3 \cos(4\tau - \xi)$$

$$- \frac{675}{128} m^2 e^3 \cos(4\tau - 3\xi)$$

$$\mathfrak{r}_2 = \frac{3}{2}\left(1 + \frac{3e^2}{8}\right)(r_0 + r_8)$$

$$\mathfrak{r}_3 = \frac{3}{2}\left(1 + \frac{3e^2}{8}\right)(r_9 + r_1) = \frac{45}{8}\, m\, e^2 + \frac{405}{32}\, m^2 e^2 - \frac{15}{4} m^2 e_{\prime}^2$$

$$\mathfrak{r}_4 = \frac{3}{2}\left(1 + \frac{3e^2}{8}\right)(r_1 + r_{10}) = \frac{45}{8}\, m\, e^2 + \frac{753}{32} m^2 e^2 - \frac{15}{4} m^2 e_{\prime}^2$$

$$\mathfrak{r}_5 = \frac{3}{2}\,(r_{14} + r_{11}) + \frac{225}{32}\, m^2 e^2 = \frac{27}{4}\, m^2 e^2$$

$$\mathfrak{r}_6 = \frac{3}{2}(r_{12} + r_{16}) + \frac{945}{128} m^2 e^2 = \frac{105}{16} m e^2 + \frac{3069}{64} m^2 e^2$$

$$\mathfrak{r}_7 = \frac{3}{2}(r_{15} + r_{13}) - \frac{945}{128} m^2 e^2 = -\frac{45}{16} m e^2 - \frac{717}{64} m^2 e^2$$

$$\mathfrak{r}_{20} = \frac{3}{2} r_8 = 3 m^2 \qquad \mathfrak{r}_{21} = \frac{3}{2} r_9 - \frac{r_1}{16} = -\frac{91}{16} m^2$$

$$\mathfrak{r}_{22} = \frac{3}{2} r_{10} - \frac{r_1}{16} = \frac{83}{16} m^2 \qquad \mathfrak{r}_{23} = \frac{3}{2} r_{11} = -\frac{63}{16} m - \frac{2511}{128} m^2$$

$$\mathfrak{r}_{25} = \frac{3}{2} r_{13} = -\frac{99}{64} m^2 \qquad \mathfrak{r}_{26} = \frac{3}{2} r_{14} = \frac{63}{16} m + \frac{3339}{128} m^2$$

$$\mathfrak{r}_{28} = \frac{3}{2} r_{16} = \frac{693}{64} m^2 \qquad \mathfrak{r}_{29} = \frac{3}{2} r_{17} = -\frac{27}{8} m^2$$

$$\mathfrak{r}_{30} = \frac{3}{2} r_{18} = \frac{51}{4} m^2 \qquad \mathfrak{r}_{31} = \frac{3}{2} r_{19} = 0$$

$$\mathfrak{r}_{32} = \frac{3}{2} r_{17} = -\frac{27}{8} m^2 \qquad \mathfrak{r}_{33} = \frac{3}{2} r_{19} = 0$$

$$\mathfrak{r}_{34} = \frac{3}{2} r_{18} = \frac{51}{4} m^2 \qquad \mathfrak{r}_{35} = 0$$

$$\mathfrak{r}_{36} = 0 \qquad \mathfrak{r}_{37} = 0$$

$$r_3 \{(1 + 3e^2)(2 - 2m - \mathrm{c})^2 - 1\} = (2 - 2m - \mathrm{c})^2 \mathfrak{r}_3$$
$$- 2\left\{\frac{2 - \mathrm{c}}{2 - 2m - \mathrm{c}} + 1\right\} \frac{m_, \mathrm{a}}{\mu} R_3$$

Let $r_3 = \frac{15}{8} m + \frac{187}{32} m^2 + A m e^2 + B m e_,^2$, then

$$\left\{\frac{15}{8} m + \frac{187}{32} m^2 + A m e^2 + B m e_,^2\right\} \{-4m + 3e^2 - 12 m e^2\}$$
$$= (1 - 4m)\left\{\frac{45}{8} m e^2 + \frac{405}{32} m^2 e^2 - \frac{15}{4} m^2 e_,^2\right\}$$
$$- 4\left\{-\frac{39}{32} m^2 e^2 - \frac{45}{8} m^2 e_,^2\right\}$$

Equating coefficients,

$$-\frac{12 \cdot 15}{8} + \frac{561}{32} - 4A = -\frac{45}{2} + \frac{405}{32} + \frac{39}{8}$$

$$-4B = -\frac{15}{4} + \frac{45}{2}$$

$$A = 0 \qquad\qquad B = -\frac{75}{16}$$

$$r_4\{(1+3e^2)(2-2m+\mathrm{c})^2 - 1\} = (2-2m+\mathrm{c})^2\,\mathfrak{r}_4$$

$$-2\left\{\frac{2+\mathrm{c}}{2-2m+\mathrm{c}} + 1\right\}\frac{m_{\prime}\mathrm{a}}{\mu}R_4$$

Let $r_4 = \frac{33}{16}m^2 + A m e^2 + B m^2 e^2 + C m^2 e_{\prime}^2$

$$\left\{\frac{33}{16}m^2 + A m e^2 + B m^2 e^2 + C m^2 e_{\prime}^2\right\}\{8 - 12m + 27e^2\}$$

$$= (9-12m)\left\{\frac{45}{8}m e^2 + \frac{753}{32}m^2 e^2 - \frac{15}{4}m^2 e_{\prime}^2\right\}$$

$$-4\left\{\frac{57}{32}m^2 e^2 + \frac{15}{8}m^2 e_{\prime}^2\right\}$$

$$A = \frac{405}{64}$$

$$\frac{33\,.\,27}{16} - \frac{12\,.\,405}{64} + 8B = \frac{9\,.\,753}{32} - \frac{12\,.\,45}{8} - \frac{4\,.\,57}{32}$$

$$8C = -\frac{9\,.\,15}{4} - \frac{4\,.\,15}{8}$$

$$B = \frac{5037}{256} \qquad\qquad C = -\frac{165}{32}.$$

The value of B according to M. Plana is $\frac{5829}{256}$.

$$r_5\{(1+3e^2)m^2 - 1\} = m^2\,\mathfrak{r}_5 - 2\frac{m_{\prime}\mathrm{a}}{\mu}R_5$$

Let $r_5 = -\frac{3}{2}m^2 + A m^2 e^2 + B m^2 e_{\prime}^2$

$$-\left\{-\frac{3}{2}m^2 + A m^2 e^2 + B m^2 e_{\prime}^2\right\} = \frac{9}{4}m^2 e^2 + \frac{27}{16}m^2 e_{\prime}^2$$

$$A = -\frac{9}{4} \qquad\qquad B = -\frac{27}{16}$$

$$r_6\{(1+e^2)(2-3m)^2 - 1\} = (2-3m)^2\,\mathfrak{r}_6$$

$$-2\left\{\frac{2}{2-3m}+1\right\}\frac{m_{,}\,a}{\mu}R_6$$

Let $r_6=\frac{7}{2}m^2+\frac{35}{4}m\,e^2+A\,m^2\,e^2+B\,m^2\,e_{,}^2$

$$\left\{\frac{7}{2}m^2+\frac{35}{4}m\,e^2+A\,m^2\,e^2+B\,m^2\,e_{,}^2\right\}\{3+12\,e^2-12\,m\}$$

$$=(4-12\,m)\left\{\frac{105}{16}m\,e^2+\frac{3069}{64}m^2\,e^2\right\}-4\left\{\frac{105}{16}m^2\,e^2+\frac{123\,.\,21}{8\,.\,56}m^2\,e_{,}^2\right\}$$

$$42-105+3\,A=\frac{3069}{16}-\frac{315}{4}-\frac{105}{4}$$

$$A=\frac{799}{16}\qquad\qquad B=-\frac{123}{16}$$

Let $r_7=-\frac{m^2}{2}-\frac{15}{4}m\,e^2+A\,m^2\,e^2+B\,m^2\,e_{,}^2$

$$\left(-\frac{m^2}{2}-\frac{15}{4}m\,e^2+A\,m^2\,e^2+B\,m^2\,e_{,}^2\right)(3-4\,m+12\,e^2)$$

$$=(4-4\,m)\left(-\frac{45}{16}m\,e^2-\frac{717}{64}m^2\,e^2\right)+4\left(\frac{15}{16}m^2\,e^2+\frac{3}{64}m^2\,e_{,}^2\right)$$

$$-6+15+3\,A=\frac{45}{4}-\frac{717}{16}+\frac{15}{4}\qquad A=-\frac{207}{16}\qquad B=\frac{1}{16}$$

Considering only the argument 20,

$\frac{d^2\,.\,r^2}{2\,d\,t^2}-\frac{\mu}{r}$ gives $\frac{9}{8}(c^2-1)\,e^3\cos 3\,\xi$, therefore,

$$\frac{9}{8}\{c^2-1\}+r_{20}\{9\,c^2-1\}=9\,c^2\,r_{20}-4\frac{m_{,}\,a}{\mu}R_{20}$$

$$r_{20}=\frac{9\,.\,119}{8\,.\,48}m^2-\frac{4}{8\,.\,16}m^2+\frac{9\,.\,3}{8\,.\,8\,.\,2}m^2=\frac{95}{32}m^2$$

Hence $\frac{a}{r}$ contains the term $\left\{\frac{9}{8}+\frac{95}{32}m^2\right\}e^3\cos 3\,\xi$. By putting for e, $e\left\{1-\frac{m^2}{2}\right\}$ in this expression, $\left\{\frac{9}{8}+\frac{41}{32}m^2\right\}e^3\cos 3\,\xi$ is obtained, which is the value given by M. Plana.

$$r_{21}=\{(2-2\,m-3\,c)^2-1\}=(2-2\,m-3\,c)^2 r_{,}$$

$$-2\left\{\frac{2-3\mathrm{c}}{2-2m-3\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{21}$$

$$r_{21}=-\frac{91}{4\,.\,16}m-\frac{4\,.\,7}{4\,.\,32}m=-\frac{105}{64}m$$

$$r_{22}=\{(2-2m+3\mathrm{c})^2-1\}=(2-2m+3\mathrm{c})^2\mathfrak{r}_{22}$$

$$-2\left\{\frac{2+3\mathrm{c}}{2-2m+3\mathrm{c}}\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{22}$$

$$r_{22}=\frac{25\,.\,83}{24\,.\,16}m^2+\frac{4\,.\,25}{24\,.\,32}m^2=\frac{2125}{384}m^2$$

M. Plana has $-\frac{1175}{384}m^2$.

$$r_{23}\{(2\mathrm{c}+m)^2-1\}=(2\mathrm{c}+m^2\mathfrak{r}_{23}-2\left\{\frac{2\mathrm{c}}{2\mathrm{c}+m}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{23}$$

Let $r_{23}=Am+Bm^2$, then

$$(Am+Bm^2)(3+4m)$$

$$=(4+4m)\left\{-\frac{63}{16}m-\frac{2511}{128}m^2\right\}-\frac{4\,.\,3}{16}m^2$$

$$A=-\frac{21}{4}$$

$$-21+3B=-\frac{63}{4}-\frac{2511}{32}-\frac{3}{4}\qquad B=-\frac{789}{32}$$

$$r_{24}\{(2-3m-2\mathrm{c})^2-1\}=(2-3m-2\mathrm{c})^2\mathfrak{r}_{24}$$

$$-2\left\{\frac{2-2\mathrm{c}}{2-3m-2\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{24}$$

$$r_{24}=-\frac{105}{8}m^2$$

$$r_{25}\{(2-m+2\mathrm{c})^2-1\}=(2-m+2\mathrm{c})^2\mathfrak{r}_{25}$$

$$-2\left\{\frac{2+2\mathrm{c}}{2-m+2\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{25}$$

$r_{25}=\frac{16\,.\,99}{15\,.\,64}-\frac{4\,.\,3}{15\,.\,8}=-\frac{7}{4}m^2$ M. Plana has $-\frac{7}{2}m^2$.

$$r_{26}\{(2\mathrm{c}-m)^2-1\} = (2\mathrm{c}-m)^2\mathrm{r}_{26} - 2\left\{\frac{2\mathrm{c}}{2\mathrm{c}-m}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{26}$$

Let $r_{26} = Am + Bm^2$

$$(Am+Bm^2)\{3-4m\} = (4-4m)\left\{\frac{63}{16}m + \frac{3339}{128}m^2\right\} - \frac{4.3}{16}m^2$$

$$A = \frac{21}{4}$$

$$-21 + 3B = \frac{3339}{32} - \frac{63}{4} - \frac{3}{4}$$

$$B = \frac{1161}{32}$$

$$r_{27}\{(2-m-2\mathrm{c})^2-1\} = (2-m-2\mathrm{c})^2\mathrm{r}_{27}$$
$$- 2\left\{\frac{2-2\mathrm{c}}{2-m-2\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{27}$$

$$r_{27} = \frac{15}{8}m^2$$

$$r_{28}\{(2-3m+2\mathrm{c})^2-1\} = (2-3m+2\mathrm{c})^2\mathrm{r}_{28}$$
$$- 2\left\{\frac{2+2\mathrm{c}}{2-3m+2\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{28}$$

$$r_{28} = \frac{16.693}{15.64}m^2 + \frac{4.21}{15.8}m^2 = \frac{49}{4}m^2$$

$$r_{29}\{(\mathrm{c}+2m)^2-1\} = (\mathrm{c}+2m)^2\mathrm{r}_{29} - 2\left\{\frac{\mathrm{c}}{\mathrm{c}+2m}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{29}$$

$$r_{29} = -\frac{27}{4.8}m - \frac{4.9}{4.8}m = -\frac{63}{32}m$$

$$r_{30}\{(2-4m-\mathrm{c})^2-1\} = (2-4m-\mathrm{c})^2 r_{30}$$
$$- 2\left\{\frac{2-\mathrm{c}}{2-4m-\mathrm{c}}+1\right\}\frac{m_{,}\mathrm{a}}{\mu}R_{30}$$

$$_{30} = -\frac{51}{8.4}m + \frac{4.153}{8.8}m = \frac{255}{32}m$$

$$\lambda_3 = -\frac{75}{8} m e^2.$$

The terms in λ_3 multiplied by $m^2 e^2$, $m^2 e_{,}^2$, depend upon a further approximation.

$$\lambda_4 = \left\{ 2 r_4 \left(1 - \frac{e^2}{2}\right) + \left(1 - \frac{e^2}{8}\right)(r_1 + r_{10}) + e^2 r_3 + \left(1 + \frac{e^2}{2}\right) \mathfrak{R}_4 \right.$$

$$\left. + \left(1 + \frac{3}{8} e^2\right)(\mathfrak{R}_1 + \mathfrak{R}_{10}) + \frac{5}{4} e^2 \mathfrak{R}_3 \right\} \frac{1}{(2 - 2m + \mathrm{c})}$$

$$= \frac{17}{8} m^2 + \left(\frac{405}{32} + \frac{15}{4} + \frac{15}{8}\right) \frac{m e^2}{3} \left(1 + \frac{2m}{3}\right)$$

$$+ \left(-\frac{33}{16} + \frac{5037}{128} - \frac{1}{8} + \frac{189}{16} + \frac{7}{2} + \frac{187}{31} + \frac{1}{4} - \frac{19}{16} + \frac{9}{32} - \frac{15}{8} \right.$$

$$\left. + \frac{3}{8} + \frac{45}{8} \right) \frac{m^2 e^2}{3} + \left(-\frac{165}{16} - \frac{5}{2} - \frac{5}{4} - \frac{15}{8} \right) \frac{m^2 e_{,}^2}{3}$$

$$= \frac{17}{8} m^2 + \frac{195}{32} m e^2 + \frac{3029}{128} m^2 e^2 - \frac{85}{16} m^2 e_{,}^2$$

M. Plana has $\frac{2655}{128} m^2 e^2$. As I have not succeeded in obtaining in this or the corresponding term in the reciprocal of the radius vector, the same figures as M. Plana, I think it worth while to give the following verification of the term,

$$\frac{3 \cdot 19}{4 \cdot 8} m^2 e^2$$

in the development of $\frac{\mathrm{a} R}{\mu}$, (see p. 30), upon which they partly depend, which may serve as another example of the method I have given in page 26 of developing the disturbing function.

$$\frac{\mathrm{d} R}{\mathrm{d} e} = \frac{r \,\mathrm{d} R}{\mathrm{d} r} \frac{\mathrm{d} r}{r \,\mathrm{d} e} + \frac{\mathrm{d} R}{\mathrm{d} \lambda} \frac{\mathrm{d} \lambda}{\mathrm{d} e}$$

Let $\frac{r \,\mathrm{d} R}{\mathrm{d} r} = A + B e + C e^2$ $\qquad \frac{\mathrm{d} r}{r \,\mathrm{d} e} = a + b e + c e^2$

$$\frac{\mathrm{d} R}{\mathrm{d} \lambda} = A' + B' e + C' e^2 \qquad \frac{\mathrm{d} R}{\mathrm{d} e} = a' + b' e + c' e^2$$

$$\frac{\mathrm{d} R}{\mathrm{d} e} = (a C + b B + c A) e^2 + (a' C' + b' B' + c' A') e^2$$

$$= -\cos \xi \left\{ -\frac{3}{2} e \cos (2 \tau + 2 \xi) - \frac{3}{2} \left(1 - \frac{5}{2} e^2\right) \cos 2 \tau \right\}$$

$$+\frac{e}{2}\left(-\frac{3}{2}e^2\cos(2\tau+\xi)\right)-\frac{3}{2}\left(\frac{9}{2}e\cos 2\xi\; e\cos(2\tau-\xi)\right)$$

$$+\frac{9}{8}e^2\cos\xi\left(-\frac{3}{2}\cos 2\tau\right)$$

$$+2\sin\xi\left\{\frac{3}{2}e^2\sin(2\tau-2\xi)+\frac{3}{2}\left(1-\frac{5}{2}e^2\right)\sin 2\tau\right\}$$

$$+\frac{5}{2}e\sin 2\xi\left(-\frac{9}{2}e\sin(2\tau-\xi)\right)-\frac{8}{4}e^2\sin\xi\left(\frac{3}{2}\sin 2\tau\right)$$

$$=\left\{\frac{3}{4}-\frac{15}{8}-\frac{3}{4}-\frac{27}{8}-\frac{27}{32}+\frac{3}{2}+\frac{15}{4}+\frac{45}{8}+\frac{9}{16}\right\}e^2\cos(2\tau+\xi)$$

$$=\frac{171}{32}e^2\cos(2\tau+\xi)$$

$\dfrac{aR}{\mu}$ contains the term $\dfrac{3.19}{4.8}e^3\cos(2\tau+\xi)$ as given in page 30.

$$\lambda_6=\left\{2r_6\left(1-\frac{e^2}{2}\right)+e^2r_{12}+e^2r_{16}+\frac{315}{64}m^3e^2\right.$$

$$\left.+\left(1+\frac{e^2}{2}\right)\mathfrak{R}_6+e^2\mathfrak{R}_{12}+e^2\mathfrak{R}_{16}\right\}\frac{1}{(2-3m)}$$

$$=\frac{77}{16}m^2+\left\{\frac{35}{2}+\frac{35}{8}\right\}\frac{me^2}{2}\left(1+\frac{3}{2}m\right)$$

$$+\left\{-\frac{7}{2}+\frac{799}{8}+\frac{1269}{64}+\frac{231}{32}+\frac{315}{64}-\frac{105}{16}+\frac{21}{16}-\frac{63}{4}+\frac{7}{4}\right\}\frac{m^2e^2}{2}$$

$$+\left\{-\frac{123}{8}-\frac{2583}{448}\right\}m^2e_{,}^2$$

$$\lambda_6=\frac{77}{16}m^2+\frac{175}{16}me^2+\frac{4541}{64}m^2e^2-\frac{1353}{128}m^2e_{,}^2$$

M. Plana has $-\dfrac{1253}{128}m^2e_{,}^2$.

$$\lambda_7=\left\{2r_7\left(1-\frac{e^2}{2}\right)+e^2r_{15}+e^2r_{13}-\frac{315}{64}m^2e^2\right.$$

$$\left.+\left(1+\frac{e^2}{2}\right)\mathfrak{R}_7+e^2\mathfrak{R}_{15}+e^2\mathfrak{R}_{13}\right\}\frac{1}{(2-m)}$$

$$=-\frac{11}{16}m^2+\left\{-\frac{15}{2}-\frac{15}{8}\right\}\frac{m}{2}\left(1+\frac{m}{2}\right)$$

$$+\left(\frac{1}{2}-\frac{207}{8}-\frac{97}{64}-\frac{33}{32}-\frac{315}{64}+\frac{15}{16}-\frac{3}{16}+\frac{9}{4}-\frac{1}{4}\right)\frac{m^2}{2}e^2$$

$$+\left(\frac{1}{8}+\frac{3}{64}\right)\frac{m^2e_{,}^2}{2}$$

$$= -\frac{11}{16}m^2 - \frac{75}{16}m\,e^2 - \frac{1113}{64}m^2 e^2 + \frac{11}{128}m^2 e_{\prime}^2$$

$$\lambda_{21} = \left\{2\,r_{21} + r_9 + r_3 + \frac{9}{8}r_1 + \mathfrak{B}_{21} + \mathfrak{B}_9\right.$$

$$\left. + \frac{5}{4}\mathfrak{B}_3 + \frac{13}{8}\mathfrak{B}_1\right\}\frac{1}{(2 - 2\,m - 3\,c)}$$

$$= -\left\{-\frac{105}{32}m + \frac{15}{8}m - \frac{15}{8}m\right\} = \frac{105}{32}m$$

$$\lambda_{22} = \left\{2\,r_{22} + r_{10} + r_4 + \frac{9}{8}r_1 + \mathfrak{B}_{22} + \mathfrak{B}_{10}\right.$$

$$\left. + \frac{5}{4}\mathfrak{B}_4 + \frac{13}{8}\mathfrak{B}_1\right\}\frac{1}{(2 - 2\,m + 2\,c)}$$

$$= \left\{\frac{2125}{192} + \frac{7}{2} + \frac{33}{16} + \frac{9}{8} + \frac{5}{16} + \frac{3}{8} + \frac{5}{8} + \frac{39}{32}\right\}\frac{m^2}{5}$$

$$= \frac{779}{192}m^2$$

M. Plana has $\frac{1093}{64}m^2$.

$$\lambda_{23} = \left\{2\,r_{23} + r_{11} + r_5 + \mathfrak{B}_{23} + \mathfrak{B}_{11} + \frac{5}{4}\mathfrak{B}_5\right\}\frac{1}{(2\,c + m)}$$

$$= \left\{-\frac{21}{2} - \frac{21}{8}\right\}m = -\frac{105}{16}m$$

$$\lambda_{24} = \left\{2\,r_{24} + r_{12} + r_6 + \mathfrak{B}_{24} + \mathfrak{B}_{12} + \frac{5}{4}\mathfrak{B}_6\right\}\frac{1}{(2 - 2\,m - 2\,c)}$$

$$= -\left\{\frac{35}{8} - \frac{35}{8}\right\}\frac{1}{3} = 0$$

$$\lambda_{25} = \left\{2\,r_{25} + r_{13} + r_7 + \mathfrak{B}_{25} + \mathfrak{B}_{13} + \frac{5}{4}\mathfrak{B}_7\right\}\frac{1}{(2 - m + 2\,c)}$$

$$= \left\{-\frac{7}{2} - \frac{33}{32} - \frac{1}{2} - \frac{3}{16} - \frac{1}{4} + \frac{15}{32}\right\}\frac{m^2}{4}$$

$$= -\frac{5}{4}m^2.$$

M. Plana has $-\frac{95}{64}m^2$.

$$\lambda_{26} = \left\{2\,r_{26} + r_{14} + r_5 + \mathfrak{R}_{26} + \mathfrak{R}_{14} + \frac{5}{4}\,\mathfrak{R}_5\right\}\frac{1}{(2\,c - m)}$$

$$= \left\{\frac{21}{2} + \frac{21}{8}\right\}\frac{m}{2} = \frac{105}{16}\,m$$

$$\lambda_{27} = \left\{2\,r_{27} + r_{15} + r_7 + \mathfrak{R}_{27} + \mathfrak{R}_{15} + \frac{5}{4}\,\mathfrak{R}_7\right\}\frac{1}{(2 - m - 2\,c)}$$

$$= -\left\{-\frac{15}{8} + \frac{15}{8}\right\} = 0$$

$$\lambda_{28} = \left\{2\,r_{28} + r_{16} + r_6 + \mathfrak{R}_{28} + \mathfrak{R}_{16} + \frac{5}{4}\,\mathfrak{R}_6\right\}\frac{1}{(2 - 3\,m + 2\,c)}$$

$$= \left\{\frac{49}{2} + \frac{231}{32} + \frac{7}{2} + \frac{21}{16} + \frac{9}{4} + \frac{105}{32}\right\}\frac{m^2}{4}$$

$$= \frac{673}{64}\,m^2$$

M. Plana has $\frac{665}{64}\,m^2$.

$$\lambda_{29} = \{2\,r_{29} + r_{17} + \mathfrak{R}_{29} + \mathfrak{R}_{17}\}\frac{1}{(c + 2\,m)} = -\frac{63}{16}\,m$$

$$\lambda_{30} = \{2\,r_{30} + r_{18} + \mathfrak{R}_{30} + \mathfrak{R}_{18}\}\frac{1}{(2 - 4\,m - c)} = \frac{255}{16}\,m$$

$$\lambda_{31} = \{2\,r_{31} + r_{19} + \mathfrak{R}_{31} + \mathfrak{R}_{19}\}\frac{1}{(2 + c)} = 0$$

$$\lambda_{32} = \{2\,r_{32} + r_{17} + \mathfrak{R}_{32} + \mathfrak{R}_{17}\}\frac{1}{(c - 2\,m)} = \frac{63}{16}\,m$$

$$\lambda_{33} = \{2\,r_{33} + r_{19} + \mathfrak{R}_{33} + \mathfrak{R}_{19}\}\frac{1}{(2 - c)} = 0$$

$$\lambda_{34} = \{2\,r_{34} + r_{18} + \mathfrak{R}_{34} + \mathfrak{R}_{18}\}\frac{1}{(2 + c)}$$

$$= \left\{\frac{561}{16} + \frac{17}{2} + \frac{17}{4} + \frac{51}{8}\right\}m^2 = \frac{289}{16}\,m^2$$

$$\lambda_{35} = \{2\,r_{35} + \mathfrak{R}_{35}\}\frac{1}{3\,m} = -\frac{53}{24}\,m$$

$$\lambda_{36} = \{2\,r_{36} + \mathfrak{R}_{36}\}\frac{1}{(2 - 5\,m)} = \left\{\frac{845}{24} + \frac{845}{64}\right\}\frac{m^2}{2}$$

$$= \frac{9295}{384}\, m^2$$

$$\lambda_{37} = \{2\, r_{37} + \mathfrak{R}_{37}\}\, \frac{1}{(2+m)} = \left\{\frac{1}{24} + \frac{1}{64}\right\} \frac{m^2}{2}$$

$$= \frac{11}{384}\, m^2$$

I shall now endeavour to determine the terms in R, multiplied by m^3, depending upon the cubes of the eccentricities, of the arguments, 2, 3, 5, 24, 29, 30, 32, 33, 35, and $4\,\tau - 3\,\xi$,* and those multiplied by m^4, of the arguments, 0, 9, 11, 12, 14, 15 and 17; but before proceeding further I shall recapitulate, for the convenience of reference, the terms in $\mathrm{a}\,\delta\,\frac{1}{r}$ and $\delta\,\lambda$ which are required for this purpose.

$$\begin{aligned}
\mathrm{a}\,\delta\,\frac{1}{r} = {} & \frac{m^2}{6} + \frac{m^2 e_{,}^2}{4} + \left\{m^2 + \frac{15}{4}\, m\, e^2 + \frac{189}{16}\, m^2 e^2 - \frac{5}{2}\, m^2 e_{,}^2\right\} \cos 2\tau \\
& + \frac{2}{3}\, m^2\, e \cos \xi + \left\{\frac{15}{8}\, m + \frac{187}{32}\, m^2 - \frac{75}{16}\, m\, e_{,}^2\right\} e \cos (2\,\tau - \xi) \\
& + \left\{\frac{33}{16}\, m^2 + \frac{405}{64}\, m\, e^2\right\} e \cos (2\,\tau + \xi) - \frac{3}{2}\, m^2\, e_{,} \cos \xi_{,} \\
& + \left\{\frac{7}{2}\, m^2 + \frac{35}{4}\, m\, e^2\right\} e_{,} \cos (2\,\tau - \xi_{,}) \\
& + \left\{-\frac{m^2}{2} - \frac{15}{4}\, m\, e^2\right\} e_{,} \cos (2\,\tau + \xi_{,}) \\
& + \frac{5}{3}\, m^2 e^2 \cos 2\,\xi - \frac{15}{4}\, m^2 e^2 \cos (2\,\tau - 2\,\xi) \\
& + \frac{7}{2}\, m^2 e^2 \cos (2\,\tau + 2\,\xi) \\
& + \left\{-\frac{21}{8}\, m - \frac{837}{64}\, m^2\right\} e\, e_{,} \cos (\xi + \xi_{,}) \\
& + \left\{\frac{35}{8}\, m + \frac{1269}{64}\, m^2\right\} e\, e_{,} \cos (2\,\tau - \xi - \xi_{,}) \\
& - \frac{33}{32}\, m^2\, e\, e_{,} \cos (2\,\tau + \xi + \xi_{,})
\end{aligned}$$

---

* For the indices of the arguments, see p. 23.

$$+\left\{\frac{21}{8}m+\frac{1113}{64}m^2\right\}e\,e_{\prime}\cos(\xi-\xi_{\prime})$$

$$+\left\{-\frac{15}{8}m-\frac{97}{64}m^2\right\}e\,e_{\prime}\cos(2\tau-\xi+\xi_{\prime})$$

$$+\frac{231}{32}m^2\,e\,e_{\prime}\cos(2\tau+\xi-\xi_{\prime})$$

$$-\frac{9}{4}m^2e_{\prime}^2\cos 2\xi_{\prime}+\frac{17}{2}m^2e_{\prime}^2\cos(2\tau-2\xi_{\prime})$$

$$+\frac{225}{64}m^2e^2\cos(4\tau-2\xi)$$

$$-\frac{105}{64}m\,e^3\cos(2\tau-3\xi)-\frac{21}{4}m\,e^2e_{\prime}\cos(2\xi+\xi_{\prime})$$

$$+\frac{21}{4}m\,e^2e_{\prime}\cos(2\xi-\xi_{\prime})$$

$$-\frac{63}{32}m\,e\,e_{\prime}^2\cos(\xi+2\xi_{\prime})+\frac{255}{32}m\,e\,e_{\prime}^2\cos(2\tau-\xi-2\xi_{\prime})$$

$$+\frac{63}{32}m\,e\,e_{\prime}^2\cos(\xi-2\xi_{\prime})+r_{33}\,e\,e_{\prime}^2\cos(2\tau-\xi+2\xi_{\prime})$$

The above expression is generally consistent with that given by M. Plana: but M. Plana's expression has the term $-\frac{45}{16}m^2e^2$ (Arg. 0), which I have not succeeded in obtaining.

$$r\,\delta\frac{1}{r}=\frac{m^2}{6}-\frac{m^2e^2}{4}+\frac{m^2e_{\prime}^2}{4}$$

$$+\left\{m^2+\frac{45}{16}m\,e^2+\frac{535}{64}m^2e^2-\frac{5}{2}m^2e_{\prime}^2\right\}\cos 2\tau$$

$$+\frac{m^2}{2}e\cos\xi$$

$$+\left\{\frac{15}{8}m+\frac{171}{32}m^2-\frac{15}{16}m\,e^2-\frac{75}{16}m\,e_{\prime}^2\right\}e\cos(2\tau-\xi)$$

$$+\left\{\frac{25}{16}m^2+\frac{225}{64}m\,e^2\right\}e\cos(2\tau+\xi)-\frac{3}{2}m^2e_{\prime}\cos\xi_{\prime}$$

$$+\left\{\frac{7}{2}m^2+\frac{105}{16}m\,e^2\right\}e_{\prime}\cos(2\tau-\xi_{\prime})$$

$$+\left\{-\frac{m^2}{2}-\frac{45}{16}m\,e^2\right\}e_{,}\cos(2\tau+\xi_{,})+\frac{5}{4}m^2e^2\cos$$

$$+\left\{-\frac{15}{16}m-\frac{443}{64}m^2\right\}e^2\cos(2\tau-2\xi)$$

$$+\frac{71}{32}m^2e^2\cos(2\tau+2\xi)$$

$$+\left\{-\frac{21}{8}m-\frac{789}{64}m^2\right\}e\,e_{,}\cos(\xi+\xi_{,})$$

$$+\left\{\frac{35}{8}m+\frac{1157}{64}m^2\right\}e\,e_{,}\cos(2\tau-\xi-\xi_{,})$$

$$-\frac{25}{32}m^2e\,e_{,}\cos(2\tau+\xi+\xi_{,})$$

$$+\left\{\frac{21}{8}m+\frac{1161}{64}m^2\right\}e\,e_{,}\cos(\xi-\xi_{,})$$

$$+\left\{-\frac{15}{8}m-\frac{81}{64}m^2\right\}e\,e_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\frac{175}{32}m^2e\,e_{,}\cos(2\tau+\xi-\xi_{,})$$

$$-\frac{9}{4}m^2e_{,}^2\cos 2\xi_{,}+\frac{17}{2}m^2e_{,}^2\cos(2\tau-2\xi_{,})$$

$$+\frac{225}{64}m^2e^2\cos(4\tau-2\xi)$$

$$-\frac{165}{64}m\,e^3\cos(2\tau-3\xi)-\frac{63}{16}m\,e^2e_{,}\cos(2\xi+\xi_{,})$$

$$-\frac{35}{16}m\,e^2e_{,}\cos(2\tau-2\xi-\xi_{,})$$

$$-\frac{21}{16}m\,e^2e_{,}\cos(2\xi-\xi_{,})+\frac{15}{16}m\,e^2e_{,}\cos(2\tau-2\xi+$$

$$-\frac{63}{32}m\,e\,e_{,}^2\cos(\xi+2\xi_{,})+\frac{255}{32}m\,e\,e_{,}^2\cos(2\tau-\xi-$$

$$+\frac{63}{32}m\,e_{,}^2\cos(\xi-2\xi_{,})+r_{33}\,e\,e_{,}^2\cos(2\tau-\xi+2\xi_{,})$$

$$\delta\lambda=\left\{\frac{11}{8}m^2+\frac{75}{16}m\,e^2+\frac{1101}{64}m^2e^2-\frac{55}{16}m^2e_{,}^2\right\}\sin 2\tau$$

$$+\frac{5}{2}m^2 e\sin\xi+\left\{\frac{15}{4}m+\frac{263}{16}m^2-\frac{75}{8}m\,e_{,}^2\right\}e\sin(2\tau-\xi)$$

$$+\left\{\frac{17}{8}m^2+\frac{195}{32}m\,e^2\right\}e\sin(2\tau+\xi)$$

$$+\left\{-3m+\lambda_5{}^*e^2-\frac{27}{8}m\,e_{,}^2\right\}e_{,}\sin\xi_{,}$$

$$+\left\{\frac{77}{16}m^2+\frac{175}{16}m\,e^2\right\}e_{,}\sin(2\tau-\xi_{,})$$

$$+\left\{-\frac{11}{16}m^2-\frac{75}{16}m\,e^2\right\}e_{,}\sin(2\tau+\xi_{,})$$

$$+\frac{43}{16}m^2e^2\sin 2\xi+\left\{\frac{45}{16}m+\lambda_9{}^*\right\}e^2\sin(2\tau-2\xi)$$

$$+\frac{95}{32}m^2e^2\sin(2\tau+2\xi)+\left\{-\frac{21}{4}m-\frac{717}{32}m^2\right\}e\,e_{,}\sin(\xi+\xi_{,})$$

$$+\left\{\frac{35}{4}m+\frac{1801}{32}m^2\right\}e\,e_{,}\sin(2\tau-\xi-\xi_{,})$$

$$-\frac{17}{16}m^2e\,e_{,}\sin(2\tau+\xi+\xi_{,})$$

$$+\left\{\frac{21}{4}m+\frac{1233}{32}m^2\right\}e\,e_{,}\sin(\xi-\xi_{,})$$

$$+\left\{-\frac{15}{4}m-\frac{173}{32}m^2\right\}e\,e_{,}\sin(2\tau-\xi+\xi_{,})$$

$$+\frac{119}{16}m^2e\,e_{,}\sin(2\tau+\xi-\xi_{,})-\frac{9}{4}m\,e_{,}^2\sin 2\xi_{,}$$

$$+\frac{187}{16}m^2e_{,}^2\sin(2\tau-2\xi_{,})$$

$$+\frac{105}{32}m\,e^3\sin(2\tau-3\xi)-\frac{105}{16}m\,e^2e_{,}\sin(2\xi+\xi_{,})$$

$$+\frac{105}{16}m\,e^2e_{,}\sin(2\tau-2\xi-\xi_{,})+\frac{105}{16}m\,e^2e_{,}\sin(2\xi-\xi_{,})$$

$$-\frac{45}{16}m\,e^2e_{,}\sin(2\tau-2\xi+\xi_{,})-\frac{63}{16}m\,e\,e_{,}^2\sin(\xi+2\xi_{,})$$

---

* The terms marked with an asterisk are left indeterminate, because they have not yet been obtained in this Treatise or verified by the Author.

$$+\frac{255}{16}\,m\,e\,e_{,}^{2}\sin(2\tau-\xi-2\xi_{,})+\frac{63}{16}\,m\,e\,e_{,}^{2}\sin(\xi-2\xi_{,})$$

$$+\lambda_{33}\,e\,e_{,}^{2}\sin(2\tau-\xi+2\xi_{,})-\frac{53}{24}\,m\,e_{,}^{3}\sin 3\xi_{,}$$

The following terms arise in $\frac{\mathrm{a}}{\mu}\ \delta R$ from the expression

$$\delta R=-\mathrm{a}\left(\frac{\mathrm{d}R}{\mathrm{d\,a}}\right)\mathrm{r}\,\delta\frac{1}{r}+\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)\delta\lambda$$

taking care to omit those combinations which do not give rise to the terms now under investigation.

$$\frac{\mathrm{a}}{\mu}\delta R=m^{2}\left\{\frac{1}{2}+\frac{3}{2}\cos 2\tau\right\}\left\{\left\{-\frac{15}{16}e^{2}-\frac{75}{16}e_{,}^{2}\right\}m\,e\cos(2\tau-\xi)\right.$$

$$+\frac{225}{64}\,m\,e^{3}\cos(2\tau+\xi)+\frac{105}{16}\,m\,e^{2}e_{,}\cos(2\tau-\xi_{,})$$

$$-\frac{45}{16}\,m\,e^{2}e_{,}\cos(2\tau+\xi_{,})-\frac{165}{64}\,m\,e^{3}\cos(2\tau-3\xi)$$

$$-\frac{63}{16}m\,e^{2}e_{,}\cos(2\xi+\xi_{,})-\frac{35}{16}m\,e^{2}e_{,}\cos(2\tau-2\xi-\xi_{,})-\frac{21}{16}m\,e^{2}e_{,}\cos(2\xi-\xi_{,})$$

$$+\frac{15}{16}m\,e^{2}\,e_{,}\cos(2\tau-2\xi+\xi_{,})-\frac{63}{32}m\,e\,e_{,}^{2}\cos(\xi+2\xi_{,})$$

$$+\frac{255}{32}\,m\,e\,e_{,}^{2}\cos(2\tau-\xi-2\xi_{,})+\frac{63}{32}\,m\,e_{,}^{2}\cos(\xi-2\xi_{,})$$

$$+r_{33}\,e\,e_{,}^{2}\cos(2\tau-\xi+2\xi_{,})-\frac{m^{2}e^{2}}{4}$$

$$+\frac{m^{2}e_{,}^{2}}{4}+\left\{\frac{535}{64}m^{2}e^{2}-\frac{5}{2}m^{2}e_{,}^{2}\right\}\cos 2\tau+\frac{5}{4}m^{2}e^{2}\cos 2\xi$$

$$-\frac{443}{64}\,m^{2}e^{2}\cos(2\tau-2\xi)+\frac{71}{32}m^{2}e^{2}\cos(2\tau+2\xi)$$

$$-\frac{789}{64}\,m^{2}e\,e_{,}\cos(\xi+\xi_{,})+\frac{1157}{64}m^{2}e\,e_{,}\cos(2\tau-\xi-\xi_{,})$$

$$-\frac{25}{32}m^{2}e\,e_{,}\cos(2\tau+\xi+\xi_{,})+\frac{1161}{64}\,m^{2}e\,e_{,}\cos(\xi-\xi_{,})$$

$$-\frac{81}{64}m^{2}e\,e_{,}\cos(2\tau-\xi+\xi_{,})+\frac{175}{32}m^{2}e\,e_{,}\cos(2\tau+\xi-\xi_{,})$$

$$\left.-\frac{9}{4}m^{2}e_{,}^{2}\cos 2\xi_{,}+\frac{17}{2}m^{2}e_{,}^{2}\cos(2\tau-2\xi_{,})+\frac{225}{64}\ m^{2}e^{2}\cos(4\tau-2\xi)\right\}$$

$$+m^{2}\left\{-e\cos\xi-\frac{9}{2}e\cos(2\tau-\xi)+\frac{3}{2}e\cos(2\tau+\xi)+\frac{3}{2}e_{,}\cos\xi_{,}\right.$$

$$+\frac{21}{4}e_{,}\cos(2\tau-\xi_{,})-\frac{3}{4}e_{,}\cos(2\tau+\xi_{,})\Big\}\;\Big\{\frac{45}{16}m\,e^2\cos 2\tau$$

$$-\frac{15}{16}m\,e^2\cos(2\tau-2\xi)-\frac{21}{8}m\,e\,e_{,}\cos(\xi+\xi_{,})+\frac{35}{8}m\,e\,e_{,}\cos(2\tau-\xi-\xi_{,})$$

$$+\frac{21}{8}m\,e\,e_{,}\cos(\xi-\xi_{,})-\frac{15}{8}m\,e\,e_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\frac{m^2}{2}e\cos\xi+\frac{171}{32}m^2e\cos(2\tau-\xi)+\frac{25}{16}m^2e\cos(2\tau+\xi)$$

$$-\frac{3}{2}m^2e_{,}\cos\xi_{,}+\frac{7}{2}m^2e_{,}\cos(2\tau-\xi_{,})-\frac{m^2}{2}e_{,}\cos(2\tau+\xi_{,})\Big\}$$

$$+m^2\Big\{\frac{3}{4}e^2+\frac{3}{4}e_{,}^2+\Big\{-\frac{15}{4}e^2-\frac{15}{4}e_{,}^2\Big\}\cos 2\tau-\frac{e^2}{4}\cos 2\xi$$

$$+\frac{15}{4}e^2\cos(2\tau-2\xi)+\frac{3}{2}e^2\cos(2\tau+2\xi)-\frac{3}{2}e\,e_{,}\cos(\xi+\xi_{,})$$

$$-\frac{63}{4}e\,e_{,}\cos(2\tau-\xi-\xi_{,})-\frac{3}{4}e\,e_{,}\cos(2\tau+\xi+\xi_{,})-\frac{3}{2}e\,e_{,}\cos(\xi-\xi_{,})$$

$$+\frac{9}{4}e\,e_{,}\cos(2\tau-\xi+\xi_{,})+\frac{21}{4}e\,e_{,}\cos(2\tau+\xi-\xi_{,})$$

$$+\frac{9}{4}e_{,}^2\cos 2\xi_{,}+\frac{51}{4}e_{,}^2\cos(2\tau-2\xi_{,})\Big\}$$

$$\Big\{\frac{15}{8}m\,e\cos(2\tau-\xi)+\frac{m^2}{6}+m^2\cos 2\tau\Big\}$$

$$+\frac{3}{2}m^2\sin 2\tau\Big\{-\frac{75}{8}m\,e_{,}^2e\sin(2\tau-\xi)+\frac{195}{32}m\,e^3\sin(2\tau+\xi)$$

$$+\Big\{\lambda_5e^2-\frac{27}{8}e_{,}^2\Big\}m\,e_{,}\sin\xi_{,}+\frac{175}{16}m\,e^2e_{,}\sin(2\tau-\xi_{,})$$

$$-\frac{75}{16}m\,e^2e_{,}\sin(2\tau+\xi_{,})+\frac{105}{32}m\,e^3\sin(2\tau-3\xi)$$

$$-\frac{105}{16}m\,e^2e_{,}\sin(2\xi+\xi_{,})+\frac{105}{16}m\,e^2e_{,}\sin(2\tau-2\xi-\xi_{,})$$

$$+\frac{105}{16}m\,e^2e_{,}\sin(2\xi-\xi_{,})-\frac{45}{16}m\,e^2e_{,}\sin(2\tau-2\xi+\xi_{,})$$

$$-\frac{63}{16}m\,e\,e_{,}^2\sin(\xi+2\xi_{,})+\frac{255}{16}m\,e\,e_{,}^2\sin(2\tau-\xi-2\xi_{,})$$

$$+\frac{63}{16}m\,e\,e_{,}^2\sin(\xi-2\xi_{,})+\lambda_{33}e\,e_{,}^2\sin(2\tau-\xi+2\xi_{,})-\frac{53}{24}m\,e_{,}^3\sin 3\xi_{,}$$

$$\vdots$$

$$+\Big\{\frac{1101}{64}m^2e^2-\frac{55}{16}m^2e_{,}^2\Big\}\sin 2\tau+\frac{43}{16}m^2e^2\sin 2\xi$$

$$+ \lambda_9\, m^2 e^2 \sin(2\tau - 2\xi) + \frac{95}{32} m^2 e^2 \sin(2\tau + 2\xi) - \frac{717}{32} m^2 e\, e_{,} \sin(\xi + \xi_{,})$$

$$+ \frac{1801}{32} m^2 e\, e_{,} \sin(2\tau - \xi - \xi_{,}) - \frac{17}{16} m^2 e\, e_{,} \sin(2\tau + \xi + \xi_{,})$$

$$+ \frac{1233}{32} m^2 e\, e_{,} \sin(\xi - \xi_{,}) - \frac{173}{32} m^2 e\, e_{,} \sin(2\tau - \xi + \xi_{,})$$

$$+ \frac{119}{16} m^2 e\, e_{,} \sin(2\tau + \xi - \xi_{,}) - \frac{9}{4} m^2 e_{,}^2 \sin 2\xi_{,} + \frac{187}{16} m^2 e_{,}^2 \sin(2\tau - 2\xi_{,}) \Big\}$$

$$+ \Big\{ -\frac{9}{2} e \sin(2\tau - \xi) + \frac{3}{2} e \sin(2\tau + \xi) + \frac{21}{4} e_{,} \sin(2\tau - \xi_{,}) - \frac{3}{4} e_{,} \sin(2\tau + \xi_{,}) \Big\}$$

$$\Big\{ \frac{75}{16} m e^2 \sin 2\tau + \frac{45}{16} m e^2 \sin(2\tau - 2\xi) - \frac{21}{4} m e\, e_{,} \sin(\xi + \xi_{,}) + \frac{35}{4} m e\, e_{,} \sin(2\tau - \xi - \xi_{,})$$

$$+ \frac{21}{4} m e\, e_{,} \sin(\xi - \xi_{,}) - \frac{15}{4} m e\, e_{,} \sin(2\tau - \xi + \xi_{,}) - \frac{9}{4} m e_{,}^2 \sin 2\xi_{,}$$

$$+ \frac{5}{2} m^2 e \sin\xi + \frac{263}{16} m^2 e \sin(2\tau - \xi) + \frac{17}{8} m^2 e \sin(2\tau + \xi)$$

$$+ \frac{77}{16} m^2 e_{,} \sin(2\tau - \xi_{,}) - \frac{11}{16} m^2 e_{,} \sin(2\tau + \xi_{,}) \Big\}$$

$$+ m^2 \Big\{ \Big\{ -\frac{15}{4} e^2 - \frac{15}{4} e_{,}^2 \Big\} \sin 2\tau + \frac{15}{4} e^2 \sin(2\tau - 2\xi) + \frac{3}{2} e^2 \sin(2\tau + 2\xi)$$

$$- \frac{63}{4} e\, e_{,} \sin(2\tau - \xi - \xi_{,}) - \frac{3}{4} e\, e_{,} \sin(2\tau + \xi + \xi_{,}) + \frac{9}{4} e\, e_{,} \sin(2\tau - \xi + \xi_{,})$$

$$+ \frac{21}{4} e\, e_{,} \sin(2\tau + \xi - \xi_{,}) + \frac{51}{4} e_{,}^2 \sin(2\tau - 2\xi_{,}) \Big\}$$

$$\Big\{ \frac{15}{4} m e \sin(2\tau - \xi) - 3 m e_{,} \sin\xi_{,} + \frac{11}{8} m^2 \sin 2\tau \Big\}$$

$$\frac{\mathrm{a}\,\delta R}{\mu} = \Big\{ -\frac{1}{8} + \frac{1605}{256} - \frac{1}{4} - \frac{1539}{128} + \frac{75}{64} + \frac{1}{8} - \frac{15}{8} + \frac{3303}{256}$$

$$- \frac{2367}{64} + \frac{51}{32} - \frac{165}{64} \Big\} m^4 e^2$$

$$+ \Big\{ \frac{1}{8} - \frac{15}{8} - \frac{9}{8} + \frac{147}{16} + \frac{3}{16} + \frac{1}{8} - \frac{15}{8} - \frac{165}{64} + \frac{1617}{128} + \frac{33}{128} - \frac{165}{64} \Big\} m^4 e_{,}$$

$$+ \Big\{ \Big\{ -\frac{45}{64} + \frac{675}{256} - \frac{405}{64} + \frac{135}{64} + \frac{135}{64} - \frac{225}{64} + \frac{225}{64} + \frac{585}{128} - \frac{675}{64}$$

$$- \frac{405}{64} + \frac{225}{64} - \frac{225}{32} + \frac{225}{82} \Big\} e^2$$

$$+ \Big\{ -\frac{225}{64} - \frac{63}{32} + \frac{63}{32} + \frac{735}{64} + \frac{45}{64} - \frac{225}{64} - \frac{225}{32} + \frac{735}{32}$$

$$+\frac{45}{32}-\frac{225}{32}\Big\}\, e_{,}^{2}\Bigg\}\, m^3\, e \cos \xi$$

$$+\Bigg\{\Big\{-\frac{15}{32}-\frac{45}{32}+\frac{15}{32}+\frac{45}{32}\Big\}\, e^2+\Big\{-\frac{75}{32}+\frac{105}{32}-\frac{45}{32}+\frac{441}{64}+\frac{63}{64}+\frac{45}{32}$$

$$+\frac{441}{32}+\frac{63}{32}-\frac{189}{8}-\frac{27}{8}\Big\}\, e_{,}^{2}\Bigg\}\, m^3\, e \cos (2\,\tau-\xi)$$

$$+\Big\{\frac{315}{64}-\frac{135}{64}+\frac{21}{16}-\frac{21}{16}-\frac{315}{32}+\frac{135}{32}-\frac{945}{64}+\frac{135}{64}+\frac{525}{64}$$

$$-\frac{225}{64}-\frac{315}{16}+\frac{135}{16}+\frac{1575}{128}-\frac{225}{128}-\frac{945}{32}+\frac{135}{32}\Big\}\, m^3\, e^2\, e_{,} \cos \xi_{,}$$

$$+\Big\{-\frac{443}{128}+\frac{15}{16}+\frac{675}{256}-\frac{171}{64}-\frac{9}{8}+\frac{5}{8}-\frac{1}{8}$$

$$+\frac{129}{64}-\frac{45}{8}\Big\}\, m^4\, e^2 \cos (2\,\tau-2\,\xi)$$

$$+\Big\{-\frac{789}{128}+\frac{3471}{256}-\frac{75}{128}+\frac{3}{4}+\frac{9}{8}+\frac{21}{8}+\frac{3}{8}+\frac{525}{128}-\frac{513}{256}-\frac{1}{4}-\frac{63}{8}-\frac{3}{8}$$

$$+\frac{5403}{128}-\frac{51}{64}+\frac{99}{64}+\frac{231}{64}+\frac{357}{64}-\frac{789}{128}-\frac{693}{64}-\frac{33}{64}\Big\}\, m^4\, e\, e_{,} \cos (\xi+\xi_{,})$$

$$+\Big\{\frac{1157}{128}-\frac{2367}{256}-\frac{7}{4}+\frac{27}{8}+\frac{513}{128}+\frac{21}{16}-\frac{21}{8}-\frac{3}{4}$$

$$-\frac{2151}{128}+\frac{105}{16}\Big\}\, m^4\, e\, e_{,} \cos (2\,\tau-\xi-\xi_{,})$$

$$+\Big\{\frac{1161}{128}-\frac{243}{256}+\frac{525}{128}+\frac{3}{4}-\frac{63}{8}-\frac{3}{8}+\frac{3}{8}+\frac{3591}{256}-\frac{75}{128}-\frac{1}{4}+\frac{9}{8}+\frac{21}{8}$$

$$-\frac{519}{128}+\frac{357}{64}-\frac{693}{64}-\frac{33}{64}+\frac{5523}{128}-\frac{51}{64}+\frac{99}{64}+\frac{231}{64}\Big\}\, m^4\, e\, e_{,} \cos (\xi-\xi_{,})$$

$$+\Big\{-\frac{81}{128}+\frac{3483}{256}+\frac{1}{4}+\frac{27}{8}+\frac{513}{128}-\frac{3}{16}+\frac{3}{8}-\frac{3}{4}$$

$$+\frac{3699}{128}-\frac{15}{16}\Big\}\, m^4\, e\, e_{,} \cos (2\,\tau-\xi+\xi_{,})$$

$$+\Big\{-\frac{9}{8}+\frac{51}{8}-\frac{9}{8}-\frac{21}{16}-\frac{21}{16}+\frac{3}{8}+\frac{51}{8}+\frac{561}{64}-\frac{231}{128}$$

$$-\frac{231}{128}+\frac{561}{64}\Big\}\, m^4\, e_{,}^{2} \cos 2\,\xi_{,}$$

$$+\Big\{-\frac{165}{128}+\frac{15}{32}-\frac{15}{64}\Big\}\, m^3\, e^3 \cos (2\,\tau-3\,\xi)$$

$$+\Big\{-\frac{35}{32}-\frac{189}{64}-\frac{45}{64}+\frac{189}{32}-\frac{35}{16}-\frac{45}{32}-\frac{315}{64}$$

$$+\frac{189}{16}-\frac{45}{8}\Big\}\, m^3\, e^2\, e_{,} \cos (2\,\tau-2\,\xi-\xi_{,})$$

$$+\left\{-\frac{63}{64}+\frac{765}{128}-\frac{63}{32}-\frac{105}{64}+\frac{765}{64}-\frac{105}{32}\right\}m^3\,e\,e_{,}^{2}\cos(\xi+2\,\xi_{,})$$

$$+\left\{\frac{255}{64}-\frac{189}{128}+\frac{105}{32}-\frac{441}{64}+\frac{135}{64}-\frac{189}{64}\right.$$

$$\left.+\frac{81}{16}-\frac{441}{32}+\frac{189}{8}\right\}m^3\,e\,e_{,}^{2}\cos(2\,\tau-\xi-2\,\xi_{,})$$

$$+\left\{\left\{\frac{63}{64}+\frac{63}{32}-\frac{315}{64}+\frac{765}{64}-\frac{315}{32}+\frac{765}{32}\right\}m^3\right.$$

$$\left.+\frac{3}{4}m^2\,r_{33}+\frac{3}{4}m^2\,\lambda_{33}\right\}e\,e_{,}^{2}\cos(\xi-2\,\xi_{,})$$

$$+\left\{\left\{\frac{189}{128}-\frac{45}{32}-\frac{63}{64}+\frac{135}{64}+\frac{189}{64}-\frac{81}{16}-\frac{63}{32}+\frac{27}{8}\right\}m^3\right.$$

$$\left.+\frac{1}{2}m^2\,r_{33}\right\}e\,e_{,}^{2}\cos(2\,\tau-\xi+2\,\xi_{,})$$

$$+\left\{-\frac{495}{256}+\frac{135}{64}+\frac{225}{64}-\frac{315}{128}+\frac{405}{64}-\frac{225}{32}\right\}m^3\,e^3\cos(4\,\tau-3\,\xi)$$

$$\frac{\mathbf{a}\,\delta R}{\mu}=-\frac{4067}{128}m^4\,e^2+\frac{799}{64}m^4\,e_{,}^{2}+\left\{-\frac{2295}{256}m^3\,e^2+\frac{495}{32}m^3\,e_{,}^{2}\right\}e\cos\xi$$

$$-\frac{39}{16}m^3\,e_{,}^{2}\,e\cos(2\,\tau-\xi)-\frac{2355}{64}m^3\,e^2\,e_{,}\cos\xi_{,}$$

$$-\frac{1739}{256}m^4\,e^2\cos(2\,\tau-2\,\xi)+\frac{2555}{64}m^4\,e\,e_{,}\cos(\xi+\xi_{,})$$

$$-\frac{1761}{256}m^4\,e\,e_{,}\cos(2\,\tau-\xi-\xi_{,})+\frac{7645}{128}m^4\,e\,e_{,}\cos(\xi-\xi_{,})$$

$$+\frac{12289}{256}m^4\,e\,e_{,}\cos(2\,\tau-\xi+\xi_{,})+\frac{1419}{64}m^4\,e_{,}^{2}\cos 2\,\xi_{,}$$

$$-\frac{135}{128}m^3\,e^3\cos(2\,\tau-3\,\xi)-\frac{75}{64}m^3\,e^2\,e_{,}\cos(2\,\tau-2\,\xi-\xi_{,})$$

$$+\frac{1287}{128}m^3\,e\,e_{,}^{2}\cos(\xi+2\,\xi_{,})+\frac{1659}{128}m^3\,e\,e_{,}\cos(2\,\tau-\xi-2\,\xi_{,})$$

$$+\left\{\frac{1539}{64}m^3+\frac{3}{4}m^2\,r_{33}+\frac{3}{4}m^2\,\lambda_{33}\right\}e\,e_{,}^{2}\cos(\xi-2\,\xi_{,})$$

$$+\left\{\frac{1}{2}m^2\,r_{33}+\frac{63}{128}m^3\right\}e\,e_{,}^{2}\cos(2\,\tau-\xi+2\,\xi_{,})+\frac{135}{256}m^3\,e^3\cos(4\,\tau-$$

In order to obtain all the terms in R multiplied by $m^4\,e\,e_{,}$, and $m^4\,e_{,}^{2}$, it is necessary to consider some of the ter $\left(\delta\frac{1}{r}\right)^2$, $\left(\delta\frac{1}{r}\right)\delta\lambda$, and $(\delta\lambda)^2$. For this purpose it is suffi to take

$$\mathrm{a}\,\delta\frac{1}{r}=\frac{15}{8}\,m\,e\cos(2\tau-\xi)$$

$$\delta\lambda=\frac{15}{4}\,m\,e\sin(2\tau-\xi)-3\,m\,e_{,}\sin\xi_{,}$$

$$\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2=\frac{225}{128}m^2e^2\{1+\cos(4\tau-2\xi)\}$$

$$\left(\delta\frac{1}{r}\right)\delta\lambda=\frac{225}{64}\,m^2e^2\sin(4\tau-2\xi)$$

$$-\frac{45}{16}m^2\,e\,e_{,}\{-\sin(2\tau-\xi-\xi_{,})+\sin(2\tau-\xi+\xi_{,})\}$$

$$(\delta\lambda)^2=\frac{225}{32}\,m^2e^2\{1-\cos(4\tau-2\xi)\}$$

$$-\frac{45}{4}m^2\,e\,e_{,}\{\cos(2\tau-\xi-\xi_{,})-\cos(2\tau-\xi+\xi_{,})\}$$

$$+\frac{9}{2}\,m^2\,e_{,}^2\{1-\cos 2\,\xi_{,}\}$$

Hence the following terms arise in $\frac{\mathrm{a}R}{\mu}$, (see p. 68,)

$$-\frac{3\,m^2}{4}(1+3\cos 2\,\tau)\,\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2$$

$$-3\,m^2\sin 2\tau\left(\mathrm{a}\,\delta\frac{1}{r}\right)\delta\lambda+\frac{3}{2}m^2\cos 2\,\tau\,(\delta\lambda)^2. \quad \text{(See p. 68.)}$$

$$=-\frac{675}{512}m^4e^2-\frac{12825}{1024}m^4e^2\cos(2\tau-2\xi)$$

$$-\frac{405}{32}m^4\,e\,e_{,}\cos(\xi+\xi_{,})+\frac{405}{32}\,m^4\,e\,e_{,}\cos(\xi-\xi_{,})$$

$$+\ \&c.$$

Uniting the terms above, multiplied by m^3 and by m^4, to others previously obtained (see pp. 30, 53, and 118),

$$\frac{\mathrm{a}R}{\mu}=-\frac{m^2}{4}+\frac{179}{96}m^4-\frac{3}{8}m^2e^2-\frac{3}{8}m^2e_{,}^2$$

$$-\frac{16943}{512}m^4e^2+\frac{799}{64}\,m^4e_{,}^2$$

[0]

$$+\left\{\frac{m^2}{2}-\frac{m^2e^2}{8}+\frac{3}{4}m^2e_{\prime}^2+\frac{135}{32}m^3+\frac{5999}{384}m^4\right.$$

$$\left.-\frac{2295}{256}m^3e^2+\frac{495}{32}m^3e_{\prime}^2\right\}e\cos\xi$$

[2]

$$+\left\{\frac{9}{4}m^2-\frac{39}{32}m^2e^2-\frac{45}{8}m^2e_{\prime}^2+\frac{15}{16}m^3\right.$$

$$\left.+\frac{235}{64}m^4-\frac{39}{16}m^3e_{\prime}^2\right\}e\cos(2\tau-\xi)$$

[3]

$$+\left\{-\frac{3}{4}m^2-\frac{9}{8}m^2e^2-\frac{27}{32}m^2e_{\prime}^2+\frac{163}{16}m^4\right.$$

$$\left.-\frac{2355}{64}m^3e^2\right\}e_{\prime}\cos\xi_{\prime}$$

[5]

$$+\left\{-\frac{15}{8}m^2+\frac{75}{16}m^2e_{\prime}^2-\frac{45}{32}m^3-\frac{8203}{512}m^{4*}\right\}e^2\cos(2\tau-2\xi)$$

[9]

$$+\left\{\frac{3}{4}m^2-\frac{3}{32}m^2e^2+\frac{27}{32}m^2e_{\prime}^2+\frac{411}{64}m^3\right.$$

$$\left.+\frac{1745}{64}m^4\right\}ee_{\prime}\cos(\xi+\xi_{\prime})$$

[11]

$$+\left\{\frac{63}{8}m^2-\frac{273}{64}m^2e^2-\frac{1107}{64}m^2e_{\prime}^2+\frac{71}{16}m^3\right.$$

$$\left.-\frac{1761}{256}m^4\right\}ee_{\prime}\cos(2\tau-\xi-\xi_{\prime})$$

[12]

$$+\left\{\frac{3}{4}m^2-\frac{3}{32}m^2e^2+\frac{27}{32}m^2e_{\prime}^2+\frac{759}{64}m^3\right.$$

---

* $\delta\lambda$ contains the term $+\frac{1125}{256}m^2e^2\sin(4\tau-2\xi)$ omitted by mistake in p. 160, hence $\frac{a\,\delta R}{\mu}$ contains the term $+\frac{3375}{1024}m^4e^2\cos(4\tau-2\xi)$ in addition to those given in p. 163.

$$\left. + \frac{9265}{128} m^4 \right\} e\, e_{,} \cos(\xi - \xi_{,})$$

[14]

$$+ \left\{ - \frac{9}{8} m^2 - \frac{39}{64} m^2 e^2 + \frac{9}{64} m^2 e_{,}^2 - \frac{3}{8} m^3 \right.$$

$$\left. + \frac{12289}{256} m^4 \right\} e\, e_{,} \cos(2\tau - \xi + \xi_{,})$$

[15]

$$+ \left\{ - \frac{9}{8} m^2 - \frac{27}{16} m^2 e^2 - \frac{7}{8} m^2 e_{,}^2 + \frac{1419}{64} m^4 \right\} e_{,}^2 \cos 2\xi_{,}$$

[17]

$$+ \left\{ \frac{7}{32} m^2 - \frac{135}{128} m^3 \right\} e^3 \cos(2\tau - 3\xi)$$

[21]

$$+ \left\{ - \frac{105}{16} m^2 - \frac{75}{64} m^3 \right\} e^2 e_{,} \cos(2\tau - 2\xi - \xi_{,})$$

[24]

$$+ \left\{ \frac{9}{8} m^2 + \frac{1287}{128} m^3 \right\} e\, e_{,}^2 \cos(\xi + 2\xi_{,})$$

[29]

$$+ \left\{ \frac{153}{8} m^2 + \frac{1659}{128} m^3 \right\} e\, e_{,}^2 \cos(2\tau - \xi - 2\xi_{,})$$

[30]

$$+ \left\{ \frac{9}{8} m^2 + \frac{1539}{64} m^3 + \frac{3}{4} m^2 r_{33} \right.$$

$$\left. + \frac{3}{4} m^2 \lambda_{33} \right\} e\, e_{,}^2 \cos(\xi - 2\xi_{,})$$

[32]

$$+ \left\{ \frac{63}{128} m^3 + \frac{1}{2} m^2 r_{33} \right\} e\, e_{,}^2 \cos(2\tau - \xi + 2\xi_{,})$$

[33]

$$+ \frac{135}{256} m^3 e^3 \cos(4\tau - 3\xi)$$

The terms in dR multiplied by $m^4 e^2$, $m^4 e\, e_{,}$ and $m^4 e_{,}^2$, which arise from differentiating R with respect to $n_{,} t$, only in as much as it is contained in the terms in r and λ due to the perturbations (see p. 54), are given by the following expression:

$$\frac{\mathrm{d}\,R}{\mathrm{d}\,t} = \left\{\frac{m^2}{2} + \frac{3}{2} m^2 \cos 2\,\tau\right\} \left\{\frac{45}{8} m^2 e^2 \sin 2\,\tau - \frac{15}{8} m^2 e^2 \sin (2\,\tau - 2\,\xi)\right.$$

$$+ \frac{21}{8} m^2 e\, e_{,} \sin (\xi + \xi_{,}) + \frac{105}{8} m^2 e\, e_{,} \sin (2\,\tau - \xi - \xi_{,})$$

$$\left. + \frac{21}{8} m^2 e\, e_{,} \sin (\xi - \xi_{,}) - \frac{15}{8} m^2 e\, e_{,} \sin (2\,\tau - \xi + \xi_{,})\right\}$$

$$+ \left\{- m^2 e \cos \xi - \frac{9}{2} m^2 e \cos (2\,\tau - \xi) + \frac{3}{2} m^2 e \cos (2\,\tau + \xi)\right.$$

$$\left. + \frac{3}{2} m^2 e_{,} \cos \xi_{,} + \frac{21}{4} m^2 e_{,} \cos (2\,\tau - \xi_{,}) - \frac{3}{4} m^2 e_{,} \cos (2\,\tau + \xi_{,})\right\}$$

$$\left\{\frac{15}{4} m^2 e \sin (2\,\tau - \xi)\right\}$$

$$+ \frac{3}{2} m^2 \sin 2\,\tau \left\{- \frac{75}{8} m^2 e^2 \cos 2\,\tau - \frac{45}{8} m^2 e^2 \cos (2\,\tau - 2\,\xi)\right.$$

$$- \frac{21}{4} m^2 e\, e_{,} \cos (\xi + \xi_{,}) - \frac{105}{4} m^2 e\, e_{,} \cos (2\,\tau - \xi - \xi_{,})$$

$$- \frac{21}{4} m^2 e\, e_{,} \cos (\xi - \xi_{,}) + \frac{15}{4} m^2 e\, e_{,} \cos (2\,\tau - \xi + \xi_{,})$$

$$\left. - \frac{9}{2} m^2 e_{,}^2 \cos 2\,\xi_{,}\right\}$$

$$+ \left\{- \frac{9}{2} m^2 e \sin (2\,\tau - \xi) + \frac{3}{2} m^2 e \sin (2\,\tau + \xi)\right.$$

$$\left. + \frac{21}{4} m^2 e_{,} \sin (2\,\tau - \xi_{,}) - \frac{3}{4} m^2 e_{,} \sin (2\,\tau + \xi_{,})\right\}$$

$$\left\{- \frac{15}{2} m^2 e \cos (2\,\tau - \xi) - 3\, m^2 e_{,} \cos \xi_{,}\right\}$$

$$= \left\{- \frac{15}{16} - \frac{15}{8}\right\} m^4 e^2 \sin (2\,\tau - 2\,\xi)$$

$$+ \left\{\frac{21}{16} - \frac{315}{32} + \frac{45}{32} - \frac{315}{16} + \frac{45}{16}\right\} m^4 e\, e_{,} \sin (\xi + \xi_{,})$$

$$+ \left\{\frac{105}{16} - \frac{63}{32} + \frac{45}{16} - \frac{63}{16} + \frac{27}{4}\right\} m^2 e\, e_{,} \sin (2\,\tau - \xi - \xi_{,})$$

$$+ \left\{\frac{21}{16} + \frac{45}{32} - \frac{315}{32} + \frac{45}{16} - \frac{315}{16}\right\} m^4 e\, e_{,} \sin (\xi - \xi_{,})$$

$$+ \left\{- \frac{15}{16} - \frac{63}{32} + \frac{45}{16} - \frac{63}{16} + \frac{27}{4}\right\} m^4 e\, e_{,} \sin (2\,\tau - \xi + \xi_{,})$$

$$\frac{\mathrm{a}}{\mu} \int \mathrm{d}\,R = - \frac{45}{32} m^3 e^3 \cos (2\,\tau - 2\,\xi) + 24\, m^4 e\, e_{,} \cos (\xi + \xi_{,})$$

$$- \frac{327}{32} m^4 e\, e_{,} \cos (2\,\tau - \xi - \xi_{,}) + 24\, m^4 e\, c_{,} \cos (\xi - \xi_{,})$$

$$-\frac{87}{32} m^4 e\, e_{,} \cos(2\tau - \xi + \xi_{,}) + \&c.$$

$$a^2 \left(\delta \frac{1}{r}\right)^2 = m^2 \Big\{ \frac{e_{,}^2}{4} + \Big\{ \frac{189}{16} e^2 - \frac{5}{2} e_{,}^2 \Big\} \cos 2\tau + \frac{5}{3} \cos 2\xi - \frac{15}{4} e^2 \cos(2\tau - 2\xi)$$

$$+ \frac{7}{2} e^2 \cos(2\tau + 2\xi) - \frac{837}{64} e\, e_{,} \cos(\xi + \xi_{,}) + \frac{1269}{64} e\, e_{,} \cos(2\tau - \xi - \xi_{,})$$

$$- \frac{33}{32} e\, e_{,} \cos(2\tau + \xi + \xi_{,}) + \frac{1113}{64} e\, e_{,} \cos(\xi - \xi_{,}) - \frac{97}{64} e\, e_{,} \cos(2\tau - \xi + \xi_{,})$$

$$+ \frac{231}{32} e\, e_{,} \cos(2\tau + \xi - \xi_{,}) - \frac{9}{4} e_{,}^2 \cos 2\xi_{,} + \frac{17}{2} e_{,}^2 \cos(2\tau - 2\xi_{,})$$

$$+ \frac{225}{64} e^2 \cos(4\tau - 2\xi) \Big\} \Big\{ \frac{15}{4} m e \cos(2\tau - \xi) + \frac{m^2}{3} + 2 m^2 \cos 2\tau \Big\}$$

$$+ 2 m^2 \Big\{ \frac{2}{3} e \cos\xi + \frac{187}{32} e \cos(2\tau - \xi) + \frac{33}{16} e \cos(2\tau + \xi) - \frac{3}{2} e_{,} \cos\xi_{,}$$

$$+ \frac{7}{2} e_{,} \cos(2\tau - \xi_{,}) - \frac{e_{,}}{2} \cos(2\tau + \xi_{,}) + \frac{19}{6} m^3 \cos 2\tau \Big\} \Big\{ \frac{15}{4} m e^2 \cos 2\tau$$

$$- \frac{21}{8} m e\, e_{,} \cos(\xi + \xi_{,}) + \frac{35}{8} m e\, e_{,} \cos(2\tau - \xi - \xi_{,})$$

$$+ \frac{21}{8} m e\, e_{,} \cos(\xi - \xi_{,}) - \frac{15}{8} m e\, e_{,} \cos(2\tau - \xi + \xi_{,}) \Big\}$$

$$+ \Big\{ \frac{2}{3} m^2 e \cos\xi + \frac{187}{32} m^2 e \cos(2\tau - \xi) + \frac{33}{16} m^2 e \cos(2\tau + \xi)$$

$$- \frac{3}{2} m^2 e_{,} \cos\xi_{,} + \frac{7}{2} m^2 e_{,} \cos(2\tau - \xi_{,}) - \frac{m^2}{2} e_{,} \cos(2\tau + \xi_{,}) \Big\}^2$$

$$+ m^2 \Big\{ \frac{1}{3} + 2 \cos 2\tau \Big\} \Big\{ - \frac{75}{16} m e_{,}^2 e \cos(2\tau - \xi) + \frac{405}{64} m e^3 \cos(2\tau + \xi)$$

$$+ \frac{35}{4} m e^2 e_{,} \cos(2\tau - \xi_{,}) - \frac{15}{4} m e^2 e_{,} \cos(2\tau + \xi_{,}) - \frac{105}{64} m e^3 \cos(2\tau - 3\xi)$$

$$- \frac{21}{4} m e^2 e_{,} \cos(2\xi + \xi_{,}) + \frac{21}{4} m e^2 e_{,} \cos(2\xi - \xi_{,}) - \frac{63}{32} m e\, e_{,}^2 \cos(\xi + 2\xi_{,})$$

$$+ \frac{255}{32} m e\, e_{,}^2 \cos(2\tau - \xi - 2\xi_{,}) + \frac{63}{32} m e\, e_{,}^2 \cos(\xi - 2\xi_{,}) + r_{33} e\, e_{,}^2 \cos(2\tau - \xi + 2\xi_{,}) \Big\}$$

$$+ \frac{15}{4} m e \cos(2\tau - \xi) \Big\{ \frac{33113}{1536} m^3 e \cos(2\tau - \xi) + \frac{101}{16} m^3 e \cos(2\tau + \xi)$$

$$+ \frac{157}{8} m^3 e_{,} \cos(2\tau - \xi_{,}) - \frac{91}{24} m^3 e_{,} \cos(2\tau + \xi_{,}) + \frac{495}{128} m^3 e \cos(4\tau - \xi) \Big\}$$

$$= \Big\{ \frac{189}{16} + \frac{95}{8} + \frac{2}{9} + \frac{34969}{2048} + \frac{1089}{512} + \frac{165565}{4096} \Big\} m^4 e^2$$

$$+\left\{\frac{1}{12}-\frac{5}{2}+\frac{9}{8}+\frac{49}{8}+\frac{1}{8}\right\} m^4 e_{,}^2$$

$$+\left\{\left\{\frac{2835}{128}-\frac{225}{32}+\frac{2805}{128}+\frac{495}{64}+\frac{405}{64}\right\} e^2+\left\{-\frac{75}{16}+\frac{63}{16}+\frac{245}{16}\right.\right.$$

$$\left.\left.-\frac{63}{16}+\frac{15}{16}-\frac{75}{16}\right\} e_{,}^2\right\} m^3 e \cos \xi$$

$$+\left\{\left\{\frac{3375}{512}+\frac{5}{2}\right\} e^2+\left\{\frac{15}{16}+\frac{21}{16}-\frac{105}{16}+\frac{147}{16}+\frac{45}{16}-\frac{25}{16}\right\} e_{,}^2\right\} m^3 e \cos (2$$

$$+\left\{\frac{19035}{512}-\frac{1455}{512}+\frac{105}{8}-\frac{15}{8}-\frac{7}{4}+\frac{6545}{256}+\frac{7}{4}-\frac{2805}{256}+\frac{35}{4}-\frac{15}{4}\right\} m^3 e^2 e$$

$$+\left\{\frac{5}{3}-\frac{5}{4}+\frac{225}{64}+\frac{187}{48}\right\} m^4 e^2 \cos (2\tau-2\xi)$$

$$+\left\{-\frac{279}{64}+\frac{1269}{64}-\frac{33}{32}+\frac{665}{48}-1-\frac{187}{64}+\frac{231}{32}-\frac{455}{64}\right\} m^4 e e_{,} \cos (\xi+\xi_{,}$$

$$+\left\{\frac{423}{64}-\frac{837}{64}-\frac{133}{16}-\frac{561}{64}+\frac{7}{3}\right\} m^4 e e_{,} \cos (2\tau-\xi-\xi_{,})$$

$$+\left\{\frac{371}{64}-\frac{97}{64}+\frac{231}{32}-\frac{95}{16}-1+\frac{1309}{64}-\frac{33}{32}+\frac{2355}{64}\right\} m^4 e e_{,} \cos (\xi-\xi_{,})$$

$$+\left\{-\frac{97}{192}+\frac{1113}{64}+\frac{133}{16}-\frac{561}{64}-\frac{1}{3}\right\} m^4 e e_{,} \cos (2\tau-\xi+\xi_{,})$$

$$+\left\{-\frac{3}{4}+\frac{17}{2}+\frac{9}{8}-\frac{7}{4}\right\} m^4 e_{,}^2 \cos 2\xi_{,}$$

$$+\left\{\frac{25}{8}-\frac{35}{64}\right\} m^3 e^3 \cos (2\tau-3\xi)$$

$$+\left\{-\frac{12555}{512}-\frac{3927}{256}+\frac{35}{12}-\frac{21}{4}\right\} m^3 e^2 e_{,} \cos (2\tau-2\xi-\xi_{,})$$

$$+\left\{\frac{63}{16}-\frac{35}{16}-\frac{21}{32}+\frac{255}{32}\right\} m^3 e e_{,}^2 \cos (\xi+2\xi_{,})$$

$$+\left\{-\frac{135}{32}-\frac{147}{16}-\frac{105}{16}+\frac{85}{32}-\frac{63}{32}\right\} m^3 e e_{,}^2 \cos (2\tau-\xi-2\xi_{,})$$

$$+\left\{\left\{\frac{255}{16}-\frac{63}{16}-\frac{105}{16}+\frac{21}{32}\right\} m^3+m^2 r_{33}\right\} e e_{,}^2 \cos (\xi-2\xi_{,})$$

$$+\left\{\left\{-\frac{135}{32}-\frac{21}{16}+\frac{45}{16}+\frac{63}{32}\right\} m^3+\frac{m^2}{3} r_{33}\right\} e e_{,}^2 \cos (2\tau-\xi+2\xi_{,})$$

$$+\left\{-\frac{225}{32}-\frac{105}{64}\right\} m^3 e^3 \cos (4\tau-3\xi)$$

$$a^2\left(\delta\frac{1}{r}\right)^2=\frac{3071151}{36864} m^4 e^2+\frac{119}{24} m^4 e_{1}^2$$

[0]

$$+\left\{\frac{1635}{32}e^2+\frac{55}{8}e_{\prime}^2\right\}m^3 e\cos\xi$$

[2]

$$+\left\{\frac{4655}{512}e^2+\frac{49}{8}e_{\prime}^2\right\}m^3 e\cos(2\tau-\xi)$$

[3]

$$+\frac{8345}{128}m^3 e^2 e_{\prime}\cos\xi_{\prime}+\frac{1503}{192}m^4 e^2\cos(2\tau-2\xi)$$

[5] [9]

$$+\frac{1175}{48}m^4 e\, e_{\prime}\cos(\xi+\xi_{\prime})$$

[11]

$$-\frac{4073}{192}m^4 e\, e_{\prime}\cos(2\tau-\xi-\xi_{\prime})$$

[12]

$$+\frac{1945}{32}m^4 e\, e_{\prime}\cos(\xi-\xi_{\prime})$$

[14]

$$+\frac{3091}{192}m^4 e\, e_{\prime}\cos(2\tau-\xi+\xi_{\prime})+\frac{57}{8}m^4 e_{\prime}^2\cos 2\xi_{\prime}$$

[15] [17]

$$+\frac{165}{64}m^3 e^3\cos(2\tau-3\xi)$$

[21]

$$-\frac{64811}{1536}m^3 e^2 e_{\prime}\cos(2\tau-2\xi-\xi_{\prime})$$

[24]

$$+\frac{145}{16}m^3 e\, e_{\prime}^2\cos(\xi+2\xi_{\prime})$$

[29]

$$-\frac{617}{32}m^3 e\, e_{\prime}^2\cos(2\tau-\xi-2\xi_{\prime})$$

[30]

$$+\left\{\frac{195}{32}m^3+m^2 r_{33}\right\}e\, e_{\prime}^2\cos(\xi-2\xi_{\prime})$$

[32]

$$+\left\{-\frac{3}{4}m^3+\frac{m^2}{3}r_{33}\right\}e\, e_{\prime}^2\cos(2\tau-\xi+2\xi_{\prime})$$

[33]

$$-\frac{555}{64}\, m^3 e^3 \cos(4\tau - 3\xi)$$

$$\frac{3}{2}\frac{\mathrm{r}^4}{\mathrm{a}^4} = \frac{3}{2} + \frac{15}{2} e^2 - 6 e \cos\xi + \frac{3}{2} e^2 \cos 2\xi$$

For the terms in $\mathrm{a}^2 \left(\delta \frac{1}{r}\right)^2$ multiplied by m^4, $m^3 e$, $m^4 e$, $m^4 e_{,}$ see p. 55, for the terms multiplied by $m^3 e^2$, and $m^3 e e_{,}$ p. 120, and hence $\frac{3\,\mathrm{r}^4}{2\,\mathrm{a}^2}\left(\delta \frac{1}{r}\right)^2$ contains the terms

$$
\begin{aligned}
-6 e \cos\xi \Bigg\{ & \frac{3765}{256} m^3 e^2 + \frac{5}{2} m^3 e^2 \cos 2\tau + \frac{2341}{288} m^4 e \cos\xi \\
& + \frac{251}{96} m^4 e \cos(2\tau - \xi) + \frac{65}{48} m^4 e \cos(2\tau + \xi) \\
& + \frac{5}{2} m^4 e_{,} \cos\xi_{,} - \frac{m^4}{3} e_{,} \cos(2\tau - \xi_{,}) - \frac{5}{3} m^4 e_{,} \cos(2\tau + \xi_{,}) \\
& + \frac{187}{32} m^4 e \cos(4\tau - \xi) + \frac{33}{16} m^4 e \cos(4\tau + \xi) \\
& + \frac{7}{2} m^4 e_{,} \cos(4\tau - \xi_{,}) - \frac{m^4}{2} e_{,} \cos(4\tau + \xi_{,}) \\
& + \frac{495}{128} m^3 e^2 \cos 2\xi + \frac{5}{4} m^8 e^2 \cos(2\tau - 2\xi) + \frac{41}{16} m^3 e e_{,} \cos(\xi + \xi_{,}) \\
& - \frac{191}{48} m^3 e e_{,} \cos(2\tau - \xi - \xi_{,}) - \frac{21}{8} m^3 e e_{,} \cos(2\tau + \xi + \xi_{,}) \\
& + \frac{89}{16} m^3 e e_{,} \cos(\xi - \xi_{,}) - \frac{13}{16} m^3 e e_{,} \cos(2\tau - \xi + \xi_{,}) \\
& + \frac{21}{8} m^3 e e_{,} \cos(2\tau + \xi - \xi_{,}) + \frac{975}{128} m^3 e^2 \cos 4\tau \\
& + \frac{2805}{256} m^3 e^2 \cos(4\tau - 2\xi) + \frac{175}{16} m^3 e e_{,} \cos(4\tau - \xi - \xi_{,}) \\
& - \frac{45}{16} m^3 e e_{,} \cos(4\tau - \xi + \xi_{,}) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
+ \left\{ \frac{15}{2} e^2 + \frac{3}{2} e^2 \cos 2\xi \right\} \Bigg\{ & \frac{19}{36} m^4 + \frac{m^4}{3} \cos 2\tau + \frac{m^4}{2} \cos 4\tau \\
& + \frac{15}{8} m^8 e \cos\xi + \frac{5}{8} m^3 e \cos(2\tau - \xi) + \frac{15}{8} m^3 e \cos(4\tau - \xi) \Bigg\}
\end{aligned}
$$

$$\frac{3\,\mathrm{r}^4}{2\,\mathrm{a}^2}\left(\delta \frac{1}{r}\right)^2 = \left\{ \left\{ \frac{4905}{64} - \frac{11295}{128} - \frac{1485}{128} + \frac{225}{16} + \frac{45}{32} \right\} e^2 + \frac{165}{16} e_{,}^2 \right\} m^3 e \,\mathrm{co}$$

$$+\left\{\left\{\frac{13965}{1024}-\frac{15}{2}-\frac{15}{4}+\frac{75}{16}\right\}e^2+\frac{147}{16}e_{\prime}^2\right\}m\quad e\cos(2\tau-\xi)$$

$$+\left\{\frac{25035}{256}-\frac{123}{16}\quad\frac{267}{16}\right\}m^3e^2e_{\prime}\cos\xi_{\prime}$$

$$+\left\{\frac{1503}{128}-\frac{251}{32}+\frac{1}{4}\right\}m^4e^2\cos(2\tau-2\xi)$$

$$+\left\{\frac{1175}{32}-\frac{15}{2}\right\}m^4ee_{\prime}\cos(\xi+\xi_{\prime})+\left\{-\frac{4073}{128}+1\right\}m^4ee_{\prime}\cos(2\tau-\xi-\xi$$

$$+\left\{\frac{5835}{64}-\frac{15}{2}\right\}m^4ee_{\prime}\cos(\xi-\xi_{\prime})+\left\{\frac{3091}{128}+5\right\}m^4ee_{\prime}\cos(2\tau-\xi+\xi_{\prime})$$

$$+\frac{171}{16}m^4e_{\prime}^2\cos2\xi_{\prime}+\left\{\frac{495}{128}-\frac{15}{4}+\frac{15}{32}\right\}m^3e^3\cos(2\tau-3\xi)$$

$$+\left\{-\frac{64811}{1024}+\frac{191}{16}\right\}m^3e^2e_{\prime}\cos(2\tau-2\xi-\xi_{\prime})$$

$$+\frac{435}{32}ee_{\prime}^2\cos(\xi+2\xi_{\prime})-\frac{1851}{64}m^3ee_{\prime}^2\cos(2\tau-\xi-2\xi_{\prime})$$

$$\left\{\frac{585}{64}m^3+\frac{3}{2}m^2r_{33}\right\}m^3ee_{\prime}^2\cos(\xi-2\xi_{\prime})$$

$$+\left\{-\frac{9}{8}m^3+\frac{m^2}{2}r_{33}\right\}m^3ee_{\prime}^2\cos(2\tau-\xi+2\xi_{\prime})$$

$$+\left\{-\frac{1665}{128}-\frac{8415}{256}+\frac{75}{32}\right\}m^3e^3\cos(4\tau-3\xi)$$

$$\frac{3r^4}{2a^2}\left(\delta\frac{1}{r}\right)^2=\left\{-\frac{497}{64}e^2+\frac{165}{16}e_{\prime}^2\right\}m^3e\cos\xi$$

[2]

$$+\left\{\frac{7245}{1024}e^2+\frac{147}{16}e_{\prime}^2\right\}m^3e\cos(2\tau-\xi)+\frac{18795}{256}m^3e^2e_{\prime}\cos\xi_{\prime}$$

[3] [5]

$$+\frac{531}{128}m^4e^2\cos(2\tau-2\xi)+\frac{935}{32}m^4ee_{\prime}\cos(\xi+\xi_{\prime})$$

[9] [11]

$$-\frac{3945}{128}m^4ee_{\prime}\cos(2\tau-\xi-\xi_{\prime})+\frac{4355}{64}m^4ee_{\prime}\cos(\xi-\xi_{\prime})$$

[12] [14]

$$+\frac{3731}{128}m^4ee_{\prime}\cos(2\tau-\xi+\xi_{\prime})+\frac{171}{16}m^4e_{\prime}^2\cos2\xi_{\prime}$$

[15] [17]

$$+ \underset{[21]}{\frac{75}{128} m^3 e^3 \cos (2\tau - 3\xi)} - \underset{[24]}{\frac{52587}{1024} m^3 e^2 e_{,} \cos (2\tau - 2\xi - \xi_{,})}$$

$$+ \underset{[29]}{\frac{435}{32} m^3 e\, e_{,}^2 \cos (\xi + 2\xi_{,})} - \underset{[30]}{\frac{1851}{64} m^3 e\, e_{,}^2 \cos (2\tau - \xi - 2\xi_{,})}$$

$$+ \underset{[32]}{\left\{\frac{585}{64} m^3 + \frac{3}{2} m^2 r_{33}\right\} e\, e_{,}^2 \cos (\xi - 2\xi_{,})}$$

$$+ \underset{[33]}{\left\{- \frac{9}{8} m^3 + \frac{m^2}{2} r_{33}\right\} e\, e_{,}^2 \cos (2\tau - \xi + 2\xi_{,})}$$

$$- \frac{11145}{256} m^3 e^3 \cos (4\tau - 3\xi)$$

$a^3 \left(\delta \frac{1}{r}\right)^3$ contains only the terms of the order and of the arguments in question given by the expression,

$$\frac{675}{128} m^2 e^2 \left\{1 + \cos (4\tau - 2\xi)\right\} \left\{\frac{m^2}{6} + m^2 \cos 2\tau + \frac{5}{8} m\, e \cos (2\tau - \xi)\right\}$$

Hence

$$- 2 \frac{r^5}{a^4} \left(\delta \frac{1}{r}\right)^3 = - \underset{[0]}{\frac{225}{128} m^4 e^2} - \underset{[3]}{\frac{10125}{1024} m^3 e^3 \cos (2\tau - \xi)}$$

$$- \underset{[9]}{\frac{675}{128} m^4 e^2 \cos (2\tau - 2\xi)}$$

The preceding expressions for $\frac{3\, r^4}{2\, a^4} \left(\delta \frac{1}{r}\right)^2$ &c. serve, as in former instances, to obtain the quantities denoted by the general symbol r.

I shall now attempt to determine other terms in the reciprocal of the radius vector, integrating as before by the method of indeterminate coefficients, the equation

$$\frac{d^2 . r^2}{2\, d t^2} - \frac{d^2 . r^3 \delta \frac{1}{r}}{d t^2} + \frac{3\, d^2 . r^4 \left(\delta \frac{1}{r}\right)^2}{2\, d t^2} - \frac{2\, d^2 . r^5 \left(\delta \frac{1}{r}\right)^3}{d t^2} + \&c.$$

$$-\frac{\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d} R + r\frac{\mathrm{d} R}{\mathrm{d} r} = 0$$

$$r_3 \doteq \frac{3}{2} m^2 + \frac{91}{16} m^3 + \frac{45}{8} m e^2 + \frac{405}{32} m^2 e^2 - \frac{15}{4} m^2 e_{\prime}^2$$

$$+ \left\{\frac{38649}{512} - \frac{675}{32} + \frac{57}{32} + \frac{7245}{1024} - \frac{10125}{1024}\right\} m^3 e^2 + \left\{-\frac{239}{8} + \frac{147}{16}\right\} m^3 e_{\prime}^2$$

$$= \frac{3}{2} m^2 + \frac{91}{16} m^3 + \frac{45}{8} m e^2 + \frac{405}{32} m^2 e^2 - \frac{15}{4} m^2 e_{\prime}^2$$

$$+ \frac{27321}{512} m^3 e^2 - \frac{331}{16} m^3 e_{\prime}^2$$

Let $r_3 = \frac{15}{8} m + \frac{187}{32} m^2 - \frac{75}{16} m e_{\prime}^2 + \frac{33113}{1536} m^3 + A m^2 e^2 + B m^2 e_{\prime}^2$

Since $\mathrm{c} = 1 - \frac{3}{4} m^2 - \frac{225}{32} m^3 + \frac{3}{8} m^2 e^2 - \frac{9}{8} m^2 e_{\prime}^2$ (See p. 80 and p. 135.)

$$\left\{\frac{15}{8} m + \frac{187}{32} m^2 - \frac{75}{16} m e_{\prime}^2 + \frac{33113}{1536} m^3 + A m^2 e^2 + B m^2 e_{\prime}^2\right\}$$

$$\left\{-4 m + \frac{11}{2} m^2 + 3 e^2 - 12 m e^2 + \frac{63}{4} m^2 e^2 + \frac{9}{4} m^2 e_{\prime}^2\right\}$$

$$= \left\{1 - 4 m + \frac{11}{2} m^2\right\}$$

$$\left\{\frac{3}{2} m^2 + \frac{91}{16} m^3 + \frac{45}{8} m e^2 + \frac{405}{32} m^2 e^2 - \frac{15}{4} m^2 e_{\prime}^2 + \frac{27321}{512} m^3 e^2 - \frac{331}{16} m^3 e_{\prime}^2\right\}$$

$$-2\{2 + 2 m\} \left\{\frac{9}{4} m^2 - \frac{39}{32} m^2 e^2 - \frac{45}{8} m^2 e_{\prime}^2 - \frac{39}{16} m^3 e_{\prime}^2\right\}$$

Equating coefficients

$$\frac{945}{32} - \frac{2244}{32} + \frac{33113}{512} - 4 A = \frac{27321}{512} - \frac{405}{8} + \frac{495}{16} + \frac{39}{8}$$

$$-\frac{825}{32} + \frac{135}{32} - 4 B = -\frac{331}{16} + 15 + \frac{39}{4} + \frac{45}{2}$$

$$A = -\frac{463}{128} \qquad\qquad B = -\frac{385}{32}$$

The terms in r_5 multiplied by $m^3 e^2 e_{\prime}$, $m^3 e_{\prime}^3$ cannot be obtained without carrying further the development of $\int \mathrm{d} R$.

Let $r_8 = \frac{5}{3} m^2 + A\, m^3$

$\mathfrak{r}_8 = m^2 + \frac{45}{256} m^3$ (See p. 121.)

$$\mathrm{c}^2 - 1 + r_8 \{4\,\mathrm{c}^2 - 1\} = 4\,\mathrm{c}^2\, \mathfrak{r}_8 - 4 \frac{m_{\prime}\, \mathrm{a}}{\mu} R_8$$

$$-\frac{3}{2} m^2 - \frac{225}{16} m^3 + 3 \left\{ \frac{5}{3} m^2 + A\, m^3 \right\}$$

$$= 4 \left\{ m^2 + \frac{45}{256} m^3 \right\} - 4 \left\{ \frac{m^2}{8} + \frac{45}{8} m^3 \right\}$$

$$-\frac{225}{16} + 3\, A = \frac{45}{64} - \frac{45}{2}$$

$$A = -\frac{45}{16}$$

$$\mathfrak{r}_{11} = -\frac{9}{4} m^2 + \frac{123}{32} m^3 + \left\{ \frac{1347}{32} + \frac{935}{32} \right\} m^4$$

$$= -\frac{9}{4} m^2 + \frac{123}{32} m^3 + \frac{1141}{16} m^4$$

Let $r_{11} = -\frac{21}{8} m - \frac{837}{64} m^2 + A\, m^3$

$$\left\{ -\frac{21}{8} m - \frac{837}{64} m^2 + A\, m^3 \right\} \left\{ 2\, m - \frac{1}{2} m^2 - \frac{249}{16} m^3 \right\}$$

$$= \left\{ 1 + 2\, m - \frac{m^2}{2} \right\} \left\{ -\frac{9}{4} m^2 + \frac{123}{32} m^3 + \frac{1141}{16} m^4 \right\}$$

$$-2 \left\{ \{2 - m + m^2\} \left\{ \frac{3}{4} m^2 + \frac{411}{64} m^3 + \frac{1745}{64} m^4 \right\} + 24\, m^4 \right\}$$

$$\frac{5229}{128} + \frac{837}{128} + 2\, A = \frac{1141}{16} + \frac{123}{16} + \frac{9}{8} - \frac{1745}{16} + \frac{411}{32} - \frac{3}{2} - 48$$

$$A = -\frac{7231}{128}$$

$$\mathfrak{r}_{12} = \frac{21}{4} m^2 + \frac{751}{32} m^2 + \left\{ \frac{3349}{32} - \frac{3945}{128} \right\} m^4$$

$$= \frac{21}{4} m^2 + \frac{751}{32} m^3 + \frac{9451}{128} m^4$$

Let $r_{12} = \frac{35}{8} m + \frac{1269}{64} m^2 + A m^3$

$$\left\{\frac{35}{8} m + \frac{1269}{64} m^2 + A m^3\right\} \left\{- 6m + \frac{21}{2} m^2 + \frac{153}{16} m^3\right\}$$

$$= \left\{1 + 6m + \frac{21}{2} m^2\right\} \left\{\frac{21}{4} m^2 + \frac{751}{32} m^3 + \frac{9451}{128} m^4\right\}$$

$$- 2 \left\{\{2 + 3m + 9m^2\} \left\{\frac{63}{8} m^2 + \frac{71}{16} m^3 - \frac{1761}{256} m^4\right\} - \frac{327}{32} m^4\right\}$$

$$\frac{5355}{128} + \frac{26649}{128} - 6A = \frac{9451}{128} - \frac{2253}{16} + \frac{441}{8} + \frac{1761}{64} - \frac{213}{8} - \frac{567}{4} + \frac{327}{16}$$

$$A = \frac{48935}{768}$$

$$r_{14} = - \frac{9}{4} m^2 + \frac{267}{32} m^3 + \left\{\frac{1347}{32} + \frac{4355}{64}\right\} m^4$$

$$= - \frac{9}{4} m^2 + \frac{267}{32} m^3 + \frac{7049}{64} m^4$$

Let $r_{14} = \frac{21}{8} m + \frac{1113}{64} m^2 + A m^3$

$$\left\{\frac{21}{8} m + \frac{1113}{64} m^2 + A m^2\right\} \left\{- 2m - \frac{1}{2} m^2 - \frac{201}{16} m^3\right\}$$

$$= \left\{1 - 2m - \frac{1}{2} m^2\right\} \left\{- \frac{9}{4} m^2 + \frac{267}{32} m^3 + \frac{7049}{64} m^4\right\}$$

$$- 2 \left\{\{2 + m + m^2\} \left\{\frac{3}{4} m^2 + \frac{759}{64} m^3 + \frac{9265}{128} m^4\right\} + 24 m^4\right\}$$

$$- \frac{4221}{128} - \frac{1113}{128} - 2A = \frac{7049}{64} - \frac{267}{16} + \frac{9}{8} - \frac{9265}{32} - \frac{759}{32} - \frac{3}{2} - 48$$

$$A = \frac{453}{4}$$

$$r_{15} = - \frac{3}{4} m^2 - \frac{221}{32} m^3 + \left\{- \frac{1265}{96} + \frac{3731}{128}\right\} m^4$$

$$= - \frac{3}{4} m^2 - \frac{221}{32} m^3 + \frac{6133}{384} m^4$$

Let $r_{15} = - \frac{15}{8} m - \frac{97}{64} m^2 + A m^3$

$$\left\{- \frac{15}{8} m - \frac{97}{64} m^2 + A m^3\right\} \left\{- 2m + \frac{5}{2} m^2 + \frac{201}{16} m^3\right\}$$

$$= \left\{1 - 2m + \frac{5}{2}m^2\right\} \left\{-\frac{3}{4}m^2 - \frac{221}{32}m^3 + \frac{6133}{384}m^4\right\}$$

$$-2\left\{\left\{2 + m + m^2\right\} \left\{-\frac{9}{8}m^2 - \frac{3}{8}m^3 + \frac{12289}{256}m^4\right\} - \frac{87}{32}m^4\right\}$$

$$-\frac{3015}{128} - \frac{485}{128} - 2A = \frac{6133}{384} + \frac{221}{16} - \frac{15}{8} - \frac{12289}{64} + \frac{3}{4} + \frac{9}{4} + \frac{87}{16}$$

$$A = \frac{49277}{768}$$

$$\mathfrak{r}_{21} = -\frac{91}{16}m^2 + \left\{-\frac{675}{32} - \frac{19}{96} + \frac{75}{128}\right\} m^3$$

$$= -\frac{91}{16}m^2 - \frac{7951}{384}m^3$$

Let $r_{21} = -\frac{105}{64}m + A\,m^2$

$$\left\{-\frac{105}{64}m + A\,m^2\right\} \left\{4m - \frac{1}{2}m^2\right\} = \left\{1 + 4m\right\} \left\{-\frac{91}{16}m^2 - \frac{7951}{384}m^3\right\}$$

$$-\left\{4 - 4m\right\} \left\{\frac{7}{32}m^2 - \frac{135}{128}m^3\right\}$$

$$\frac{105}{128} + 4A = -\frac{7951}{384} - \frac{91}{4} + \frac{135}{32} + \frac{7}{8}$$

$$A = -\frac{7523}{768}$$

$$\mathfrak{r}_{29} = -\frac{27}{8}m^2 + \frac{435}{32}m^3$$

Let $r_{29} = -\frac{63}{32}m + Am^2$

$$\left\{-\frac{63}{32}m + Am^2\right\} \left\{4m + \frac{5}{2}m^2\right\} = \left\{1 + 4m\right\} \left\{-\frac{27}{8}m^2 + \frac{435}{32}m^3\right\}$$

$$-\left\{4 - 4m\right\} \left\{\frac{9}{8}m^2 + \frac{1287}{128}m^3\right\}$$

$$-\frac{315}{64} + 4A = -\frac{27}{2} + \frac{435}{32} - \frac{1287}{32} + \frac{9}{2}$$

$$A = -\frac{1965}{256}$$

$$\mathfrak{r}_{30} = \frac{51}{4}m^2 + \left\{\frac{799}{8} - \frac{1851}{64}\right\} m^3 = \frac{51}{4}m^2 + \frac{4541}{64}m^3$$

Let $r_{30} = \frac{255}{32} m + A m^2$

$$\left\{\frac{255}{32} m + A m^2\right\} \left\{-8 m + \frac{35}{2} m^2\right\} = \left\{1 - 8 m\right\} \left\{\frac{51}{4} m^2 + \frac{4541}{64} m^3\right\}$$

$$- \left\{4 + 8 m\right\} \left\{\frac{153}{8} m^2 + \frac{1659}{128} m^3\right\}$$

$$\frac{8925}{64} - 8 A = \frac{4541}{64} - 102 - \frac{1659}{32} - 153$$

$$A = \frac{12011}{256}$$

$$r_{33} = \left\{-\frac{9}{8} - \frac{9}{8}\right\} m^3 + \frac{m^2}{2} r_{33}$$

$$\frac{3}{2} m^2 r_{33} = -\frac{9}{4} m^3 + \frac{m^2}{2} r_{33} - \frac{63}{32} m^3 - 2 m^2 r_{33}$$

$$r_{33} = -\frac{45}{32} m \qquad \lambda_{33} = -\frac{45}{16} m$$

$$r_{32} = -\frac{27}{8} m^2 + \frac{585}{64} m^3 + \frac{3}{2} m^2 r_{33} = -\frac{27}{8} m^2 + \frac{225}{32} m^3$$

$$\frac{1539}{64} m^3 + \frac{3}{4} m^2 r_{33} + \frac{3}{4} m^2 \lambda_{33} = \frac{2673}{128} m^3$$

Let $r_{32} = \frac{63}{32} m + A m^2$

$$\left\{\frac{63}{32} m + A m^2\right\} \left\{-4 m + \frac{5}{2} m^2\right\} = \left\{1 - 4 m\right\} \left\{-\frac{27}{8} + \frac{225}{32} m^3\right\}$$

$$- \left\{4 + 4 m\right\} \left\{\frac{9}{8} m^2 + \frac{2673}{128} m^3\right\}$$

$$\frac{315}{} - 4 A = \frac{225}{32} + \frac{27}{2} - \frac{2673}{32} \quad \frac{9}{2}$$

$$A = \frac{4635}{256}$$

If $a \delta \frac{1}{r} = A m^2 e^3 \cos (4 \tau - 3 \xi)$

$$- 8 A m^3 = \left\{\frac{119385}{128} - \frac{11145}{256}\right\} m^3 - \frac{135}{64} m^3$$

$A = -\frac{227085}{2048}$ This term is affected by the value of r_{130}, (see p. 128.)

M. Plana has the terms

$$\frac{895}{128} m^2 e^3 \cos (2\tau - \xi)$$

[3]

$$+\left\{-\frac{21}{8} m - \frac{837}{64} m^2 - \frac{7063}{128} m^3\right\} e\, e_{\prime} \cos (\xi + \xi_{\prime})$$

[11]

$$+\left\{\frac{35}{8} m + \frac{1269}{64} m^2 + \frac{47255}{768} m^3\right\} e\, e_{\prime} \cos (2\tau - \xi - \xi_{\prime})$$

[12]

$$+\left\{\frac{21}{8} m + \frac{1113}{64} m^2 + \frac{833}{8} m^3\right\} e\, e_{\prime} \cos (\xi - \xi_{\prime})$$

[14]

$$+\left\{-\frac{15}{8} m - \frac{97}{64} m^2 + \frac{16943}{256} m^3\right\} e\, e_{\prime} \cos (2\tau - \xi + \xi_{\prime})$$

[15]

$$+\left\{-\frac{105}{64} m - \frac{7703}{768} m^2\right\} e^3 \cos (2\tau - 3\xi)$$

[21]

$$+\frac{675}{512} m^2 e^3 \cos (4\tau - 3\xi)$$

By substituting $e\left(1 + \frac{m^2}{2}\right)$ for e in this expression, the following terms arise in lieu of those given by M. Plana, and with these mine ought to agree.

$$-\frac{7231}{128} m^3 e\, e_{\prime} \cos (\xi + \xi_{\prime}) + \frac{48935}{768} m^3 e\, e_{\prime} \cos (2\tau - \xi - \xi_{\prime})$$

[11] [12]

$$+\frac{1687}{16} m^3 e\, e_{\prime} \cos (\xi - \xi_{\prime}) + \frac{16703}{256} m^3 e\, e_{\prime} \cos (2\tau - \xi + \xi_{\prime})$$

[14] [15]

I shall now endeavour to show how other terms in the expression for the longitude may be obtained from the equation

$$\frac{d\lambda}{dt} = \frac{h}{r^2} - \frac{1}{r^2}\int \frac{dR}{d\lambda} dt$$

$$\frac{a}{\mu} \delta . \frac{dR}{d\lambda} = -a \frac{d . \left(\frac{dR}{d\lambda}\right)}{da} r\, \delta \frac{1}{r} + \frac{d . \left(\frac{dR}{d\lambda}\right)}{d\lambda} \delta\lambda$$

$$\frac{\mathrm{a}}{\mu}\,\delta\,\frac{\mathrm{d}R}{\mathrm{d}\lambda} = -\,3\,m^2 \sin 2\,\tau \left\{\frac{45}{16} m\,e^2 \cos 2\,\tau - \frac{15}{16} m\,e^2 \cos (2\,\tau - 2\,\xi)\right.$$

$$-\,\frac{21}{8} m\,e\,e_{,} \cos (\xi + \xi_{,}) + \frac{35}{8} m\,e\,e_{,} \cos (2\,\tau - \xi - \xi_{,})$$

$$+\,\frac{21}{8} m\,e\,e_{,} \cos (\xi - \xi_{,}) - \frac{15}{8} m\,e\,e_{,} \cos (2\,\tau - \xi + \xi_{,})$$

$$+\,\left\{9\,m^2 e \sin (2\,\tau - \xi) - 3\,m^2 e \sin (2\,\tau + \xi)\right.$$

$$\left.-\,\frac{21}{2} m^2 e_{,} \sin (2\,\tau - \xi_{,}) + \frac{3}{2} m^2 e_{,} \sin (2\,\tau + \xi_{,})\right\}$$

$$\left\{\frac{15}{8} m\,e \cos (2\,\tau - \xi)\right\}$$

$$+\,3\,m^2 \cos 2\,\tau \left\{\frac{75}{16} m\,e^2 \sin 2\,\tau + \frac{45}{16} m\,e^2 \sin (2\,\tau - 2\,\xi)\right.$$

$$-\,\frac{21}{4} m\,e\,e_{,} \sin (\xi + \xi_{,}) + \frac{35}{4} m\,e\,e_{,} \sin (2\,\tau - \xi - \xi_{,})$$

$$\left.+\,\frac{21}{4} m\,e\,e_{,} \sin (\xi - \xi_{,}) - \frac{15}{4} m\,e\,e_{,} \sin (2\,\tau - \xi + \xi_{,}) - \frac{9}{4} m\,e_{,}^{2} \sin 2\,\xi_{,}\right\}$$

$$+\,\left\{-\,9\,m^2 e \cos (2\,\tau - \xi) + 3\,m^2 e \cos (2\,\tau + \xi)\right.$$

$$\left.+\,\frac{21}{2} m^2 e_{,} \cos (2\,\tau - \xi_{,}) - \frac{3}{2} m^2 e_{,} \cos (2\,\tau + \xi_{,})\right\}$$

$$\left\{\frac{15}{4} m\,e \sin (2\,\tau - \xi) - 3\,m\,e_{,} \sin \xi_{,}\right\}$$

$$= \left\{-\,\frac{9}{4} - \frac{63}{4}\right\} m^3 e_{,}^{2} \sin 2\,\tau + \left\{-\,\frac{45}{16} + \frac{45}{32} - \frac{45}{8} - \frac{135}{32}\right\} m^3 e^2 \sin 2\,\xi$$

$$+\,\left\{\frac{45}{32} - \frac{105}{16} + \frac{45}{16} - \frac{105}{8}\right\} m^3 e\,e_{,} \sin (\xi + \xi_{,})$$

$$+\,\left\{\frac{63}{16} - \frac{27}{2} + \frac{63}{8}\right\} m^3 e\,e_{,} \sin (2\,\tau - \xi - \xi_{,})$$

$$+\,\left\{\frac{63}{16} - \frac{9}{2} - \frac{63}{8}\right\} m^3 e\,e_{,} \sin (2\,\tau + \xi + \xi_{,})$$

$$+\,\left\{-\,\frac{315}{32} + \frac{45}{16} - \frac{315}{16} + \frac{45}{8}\right\} m^3 e\,e_{,} \sin (\xi - \xi_{,})$$

$$+\,\left\{-\,\frac{63}{16} + \frac{27}{2} - \frac{63}{8}\right\} m^3 e\,e_{,} \sin (2\,\tau - \xi + \xi_{,})$$

$$+\left\{-\frac{63}{16}+\frac{9}{2}+\frac{63}{8}\right\} m^3 e\, e_{,} \sin(2\,\tau+\xi-\xi_{,})$$

$$+\left\{\frac{63}{4}+\frac{27}{8}\right\} m^3 e_{,}^2 \sin(2\,\tau-2\,\xi_{,})+\left\{\frac{9}{4}-\frac{27}{8}\right\} m^3 e_{,}^2 \sin(2\,\tau+$$

$$+\left\{-\frac{135}{32}-\frac{45}{16}+\frac{225}{32}+\frac{45}{8}\right\} m^3 e^2 \sin 4\,\tau$$

$$+\left\{\frac{135}{16}+\frac{45}{32}-\frac{135}{8}+\frac{135}{32}\right\} m^3 e^2 \sin(4\,\tau-2\,\xi)$$

$$+\left\{-\frac{315}{32}-\frac{105}{16}+\frac{315}{16}+\frac{105}{8}\right\} m^3 e\, e_{,} \sin(4\,\tau-\xi-\xi_{,})$$

$$+\left\{\frac{45}{32}+\frac{45}{16}-\frac{45}{16}-\frac{45}{8}\right\} m^3 e\, e_{,} \sin(4\,\tau-\xi+\xi_{,})$$

$$= -18\, m^3 e_{,}^2 \sin 2\,\tau - \frac{45}{4} m^3 e^2 \sin 2\,\xi - \frac{495}{32} m^3 e\, e_{,} \sin(\xi+\xi_{,})$$

[1] [8] [11]

$$-\frac{27}{16} m^3 e\, e_{,} \sin(2\,\tau-\xi-\xi_{,}) - \frac{135}{16} m^3 e\, e_{,} \sin(2\,\tau+\xi+\xi_{,})$$

[12] [13]

$$-\frac{675}{32} m^3 e\, e_{,} \sin(\xi-\xi_{,}) + \frac{27}{16} m^3 e\, e_{,} \sin(2\,\tau-\xi+\xi_{,})$$

[14] [15]

$$+\frac{135}{16} m^3 e\, e_{,} \sin(2\,\tau+\xi-\xi_{,}) + \frac{153}{8} m^3 e_{,}^2 \sin(2\,\tau-2\,\xi_{,})$$

[16] [18]

$$-\frac{9}{8} m^3 e_{,}^2 \sin(2\,\tau+2\,\xi_{,}) + \frac{45}{8} m^3 e^2 \sin 4\,\tau$$

[19] [131]

$$-\frac{45}{16} m^3 e^2 \sin(4\,\tau-2\,\xi)$$

[136]

$$+\frac{525}{32} m^3 e\, e_{,} \sin(4\,\tau-\xi-\xi_{,}) - \frac{135}{32} m^3 e\, e_{,} \sin(4\,\tau-\xi+\xi_{,})$$

[138] [140]

$$-\frac{\mathrm{a}}{\mathrm{n}\,\mu}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t = \left\{\frac{3}{4}\left(1-\frac{5}{2}e^2-\frac{5}{2}e_{,}^2\right)m^2 - 9\,m^3 e_{,}^2\right\}(1+m)\cos 2\,\tau$$

$$-\frac{135}{16} m^3 e \cos\xi - \frac{9}{2} m^2 e\,(1+2\,m)\cos(2\,\tau-\xi)$$

$$+\frac{m^2}{2}e\left(1+\frac{2}{3}m\right)\cos(2\tau+\xi)$$

$$+\left(\frac{21}{8}m^2+\frac{9}{4}m^3\right)\left(1+\frac{3}{2}m\right)e_{,}\cos(2\tau-\xi_{,})$$

$$+\left(-\frac{3}{8}m^2-\frac{9}{4}m^3\right)e_{,}\left(1+\frac{m}{2}\right)\cos(2\tau+\xi_{,})-\frac{45}{8}m^3e^2\cos 2\xi$$

$$-\frac{15}{8}m\,e^2\cos(2\tau-2\xi)+\frac{3}{8}m^2e^2\cos(2\tau+2\xi)$$

$$-\frac{495}{32}m^3e\,e_{,}\cos(\xi+\xi_{,})$$

$$+\left(-\frac{63}{4}m^2-\frac{27}{16}m^3\right)(1+3m)\,e\,e_{,}\cos(2\tau-\xi-\xi_{,})$$

$$+\left(-\frac{1}{4}m^2-\frac{45}{16}m^3\right)\left(1+\frac{m}{3}\right)e\,e_{,}\cos(2\tau+\xi+\xi_{,})$$

$$-\frac{675}{32}m^3e\,e_{,}\cos(\xi-\xi_{,})$$

$$+\left(-\frac{9}{4}m^2+\frac{27}{16}m^3\right)(1+m)\,e\,e_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\left(\frac{7}{4}m^2+\frac{45}{16}m^3\right)(1+m)\,e\,e_{,}\cos(2\tau+\xi-\xi_{,})$$

$$+\left(\frac{51}{8}m^2+\frac{135}{16}m^3\right)(1+2m)\,e_{,}^2\cos(2\tau-2\xi_{,})$$

$$-\frac{9}{16}m^3e_{,}^2\cos(2\tau+2\xi_{,})+\frac{45}{32}m^3e^2\cos 4\tau+\frac{45}{8}m^3e\cos(4\tau-\xi)$$

$$-\frac{45}{32}m^3e^2\cos(4\tau-2\xi)+\frac{175}{32}m^3e\,e_{,}\cos(4\tau-\xi-\xi_{,})$$

$$-\frac{45}{32}m^3e\,e_{,}\cos(4\tau-\xi+\xi_{,})$$

$$-\frac{\mathrm{a}}{\mathrm{n}\mu}\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t=\left\{\frac{3}{4}m^2+\frac{3}{4}m^3-\frac{15}{8}m^2e^2-\frac{15}{8}m^2e_{,}^2\right.$$

$$\left.-\frac{15}{8}m^3e^2-\frac{87}{8}m^3e_{,}^2\right\}\cos 2\tau-\frac{135}{16}m^3e\cos\xi$$

$$+\left\{-\frac{9}{2}m^2-9m^3+\frac{39}{16}m^2e^2\right.$$

$$\left.+\frac{45}{4}m^2e_{,}^2\right\}e\cos(2\tau-\xi)$$

$$+\left\{\frac{m^2}{2}+\frac{m^3}{3}-\frac{19}{16}m^2e^2-\frac{5}{4}m^2e_{,}^2\right\}e\cos(2\tau+\xi)$$

$$+\left\{\frac{21}{8}m^2+\frac{99}{16}m^3-\frac{105}{16}m^2e^2-\frac{369}{64}m^2e_{,}^2\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{-\frac{3}{8}m^2-\frac{39}{16}m^3+\frac{15}{16}m^2e^2+\frac{3}{64}m^2e_{,}^2\right\}e_{,}\cos(2\tau+\xi_{,})$$

$$+\left\{-\frac{15}{8}m-\frac{45}{32}m^2\right\}e^2\cos(2\tau-2\xi)$$

$$+\left\{\frac{3}{8}m^2+\frac{3}{16}m^3\right\}e^2\cos(2\tau+2\xi)$$

$$-\frac{495}{32}m^3ee_{,}\cos(\xi+\xi_{,})$$

$$+\left\{-\frac{63}{4}m^2-\frac{783}{16}m^3\right\}ee_{,}\cos(2\tau-\xi-\xi_{,})$$

$$+\left\{-\frac{m^2}{4}-\frac{139}{48}m^3\right\}ee_{,}\cos(2\tau+\xi+\xi_{,})$$

$$-\frac{675}{32}m^3ee_{,}\cos(\xi-\xi_{,})$$

$$+\left\{-\frac{9}{4}m^2+\frac{9}{16}m^3\right\}ee_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\left\{\frac{7}{4}m^2+\frac{73}{16}m^3\right\}ee_{,}\cos(2\tau+\xi-\xi_{,})$$

$$+\left\{\frac{51}{8}m^2+\frac{357}{16}m^3\right\}e_{,}^2\cos(2\tau-2\xi_{,})$$

$$-\frac{9}{16}m^3e_{,}^2\cos(2\tau+2\xi_{,})$$

$$+\frac{7}{16}m^2e^3\cos(2\tau-3\xi)+\frac{5}{16}m^2e^3\cos(2\tau+3\xi)$$

$$-\frac{35}{8}me^2e_{,}\cos(2\tau-2\xi-\xi_{,})$$

$$-\frac{3}{16}m^2e^2e_{,}\cos(2\tau+2\xi-\xi_{,})$$

$$+\frac{15}{8} m e^2 e_{,} \cos(2\tau - 2\xi + \xi_{,})$$

$$+\frac{21}{16} m^2 e^2 e_{,} \cos(2\tau + 2\xi - \xi_{,})$$

$$-\frac{153}{4} m^2 e e_{,}^2 \cos(2\tau - \xi - \xi_{,})$$

$$+\frac{17}{4} m^2 e e_{,}^2 \cos(2\tau + \xi - 2\xi_{,})$$

$$+\frac{845}{64} m^2 e_{,}^3 \cos(2\tau - 3\xi_{,}) + \frac{1}{64} m^2 e_{,}^3 \cos(2\tau + 3\xi_{,})$$

$$+\frac{45}{32} m^3 e^2 \cos 4\tau + \frac{15}{16} m^3 e \cos(4\tau - \xi)$$

$$-\frac{45}{32} m^3 e^2 \cos(4\tau - 2\xi) + \frac{175}{32} m^3 e e_{,} \cos(4\tau - \xi - \xi_{,})$$

$$-\frac{45}{32} m^3 e e_{,} \cos(4\tau - \xi + \xi_{,})$$

expression for the longitude is to be obtained, as before, ne equation

$$\frac{d\lambda}{dt} = \frac{h}{r^2} - \frac{1}{r^2} \int \frac{dR}{d\lambda} dt.$$

$$\mathfrak{a} + \mathfrak{b} m + \mathfrak{c} m^2 + \mathfrak{d} m^3 = \frac{a^2}{r^2} + \frac{2a}{r} \delta \frac{1}{r} + a^2 \left(\delta \frac{1}{r}\right)^2$$

t

$$h - \int \frac{dR}{d\lambda} dt = \sqrt{a\mu} \{\mathfrak{A} + \mathfrak{B} m + \mathfrak{C} m^2 + \mathfrak{D} m^3\}$$

$$\sqrt{a\mu} \left\{1 - \frac{e^2}{2} - \frac{m^2}{3} - \frac{803}{384} m^2 e^2 - \frac{m^2 e_{,}^2}{2}\right\}.$$ See Phil. Trans. 1834, p. 139.

$$= \{\mathfrak{a}\mathfrak{C} + \mathfrak{b}\mathfrak{B} + \mathfrak{c}\mathfrak{A}\} m^2 + \{\mathfrak{a}\mathfrak{D} + \mathfrak{b}\mathfrak{C} + \mathfrak{c}\mathfrak{B} + \mathfrak{d}\mathfrak{A}\} m^3$$

: terms in $\frac{d\lambda}{n\, dt}$ multiplied by $m^2 e^2$, $m^2 e e_{,}$, and $m^2 e_{,}^2$, are in p. 131. I shall endeavour now to obtain those multiplied e^3, $m^2 e^2 e_{,}$, $m^2 e e_{,}^2$, $m^2 e_{,}^3$, $m^3 e^2$, $m^3 e e_{,}$, and $m^3 e_{,}^2$. As ntains no terms independent of the eccentricities, the terms nultiplied by the cubes of the eccentricities are not required duce the terms in question.

$$\mathfrak{a} = \frac{\mathrm{a}^2}{\mathrm{r}^2} = 1 + \frac{e^2}{2} + 2\,e\left(1 + \frac{3}{8}\,e^2\right)\cos\xi + \frac{5}{2}e^2\cos 2\,\xi + \frac{13}{4}e^3 \mathrm{c}$$

$$\frac{\mathrm{a}}{\mathrm{r}} = 1 + e\left(1 - \frac{e^2}{8}\right)\cos\xi + e^2\cos 2\,\xi + \frac{9}{8}\,e^3\cos 3\,\xi$$

Let $\mathfrak{a} = 1 + \mathfrak{a}_{(1)}\,e + \mathfrak{a}_{(2)}\,e^2 + \mathfrak{a}_{(3)}\,e^3$

$\mathfrak{b} = \mathfrak{b}_{(1)}\,e + \mathfrak{b}_{(2)}\,e^2 + \mathfrak{b}_{(3)}\,e^3$

$\mathfrak{c} = \mathfrak{c} + \mathfrak{c}_{(1)}\,e + \mathfrak{c}_{(2)}\,e^2 + \mathfrak{c}_{(3)}\,e^3$

$\mathfrak{d} = \mathfrak{d} + \mathfrak{d}_{(1)}\,e + \mathfrak{d}_{(2)}\,e^2$

$\mathfrak{A} = 1 - \frac{e^2}{2} \qquad \mathfrak{B} = \mathfrak{B}_{(2)}\,e^2 + \mathfrak{B}_{(3)}\,e^3$

$\mathfrak{C} = \mathfrak{C} + \mathfrak{C}_{(1)}\,e + \mathfrak{C}_{(2)}\,e^2 + \mathfrak{C}_{(3)}\,e^3$

$\mathfrak{D} = \mathfrak{D} + \mathfrak{D}_{(1)}\,e + \mathfrak{D}_{(2)}\,e^2$

$\mathfrak{a}\,\mathfrak{C} = \{\,\mathfrak{C}_{(3)} + \mathfrak{a}_{(1)}\,\mathfrak{C}_{(2)} + \mathfrak{a}_{(2)}\,\mathfrak{C}_{(1)} + \mathfrak{a}_{(3)}\,\mathfrak{C}\,\}\,e^3$

$\mathfrak{b}\,\mathfrak{B} = \mathfrak{b}_{(1)}\,\mathfrak{B}_{(2)}\,e^3 \qquad \mathfrak{c}\,\mathfrak{A} = \mathfrak{c}'_{(3)}\,e^3 - \frac{1}{2}\,\mathfrak{C}_1\,e^3$

$\mathfrak{a}\,\mathfrak{D} = \mathfrak{D}_2\,e^2 + \mathfrak{a}_{(1)}\,\mathfrak{D}_{(1)}\,e^2 + \mathfrak{a}_{(2)}\,\mathfrak{D}\,e^2$

$\mathfrak{b}\,\mathfrak{C} = \mathfrak{b}_{(1)}\,\mathfrak{C}_{(1)}\,e^2 + \mathfrak{b}_{(2)}\,\mathfrak{C}\,e^3 \qquad \mathfrak{c}\,\mathfrak{B} = \mathfrak{c}\,\mathfrak{B}_2\,e^2$

$\mathfrak{d}\,\mathfrak{A} = \mathfrak{d}_{(2)}\,e^2 - \frac{e^2}{2}\,\mathfrak{d}$

Making m and e equal to unity,

$$\frac{\mathrm{d}\lambda}{\mathrm{n\,d}t} = \mathfrak{C}_3 + \mathfrak{D}_2 + \mathfrak{a}_{(1)}\,\{\mathfrak{C}_{(2)} + \mathfrak{D}_{(1)}\} + \mathfrak{a}_{(2)}\,\{\mathfrak{C}_{(1)} + \mathfrak{D}\} + \mathfrak{a}$$
$$+ \mathfrak{b}_{(1)}\,\{\mathfrak{B}_{(2)} + \mathfrak{C}_{(1)}\} + \mathfrak{b}_2\,\mathfrak{C} + \mathfrak{c}\,\mathfrak{B}_2 + \mathfrak{A}\,\{\mathfrak{c} + \mathfrak{d}\}$$

The last equation merely serves as a guide to show the comb
tions which enter into the formation of the terms now under in
tigation, of the order $m^2\,e^3$ and $m^3\,e^2$.

$$\frac{\mathrm{d}\lambda}{\mathrm{n\,d}t} = \left\{-\frac{15}{8}m^3 e^2 - \frac{87}{8}m^3 e_{\prime}^2\right\}\cos 2\,\tau + \left\{\frac{39}{16}m^2 e^2 + \frac{45}{4}m^2 e_{\prime}^2\right\} e\cos(2$$
$$+ \left\{\frac{19}{16}m^2 e^2 + \frac{5}{4}m^2 e_{\prime}^2\right\} e\cos(2\,\tau + \xi) + \mathfrak{R}_5\,e_{\prime}\cos\xi_{\prime}$$

$$+\left\{-\frac{105}{16}m^2e^2-\frac{369}{64}m^2e_{,}^2\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{\frac{15}{16}m^2e^2+\frac{3}{64}m^2e_{,}^2\right\}e_{,}\cos(2\tau+\xi_{,})$$

$$+\mathfrak{R}\,e^2\cos(2\tau-2\xi)+\frac{3}{16}m^2e^2\cos(2\tau+2\xi)$$

$$-\frac{495}{32}m^3ee_{,}\cos(\xi+\xi_{,})-\frac{783}{16}m^3ee_{,}\cos(2\tau-\xi-\xi_{,})$$

$$-\frac{139}{48}m^3ee_{,}\cos(2\tau+\xi+\xi_{,})-\frac{675}{32}m^3ee_{,}\cos(\xi-\xi_{,})$$

$$+\frac{9}{16}m^3ee_{,}\cos(2\tau-\xi+\xi_{,})+\frac{73}{16}m^3ee_{,}\cos(2\tau+\xi-\xi_{,})$$

$$+\frac{357}{16}m^3e_{,}^2\cos(2\tau-2\xi_{,})-\frac{9}{16}m^3e_{,}^2\cos(2\tau+2\xi_{,})$$

$$+\frac{7}{16}m^2e^3\cos(2\tau-3\xi)+\frac{5}{16}m^2e^3\cos(2\tau+3\xi)$$

$$+\mathfrak{R}_{24}e^2e_{,}\cos(2\tau-2\xi-\xi_{,})-\frac{3}{16}m^2e^2e_{,}\cos(2\tau+2\xi+\xi_{,})$$

$$+\mathfrak{R}_{27}e^2e_{,}\cos(2\tau-2\xi+\xi_{,})+\frac{21}{16}m^2e^2e_{,}\cos(2\tau+2\xi-\xi_{,})$$

$$-\frac{153}{4}m^2ee_{,}\cos(2\tau-\xi-2\xi_{,})+\frac{17}{4}m^2ee_{,}^2\cos(2\tau+\xi-2\xi_{,})$$

$$+\frac{845}{64}m^2e_{,}^3\cos(2\tau-3\xi_{,})+\frac{1}{64}m^2e_{,}^3\cos(2\tau+3\xi_{,})$$

$$+\frac{45}{32}m^3e^2\cos 4\tau-\frac{45}{32}m^3e^2\cos(4\tau-2\xi)$$

$$+\frac{175}{32}m^3ee_{,}\cos(4\tau-\xi-\xi_{,})-\frac{45}{32}m^3ee_{,}\cos(4\tau-\xi+\xi_{,})$$

$$+2e\cos\xi\left\{-\frac{803}{384}m^2e^2-\frac{m^2}{2}e_{,}^2+\left\{-\frac{15}{8}m^2e^2-\frac{15}{8}m^2e_{,}^2\right\}\cos 2\tau\right.$$

$$-\frac{135}{16}m^2e\cos\xi-9m^3e\cos(2\tau-\xi)+\frac{m^3}{3}e\cos(2\tau+\xi)$$

$$+\frac{99}{16}m^3e_{,}\cos(2\tau-\xi_{,})-\frac{39}{16}m^3e_{,}\cos(2\tau+\xi_{,})-\frac{45}{32}m^2e^2\cos(2\tau-2\xi)$$

$$+\frac{3}{8}m^2e^2\cos(2\tau+2\xi)-\frac{63}{4}m^2ee_{,}\cos(2\tau-\xi-\xi_{,})$$

$$-\frac{m^2}{4}ee_{,}\cos(2\tau+\xi+\xi_{,})-\frac{9}{4}m^2ee_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\frac{7}{4}m^2ee_{,}\cos(2\tau+\xi-\xi_{,})+\frac{51}{8}m^2e_{,}^2\cos(2\tau-2\xi_{,})$$

$$+\frac{15}{16}m^3 e\cos(4\tau-\xi)$$

$$+\left\{\frac{e^2}{2}+\frac{5}{2}e^2\cos 2\xi\right\}\left\{-\frac{9}{2}m^2 e\cos(2\tau-\xi)+\frac{m^2}{2}e\cos(2\tau+\xi)\right.$$

$$\left.+\frac{21}{8}m^2 e_{,}\cos(2\tau-\xi_{,})-\frac{3}{8}m^2 e_{,}\cos(2\tau+\xi_{,})+\frac{3}{4}m^3\cos 2\tau\right\}$$

$$+\left\{\frac{13}{4}e^3\cos 3\xi+\frac{3}{4}e^3\cos\xi\right\}\left\{-\frac{m^2}{3}+\frac{3}{4}m^2\cos 2\tau\right\}$$

$$+\frac{15}{4}m e\cos(2\tau-\xi)\left\{-\frac{15}{8}m e^2\cos(2\tau-2\xi)-\frac{9}{2}m^2 e\cos(2\tau-\xi)\right.$$

$$\left.+\frac{m^2}{2}e\cos(2\tau+\xi)+\frac{21}{8}m^2 e_{,}\cos(2\tau-\xi_{,})-\frac{3}{8}m^2 e_{,}\cos(2\tau+\xi_{,})\right\}$$

$$+\left\{\frac{75}{4}m e^2\cos 2\tau-\frac{45}{8}m e^2\cos(2\tau-2\xi)-\frac{21}{4}m e e_{,}\cos(\xi+\xi_{,})\right.$$

$$\left.+\frac{35}{4}m e e_{,}\cos(2\tau-\xi-\xi_{,})+\frac{21}{4}m e e_{,}\cos(\xi-\xi_{,})-\frac{15}{4}m e e_{,}\cos(2\tau-\xi+\xi_{,})\right\}$$

$$\left\{-\frac{m^2}{3}+\frac{3}{4}m^2\cos 2\tau\right\}$$

$$+\frac{3765^*}{256}m^3 e^2+\frac{5}{2}m^3 e^2\cos 2\tau+\frac{225}{32}m^2 e^3\cos\xi$$

$$+\frac{75}{16}m^2 e^2 e_{,}\cos\xi_{,}+\frac{315}{64}m^2 e^2 e_{,}\cos(2\tau-\xi_{,})-\frac{315}{64}m^2 e^2 e_{,}\cos(2\tau+\xi_{,})$$

$$+\frac{495}{128}m^3 e^2\cos 2\xi+\frac{5}{4}m^3 e^2\cos(2\tau-2\xi)+\frac{41}{16}m^3 e e_{,}\cos(\xi+\xi_{,})$$

$$-\frac{191}{48}m^3 e e_{,}\cos(2\tau-\xi-\xi_{,})-\frac{21}{8}m^3 e e_{,}\cos(2\tau+\xi+\xi_{,})$$

$$+\frac{89}{16}m^3 e e_{,}\cos(\xi-\xi_{,})-\frac{13}{16}m^3 e e_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\frac{21}{8}m^3 e e_{,}\cos(2\tau+\xi-\xi_{,})-\frac{315}{64}m^2 e^2 e_{,}\cos(2\tau-2\xi+\xi_{,})$$

$$+\frac{975}{128}m^3 e^2 e_{,}\cos 4\tau+\frac{2805}{256}m^3 e^2\cos(4\tau-2\xi)$$

$$+\frac{175}{16}m^3 e e_{,}\cos(4\tau-\xi-\xi_{,})-\frac{45}{16}m^3 e e_{,}\cos(4\tau-\xi+\xi_{,})$$

---

* For these terms, see p. 120 and p. 144.

$$+\frac{525}{64}m^2e^2e_{,}\cos(4\tau-2\xi-\xi_{,})-\frac{225}{64}m^2e^2e_{,}\cos(4\tau-2\xi+\xi_{,})$$

$$+\left\{\frac{12883}{128}m^3e^2-\frac{239}{6}m^3e_{,}^2\right\}\cos 2\tau+2r_2\iota^3\cos\xi$$

$$+\left\{-\frac{463}{64}m^2e^2-\frac{385}{16}m^3e_{,}^2\right\}e\cos(2\tau-\xi)$$

$$+\left\{\frac{5037}{128}m^2e^2-\frac{165}{16}m^2e_{,}^2\right\}e\cos(2\tau+\xi)$$

$$+\left\{-\frac{9}{2}m^2e^2-\frac{27}{8}m^2e_{,}^2\right\}e_{,}\cos\xi_{,}+\left\{\frac{799}{8}m^2e^2-\frac{123}{8}m^2e_{,}^2\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{-\frac{207}{8}m^2e^2+\frac{1}{8}m^2e_{,}^2\right\}e_{,}\cos(2\tau+\xi_{,})-\frac{45}{8}m^3e^2\cos 2\xi$$

$$-\frac{225}{8}m^3e^2\cos(2\tau-2\xi)+\frac{127}{6}m^3e^2\cos(2\tau+2\xi)$$

$$-\frac{7231}{64}m^3e\,e_{,}\cos(\xi+\xi_{,})+\frac{48935}{384}m^3e\,e_{,}\cos(2\tau-\xi-\xi_{,})$$

$$-\frac{1727}{64}m^3e\,e_{,}\cos(2\tau+\xi+\xi_{,})+\frac{453}{2}m^3e\,e_{,}\cos(\xi-\xi_{,})$$

$$+\frac{49277}{384}m^3e\,e_{,}\cos(2\tau-\xi+\xi_{,})+\frac{5727}{64}m^3e\,e_{,}\cos(2\tau+\xi-\xi_{,})$$

$$+\frac{799}{6}m^3e_{,}^2\cos(2\tau-2\xi_{,})+\frac{59}{32}m^2e^3\cos 3\xi$$

$$-\frac{7523}{384}m^2e^3\cos(2\tau-3\xi)+\frac{2125}{192}m^2e^3\cos(2\tau+3\xi)$$

$$-\frac{789}{16}m^2e^2e_{,}\cos(2\xi+\xi_{,})-\frac{105}{4}m^2e^2e_{,}\cos(2\tau-2\xi-\xi_{,})$$

$$-\frac{7}{2}m^2e^2e_{,}\cos(2\tau+2\xi+\xi_{,})+\frac{1161}{16}m^2e^2e_{,}\cos(2\xi-\xi_{,})$$

$$+\frac{15}{4}m^2e^2e_{,}\cos(2\tau-2\xi+\xi_{,})+\frac{49}{2}m^2e^2e_{,}\cos(2\tau+2\xi-\xi_{,})$$

$$-\frac{1965}{128}m^2e\,e_{,}^2\cos(\xi+2\xi_{,})+\frac{12011}{128}m^2e\,e_{,}^2\cos(2\tau-\xi-2\xi_{,})$$

$$+\frac{4635}{128}m^2e\,e_{,}^2\cos(\xi-2\xi_{,})+2r_{33}m^2e\,e_{,}^2\cos(2\tau-\xi+2\xi_{,})$$

$$+\frac{561}{16}m^2e\,e_{,}^2\cos(2\tau+\xi-2\xi_{,})-\frac{53}{8}m^2e_{,}^3\cos 3\xi_{,}+\frac{845}{24}m^2e_{,}^3\cos(2\tau-3\xi_{,})$$

$$+\frac{m^2}{24}e_{,}^3\cos(2\tau+3\xi_{,})+\frac{105}{4}m^3e^2\cos 4\tau+\frac{39795}{32}m^3e^2\cos(4\tau-2\xi)$$

$$+\frac{5775}{128}m^3e\,e_{,}\cos(4\tau-\xi-\xi_{,})-\frac{1485}{128}m^3e\,e_{,}\cos(4\tau-\xi+\xi_{,})$$

$$- \frac{225}{16} m^2 e^2 e_{,} \cos (4 \tau - 2 \xi + \xi_{,}) + \frac{525}{16} m^2 e^2 e_{,} \cos (4 \tau - 2 \xi - \xi_{,})$$

$$+ 2 e \cos \xi \left\{ \frac{m^2}{4} e_{,}^2 + \left\{ \frac{189}{16} m^2 e^2 - \frac{5}{2} m^2 e_{,}^2 \right\} \cos 2 \tau \right.$$

$$+ \frac{33113}{1536} m^3 e \cos (2 \tau - \xi) + \frac{101}{16} m^3 e \cos (2 \tau + \xi)$$

$$+ \frac{157}{8} m^3 e_{,} \cos (2 \tau - \xi_{,}) - \frac{91}{24} m^3 e_{,} \cos (2 \tau + \xi_{,})$$

$$+ \frac{5}{3} m^2 e^2 \cos 2 \xi - \frac{15}{4} m^2 e^2 \cos (2 \tau - 2 \xi) + \frac{7}{2} m^2 e^2 \cos (2 \tau + 2 \xi)$$

$$- \frac{837}{64} m^2 e e_{,} \cos (\xi + \xi_{,}) + \frac{1269}{64} m^2 e e_{,} \cos (2 \tau - \xi - \xi_{,})$$

$$- \frac{33}{32} m^2 e e_{,} \cos (2 \tau + \xi + \xi_{,}) + \frac{1113}{64} m^2 e e_{,} \cos (\xi - \xi_{,})$$

$$- \frac{97}{64} m^2 e e_{,} \cos (2 \tau - \xi + \xi_{,}) + \frac{231}{32} m^2 e e_{,} \cos (2 \tau + \xi - \xi_{,})$$

$$\left. - \frac{9}{4} m^2 e_{,}^2 \cos 2 \xi_{,} + \frac{17}{2} m^2 e_{,}^2 \cos (2 \tau - 2 \xi_{,}) + \frac{495}{128} m^3 e \cos (4 \tau - \xi) \right\}$$

$$+ \left\{ 2 e^2 \cos 2 \xi - e^2 \right\} \left\{ \frac{19}{6} m^3 \cos 2 \tau + \frac{2}{3} m^2 e \cos \xi + \frac{187}{32} m^2 e \cos (2 \tau - \xi) \right.$$

$$+ \frac{33}{16} m^2 e \cos (2 \tau + \xi) - \frac{3}{2} m^2 e_{,} \cos \xi_{,} + \frac{7}{2} m^2 e_{,} \cos (2 \tau - \xi_{,})$$

$$\left. - \frac{m^2}{2} e_{,} \cos (2 \tau + \xi_{,}) \right\}$$

$$+ \left\{ \frac{9}{4} e^3 \cos 3 \xi - \frac{5}{4} e^3 \cos \xi \right\} \left\{ \frac{m^2}{6} + m^2 \cos 2 \tau \right\}$$

Omitting terms which have been obtained through previous developments (see p. 151.), and others where the divisor introduced by integration is a multiple of m, which I shall endeavour hereafter to treat by a different method;

$$\frac{d \lambda}{n\, d t} = \left\{ \left\{ - \frac{15}{8} - 9 + \frac{1}{3} - \frac{3}{8} - \frac{25}{4} + \frac{5}{2} + \frac{12883}{128} + \frac{33113}{1536} + \frac{101}{16} - \frac{19}{6} \right\} m^3 e^2 \right.$$

$$\left. + \left\{ - \frac{87}{8} - \frac{239}{6} \right\} m^3 e_{,}^2 \right\} \cos 2 \tau$$

$$+ \left\{ \left\{ - \frac{803}{384} - \frac{1}{4} - \frac{225}{64} + \frac{225}{32} + 2 r_2 + \frac{5}{3} + \frac{2}{3} - \frac{2}{3} - \frac{5}{24} \right\} m^2 e^2 \right.$$

$$\left. + m^2 e_{,}^2 \right\} e \cos \xi$$

$$+\left\{\left\{\frac{39}{16}-\frac{15}{8}-\frac{45}{32}-\frac{9}{4}+\frac{5}{8}+\frac{9}{32}-\frac{463}{64}+\frac{189}{16}-\frac{15}{4}-\frac{187}{32}\right.\right.$$

$$\left.\left.-\frac{5}{8}+\frac{33}{16}\right\} m^2 e^3+\left\{\frac{45}{4}-\frac{15}{8}-\frac{385}{16}-\frac{5}{2}\right\} m^2 e_{,}^2\right\} e \cos (2 \tau-\xi)$$

$$+\left\{-\frac{135}{16}+\frac{15}{16}-\frac{135}{64}+\frac{495}{128}-\frac{165}{32}\right\} m^3 e^2 \cos 2 \xi$$

$$+\left\{\frac{3}{16}+\frac{1}{3}+\frac{15}{16}+\frac{127}{6}+\frac{101}{16}+\frac{19}{6}\right\} m^3 e^2 \cos (2 \tau+2 \xi)$$

$$+\left\{-\frac{495}{32}-\frac{45}{64}+\frac{7}{4}+\frac{105}{32}+\frac{41}{16}-\frac{7231}{64}\right\} m^3 e\, e_{,} \cos (\xi+\xi_{,})$$

$$+\left\{-\frac{783}{16}+\frac{99}{16}-\frac{35}{12}-\frac{63}{32}-\frac{191}{48}+\frac{48935}{384}+\frac{157}{8}\right\} m^3 e\, e_{,} \cos (2 \tau-\xi-\xi_{,})$$

$$+\left\{-\frac{139}{48}-\frac{39}{16}-\frac{63}{32}-\frac{21}{8}-\frac{1687}{64}-\frac{91}{24}\right\} m^3 e\, e_{,} \cos (2 \tau+\xi+\xi_{,})$$

$$+\left\{-\frac{675}{32}+\frac{315}{64}-\frac{7}{4}-\frac{45}{32}+\frac{89}{16}+\frac{453}{2}\right\} m^3 e\, e_{,} \cos (\xi-\xi_{,})$$

$$+\left\{\frac{9}{16}-\frac{39}{16}+\frac{5}{4}+\frac{63}{32}-\frac{13}{16}+\frac{49277}{384}-\frac{91}{24}\right\} m^3 e\, e_{,} \cos (2 \tau-\xi+\xi_{,})$$

$$+\left\{\frac{73}{16}+\frac{99}{16}+\frac{63}{32}+\frac{21}{8}+\frac{5727}{64}+\frac{157}{8}\right\} m^3 e\, e_{,} \cos (2 \tau+\xi-\xi_{,})$$

$$+\left\{\frac{357}{16}+\frac{799}{6}\right\} m^3 e_{,}^2 \cos (2 \tau-2 \xi_{,})$$

$$+\left\{-\frac{9}{16}-\frac{3}{2}\right\} m^3 e_{,}^2 \cos (2 \tau+2 \xi_{,})$$

$$+\left\{-\frac{13}{12}+\frac{95}{16}+\frac{5}{3}+\frac{2}{3}+\frac{3}{8}\right\} m^2 e^3 \cos 3 \xi$$

$$+\left\{\frac{7}{16}-\frac{45}{32}-\frac{45}{8}+\frac{39}{32}-\frac{7523}{384}-\frac{15}{4}+\frac{187}{32}+\frac{9}{8}\right\} m^2 e^3 \cos (2 \tau-3 \xi)$$

$$+\left\{-\frac{789}{16}-\frac{837}{64}-\frac{3}{2}\right\} m^2 e^2 e_{,} \cos (2 \xi+\xi_{,})$$

$$+\left\{\frac{1161}{16}+\frac{1113}{64}-\frac{3}{2}\right\} m^2 e^2 e_{,} \cos (2 \xi-\xi_{,})$$

$$+\left\{-\frac{1965}{128}-\frac{9}{4}\right\} m^2 e\, e_{,}^2 \cos (\xi+2 \xi_{,})$$

$$+\left\{-\frac{153}{4}+\frac{51}{8}+\frac{12011}{128}+\frac{17}{2}\right\} m^2 e\, e_{,}^2 \cos (2 \tau-\xi-2 \xi_{,})$$

$$+\left\{\frac{4635}{128}-\frac{9}{4}\right\} m^2 e\, e_{,}^2 \cos (\xi-2 \xi_{,})$$

$$+\left\{\frac{45}{32}+\frac{15}{16}+\frac{15}{16}+\frac{225}{32}+\frac{975}{128}+\frac{105}{4}+\frac{495}{128}\right\} m^3 e^2 \cos 4 \tau$$

$$+\left\{-\frac{45}{32}+\frac{15}{16}-\frac{135}{16}-\frac{135}{64}+\frac{2805}{256}+\frac{39795}{32}+\frac{495}{128}\right\} m^3 c^2 \cos(4\tau-2\xi)$$

$$+\left\{\frac{175}{32}+\frac{315}{64}+\frac{105}{32}+\frac{175}{16}+\frac{5775}{128}\right\} m^3 e e_, \cos(4\tau-\xi-\xi_,)$$

$$+\left\{-\frac{45}{32}-\frac{45}{64}-\frac{45}{32}-\frac{45}{16}-\frac{1485}{128}\right\} m^3 e e_, \cos(4\tau-\xi+\xi_,)$$

$$+\left\{\frac{525}{64}+\frac{525}{16}\right\} m^2 e^2 e_, \cos(4\tau-2\xi-\xi_,)$$

$$+\left\{-\frac{225}{64}-\frac{225}{16}\right\} m^2 e^2 e_, \cos(4\tau-2\xi-\xi_,)$$

$$-\frac{227085}{1024} m^2 e^3 \cos(4\tau-3\xi_,)$$

$$\lambda_1=\left\{\frac{75}{8}m+\frac{801}{32}m^2+\frac{57055}{512}m^3\right\}\frac{e^2}{2}\{1+m+m^2\}$$

$$+\left\{-\frac{55}{8}m^2-\frac{1217}{24}m^3\right\}\frac{e_,^2}{2}\{1+m\}$$

$$=\left\{\frac{75}{16}m+\frac{1101}{64}m^2+\frac{74671}{1024}m^3\right\}e^2$$

$$+\left\{-\frac{55}{16}m^2-\frac{691}{24}m^3\right\}e_,^2$$

$$\lambda_3=-\frac{369}{64}m^2e^2+\left\{-\frac{75}{8}m-\frac{275}{16}m^2\right\}e_,^2\{1+2m\}$$

$$=-\frac{369}{64}m^2e^2+\left\{-\frac{75}{8}m-\frac{575}{16}m^2\right\}e_,^2$$

For λ_4, λ_6, and λ_7, see p. 152.

$$\lambda_8=\left\{\frac{5}{2}+\frac{21}{6}m^2-\frac{1425}{128}m^3\right\}\frac{1}{2}\left\{1+\frac{3}{4}m^2+\frac{225}{32}m^3\right\}$$

$$=\frac{5}{4}+\frac{43}{16}m^2+\frac{2765}{256}m^3$$

$$\lambda_{10}=\left\{\frac{285}{32}m^2+\frac{1541}{48}m^3\right\}\frac{1}{3}\left\{1+\frac{m}{2}\right\}$$

$$=\frac{95}{32}m^2+\frac{913}{96}m^3$$

$$\lambda_{11}=\left\{-\frac{21}{4}m-\frac{805}{32}m^2-\frac{1945}{16}m^3\right\}\left\{1-m+\frac{7}{4}m^2\right\}$$

$$= -\frac{21}{4}m - \frac{717}{32}m^2 - \frac{3379}{32}m^3$$

$$\lambda_{12} = \left\{\frac{35}{4}m + \frac{961}{32}m^2 + \frac{12217}{128}m^3\right\}\left\{1 + 3m + \frac{33}{4}m^2\right\}$$

$$= \frac{35}{4}m + \frac{1801}{32}m^2 + \frac{32989}{128}m^3$$

$$\lambda_{13} = \left\{-\frac{51}{16}m^2 - \frac{2565}{64}m^2\right\}\frac{1}{3}\left\{1 + \frac{m}{3}\right\}$$

$$= -\frac{17}{16}m^2 - \frac{2633}{192}m^3$$

$$\lambda_{14} = \left\{\frac{21}{4}m + \frac{1065}{32}m^2 + \frac{12615}{64}m^3\right\}\left\{1 + m + \frac{7}{4}m^2\right\}$$

$$= \left\{\frac{21}{4}m + \frac{1233}{32}m^2 + \frac{15333}{64}m^3\right.$$

$$\lambda_{15} = \left\{-\frac{15}{4}m - \frac{53}{32}m^2 + \frac{49321}{384}m^3\right\}\left\{1 + m + \frac{m^2}{4}\right\}$$

$$= -\frac{15}{4}m - \frac{173}{32}m^2 + \frac{48325}{384}m^3$$

$$\lambda_{16} = \left\{\frac{357}{16}m^2 + \frac{7965}{64}m^3\right\}\frac{1}{3}\{1 + m\}$$

$$= \frac{119}{16}m^2 + \frac{3131}{64}m^3$$

$$\lambda_{18} = \left\{\frac{187}{8}m^2 + \frac{8534}{48}m^3\right\}\frac{1}{2}\{1 + 2m\}$$

$$= \frac{187}{16}m^2 + \frac{9707}{96}m^3$$

$$\lambda_{19} = \left\{-\frac{9}{16} - \frac{3}{2}\right\}\frac{m^3}{2} = -\frac{33}{32}m^3$$

$${}_{20} = \left\{\frac{13}{4} + \frac{121}{16}m^2\right\}\frac{1}{3}\left\{1 + \frac{3}{4}m^2\right\}$$

$$= \frac{13}{12} + \frac{10}{3}m^2$$

All the discrepancies between the values given in this page and those of M. Plana, except that of λ_{11}, are removed as in former i stances by sustituting $e\left\{1-\frac{m^2}{2}\right\}$ for e in my expression.

$$\lambda_{21} = -\left\{-\frac{105}{32}m - \frac{8531}{384}m^2\right\}\left\{1-2m\right\}$$

$$= \frac{105}{32}m + \frac{6511}{384}m^2$$

For λ_{22}, λ_{25}, λ_{28}, λ_{34}, λ_{36}, and λ_{37}, see p. 154.

$$\lambda_{23} = \left\{-\frac{105}{8}m - \frac{4089}{64}m^2\right\}\frac{1}{2}\left\{1-\frac{m}{2}\right\}$$

$$= -\frac{105}{16}m - \frac{3669}{128}m^2$$

$$\lambda_{26} = \left\{\frac{105}{8}m + \frac{5661}{64}m^2\right\}\frac{1}{2}\left\{1+\frac{m}{2}\right\}$$

$$= \frac{105}{16}m + \frac{6081}{128}m^2$$

$$\lambda_{29} = \left\{-\frac{63}{16}m - \frac{2253}{128}m^2\right\}\left\{1-2m\right\}$$

$$= -\frac{63}{16}m - \frac{1245}{128}m^2$$

$$\lambda_{30} = \left\{\frac{225}{16}m + \frac{9019}{128}m^2\right\}\left\{1+4m\right\}$$

$$= \frac{255}{16}m + \frac{17179}{128}m^2$$

$$\lambda_{32} = \left\{\frac{63}{16}m + \frac{4347}{128}m^2\right\}\left\{1+2m\right\}$$

$$= \frac{63}{16}m + \frac{5355}{128}m^2$$

$$\lambda_{131} = \frac{3075}{256}m^3e^2$$

$$\lambda_{136} = \left\{\frac{1125}{128}m^2 + \frac{319335}{256}m^3\right\}\frac{1}{2}\left\{1+2m\right\}$$

$$= \frac{1125}{256} m^2 + \frac{321585}{512} m^3$$

$$\lambda_{138} = \frac{2975}{128} m^3 \qquad \lambda_{140} = -\frac{765}{128} m^3$$

And the longitude contains the following terms, to which I have not assigned any index:

$$-\frac{227085}{1024} m^2 e^3 \sin(4\tau - 3\xi) - \frac{1125}{728} m^2 e^2 e_{,} \sin(4\tau - 2\xi + \xi_{,})$$

$$+ \frac{2625}{120} m^2 e^2 e_{,} \sin(4\tau - 2\xi - \xi_{,})$$

M. de Pontécoulant, to whom I communicated these sheets in their progress through the press, kindly favoured me with the notice of some numerical corrections (see p. xix), which have enabled me to obtain values of λ_{13}, λ_{14}, λ_{15}, and λ_{21}, agreeing with those given by M. Plana.

I shall next attempt to arrive at the terms in the longitude multiplied by $m^3 e$, those multiplied by $m^3 e_{,}$ have already been obtained in p. 64.

$$\mathfrak{a}^2 \left(\delta \frac{1}{r}\right)^2 = \frac{15}{8} m^3 e \cos\xi + \frac{5}{8} m^3 e \cos(2\tau - \xi)$$

$$+ \frac{15}{8} m^3 e \cos(4\tau - \xi) \qquad \text{(See p. 55.)}$$

$$\mathfrak{a}\left(\delta \frac{1}{r}\right) = \frac{m^2}{6} + \left(m^2 + \frac{19}{6} m^3\right) \cos 2\tau + \frac{2}{3} m^2 e \cos\xi$$

$$+ \left\{\frac{15}{8} m + \frac{187}{32} m^2 + \frac{33113}{1536} m^3\right\} e \cos(2\tau - \xi)$$

$$+ \left\{\frac{33}{16} m^2 + \frac{101}{16} m^3\right\} e \cos(2\tau + \xi) + \frac{495}{128} m^3 e \cos(4\tau - \xi)$$

$$\frac{d\lambda}{n\,dt} = \{\mathfrak{a}\,\mathfrak{D} + \mathfrak{b}\,\mathfrak{C} + \mathfrak{c}\,\mathfrak{B} + \mathfrak{d}\,\mathfrak{A}\}\, m^2 \qquad \text{(See p. 134.)}$$

$$\mathfrak{a} = 1 + 2 e \cos\xi \qquad \mathfrak{b} = \frac{15}{4} m e \cos(2\tau - \xi)$$

$$\mathfrak{B} = 0 \qquad \mathfrak{A} = 1$$

$$\frac{\mathrm{d}\,\lambda}{\mathrm{n}\,\mathrm{d}\,t} = -\frac{135}{16} m^3 e \cos \xi - 9\, m^3 e \cos (2\,\tau - \xi) + \frac{m^3}{3} e \cos (2\,\tau + \xi)$$

$$+ \frac{15}{16} m^3 e \cos (4\,\tau - \xi) + 2\, e \cos \xi \left(\frac{3}{4} m^3 \cos 2\,\tau\right)$$

$$+ \frac{15}{4} m\, e \cos (2\,\tau - \xi) \left\{1 - \frac{m^2}{3} + \frac{3}{4} m^2 \cos 2\,\tau\right\}$$

$$+ \frac{33113}{768} m^3 e \cos (2\,\tau - \xi) + \frac{101}{8} m^3 e \cos (2\,\tau + \xi) + \frac{495}{64} m^3 e \cos (4\,\tau - \xi)$$

$$+ \frac{19}{6} m^3 e \cos (2\,\tau - \xi) + \frac{19}{6} m^3 e \cos (2\,\tau + \xi) + \frac{15}{8} m^3 e \cos \xi$$

$$+ \frac{5}{8} m^3 e \cos (2\,\tau - \xi) + \frac{15}{8} m^3 e \cos (4\,\tau - \xi)$$

$$= \left\{-\frac{135}{16} + \frac{45}{32} + \frac{15}{8}\right\} m^3 e \cos \xi$$

$$+ \left\{-9 + \frac{3}{4} - \frac{5}{4} + \frac{33113}{768} + \frac{19}{6} + \frac{5}{8}\right\} m^3 e \cos (2\,\tau - \xi)$$

$$+ \left\{\frac{1}{3} + \frac{3}{4} + \frac{101}{8} + \frac{19}{6}\right\} m^3 e \cos (2\,\tau + \xi)$$

$$+ \left\{\frac{15}{16} + \frac{45}{32} + \frac{495}{64} + \frac{15}{8}\right\} m^3 e \cos (4\,\tau - \xi)$$

Uniting to these, terms found before,

$$\frac{\mathrm{d}\,t}{\mathrm{n}\,\mathrm{d}\,t} = 1 + \left\{2 + m^2 - \frac{165}{32} m^3\right\} e \cos \xi$$

$$+ \left\{\frac{15}{4} m + \frac{143}{16} m^2 + \frac{28729}{768} m^3\right\} e \cos (2\,\tau - \xi)$$

$$+ \left\{\frac{51}{8} m^2 + \frac{135}{8} m^3\right\} e \cos (2\,\tau + \xi)$$

$$+ \frac{765}{64} m^3 e \cos (4\,\tau - \xi)$$

Integrating,

$$\lambda = \mathrm{n}\, t + \left\{2 + \frac{5}{2} m^2 + \frac{285}{32} m^3\right\} e \sin \xi$$

[2]

$$+ \left\{\frac{15}{4} m + \frac{263}{16} m^2 + \frac{51817}{768} m^3\right\} e \sin (2\,\tau - \xi)$$

[3]

$$+\left\{\frac{17}{8}m^2+\frac{169}{24}m^3\right\}e\cos(2\tau+\xi)$$

[4]

$$+\frac{255}{64}m^3 e\cos(4\tau-\xi)$$

[132]

The coefficient of $e\sin\xi$ in the expression for the longitude, according to M. Plana, is

$$2+\frac{3}{2}m^2-\frac{75}{64}m^3$$

and of $e\cos\xi$ in that for $a\,\delta\frac{1}{r}$ is

$$1+\frac{m^2}{6}-\frac{645}{128}m^3$$

In order that these values may be *consistent* with those found in this Treatise, it is evident that the following equation ought to obtain,

$$\frac{1+\frac{2}{3}m^2}{1+\frac{m^2}{6}-\frac{645}{128}m^3}=\frac{2+\frac{5}{2}m^2+\frac{285}{32}m^3}{2+\frac{3}{2}m^2-\frac{75}{64}m^3}$$

which, however, is not the case.

M. Plana has

$$\frac{50377}{768}m^3 \text{ instead of } \frac{51817}{768}m^3 \quad (\text{Arg. } 3.)$$

these figures may be arrived at, as in former instances, by substituting $e\left(1-\frac{m^2}{2}\right)$ for e in my expression.

I shall now endeavour to show how the terms in the reciprocal of the radius vector and of the longitude may be obtained, which are multiplied by $m^4 e$. For this purpose it is necessary to find the terms in the disturbing function (Arg. 3,) multiplied by m^5.

Considering only the terms multiplied by m^3,

$$a\,\delta\frac{1}{r}=\frac{19}{6}m^3\cos 2\tau+\frac{33113}{1536}m^3 e\cos(2\tau-\xi)$$

$$+\frac{101}{16}m^3 e\cos(2\tau+\xi)+\frac{495}{128}m^3 e\cos(4\tau-\xi)$$

$$\frac{\mathrm{r}}{\mathrm{a}} = 1 - e\cos\xi$$

$$\mathrm{r}\,\delta\frac{1}{r} = \frac{19}{6}m^3\cos 2\tau + \left\{-\frac{19}{12}+\frac{33113}{1536}\right\}m^3 e\cos(2\tau-\xi)$$

$$+\left\{-\frac{19}{12}+\frac{101}{16}\right\}m^3 e\cos(2\tau+\xi)+\frac{495}{128}m^3 e\cos(4\tau-\xi)$$

$$= \frac{19}{6}\,m^3\cos 2\tau + \frac{10227}{512}\,m^3 e\cos(2\tau-\xi)$$

$$+\frac{227}{48}m^3 e\cos(2\tau+\xi)+\frac{495}{128}\,m^3 e\cos(4\tau-\xi)$$

$$\delta\lambda = \frac{59}{12}m^3\sin 2\tau + \lambda_2 m^3 e\sin\xi + \frac{51817}{768}m^3 e\sin(2\tau-\xi)$$

$$+\frac{169}{24}m^3 e\cos(2\tau+\xi)+\frac{255}{64}m^3 e\cos(4\tau-\xi)$$

Considering only those combinations in the expression

$$-\,\mathrm{a}\left(\frac{\mathrm{d}R}{\mathrm{d\,a}}\right)\mathrm{r}\,\delta\frac{1}{r}+\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)\delta\lambda$$

which can give rise to the coefficient of the argument in question,

$$\frac{\mathrm{a}}{\mu}\delta R = m^2\left\{\frac{1}{2}+\frac{3}{2}\cos 2\tau\right\}\left\{r_2\,m^3 e\cos\xi+\frac{10227}{512}m^3 e\cos(2\tau-\xi)\right.$$

$$\left.+\frac{495}{128}m^3 e\cos(4\tau-\xi)\right\}$$

$$-\,m^2 e\cos\xi\left\{\frac{19}{6}\,m^3\cos 2\tau\right\}$$

$$+\frac{3}{2}\,m^2\sin 2\tau\left\{\lambda_2 m^3 e\sin\xi+\frac{255}{64}m^3 e\cos(4\tau-\xi)\right\}$$

$\frac{\mathrm{a}}{\mu}\delta R$ contains the term

$$\left\{\frac{10227}{1024}+\frac{3}{4}r_2+\frac{1485}{512}-\frac{19}{12}+\frac{3}{4}\lambda_2{}^{*}-\frac{765}{256}\right\}m^5 e\cos(2\tau-\xi)$$

If $r_2 = 0$, $\lambda_2 = \frac{285}{32}m^3$

---

[*] These terms are left indeterminate, in order to facilitate any correction which n hereafter become necessary from the discrepancy noticed in p. 197.

$$= \frac{43667}{3072} m^5 e \cos (2\tau - \xi)$$

Considering only the terms multiplied by m^3,

$$2\left(\delta \frac{1}{r}\right)^2 = \frac{15}{8} m^3 e \cos \xi + \frac{5}{8} m^3 e \cos (2\tau - \xi) + \frac{15}{8} m^3 e \cos (4\tau - \xi)$$

(See p. 55.)

$$\delta \frac{1}{r} \delta \lambda = \left\{ \frac{m^2}{6} + m^2 \cos 2\tau \right\} \frac{15}{4} m e \sin (2\tau - \xi)$$

$$+ \frac{15}{8} m e \cos (2\tau - \xi) \frac{11}{8} m^2 \sin 2\tau$$

$$= \left\{ -\frac{15}{8} + \frac{165}{128} \right\} m^3 e \sin \xi + \frac{5}{8} m^3 e \sin (2\tau - \xi)$$

$$+ \left\{ \frac{15}{8} + \frac{165}{128} \right\} m^3 e \sin (4\tau - \xi)$$

$$= -\frac{75}{128} m^3 e \sin \xi + \frac{5}{8} m^3 e \sin (2\tau - \xi)$$

$$+ \frac{405}{126} m^3 e \sin (4\tau - \xi)$$

$$(\delta \lambda)^2 = \frac{165}{64} m^3 e \cos \xi - \frac{165}{64} m^3 e \cos (4\tau - \xi)$$

The disturbing function contains the terms, (see p. 68,)

$$-\frac{3}{4} m^2 \{1 + 3 \cos 2\tau\} \left\{ \frac{15}{8} m^3 e \cos \xi \right.$$

$$\left. + \frac{5}{8} m^3 e \cos (2\tau - \xi) + \frac{15}{8} m^3 e \cos (4\tau - \xi) \right\}$$

$$-3 m^2 \sin 2\tau \left\{ -\frac{75}{128} m^3 e \sin \xi \right.$$

$$\left. + \frac{5}{8} m^3 e \sin (2\tau - \xi) + \frac{405}{128} m^3 e \sin (4\tau - \xi) \right\}$$

$$\cdot \frac{3}{2} m^2 \cos 2\tau \left\{ \frac{165}{64} m^3 e \cos \xi - \frac{165}{64} m^3 e \cos (4\tau - \xi) \right\}$$

Hence $\frac{a}{\mu} R$ contains the term

$$\left. -\frac{15}{32} - \frac{135}{64} - \frac{135}{64} + \frac{225}{256} - \frac{1215}{256} + \frac{495}{256} - \frac{495}{256} \right\} m^5 e \cos (2\tau - \xi)$$

$$= -\frac{1095}{128} m^5 e \cos(2\tau - \xi)$$

Hence $$\frac{\mathrm{a} R}{\mu} = \left\{\frac{9}{4} m^2 + \frac{15}{16} m^3 + \frac{235}{64} m^4 + \frac{19787}{3072} m^5\right\} e \cos(2\tau - \xi) \qquad [3]$$

In the equation which serves to determine the expression for the radius vector, d R signifies the differential of R, only as regards the coordinates of the moon. Hence d R = the differential of R, supposing only n t variable + the differential of R with regard to $\mathrm{n}_{\prime} t$, only in as much as it is contained in the terms in r, λ and s due to the perturbations; hence, neglecting γ^2, d R = the differential of R, supposing only n t variable

$$+ \left(\frac{\mathrm{d} R}{\mathrm{d} r}\right) \mathrm{d}\, \delta r + \left(\frac{\mathrm{d} R}{\mathrm{d} \lambda}\right) \mathrm{d}\,.\,\delta\lambda$$

d . δr, and d $\delta\lambda$ being restrained to mean the differentials of those quantities with regard to $\mathrm{n}_{\prime} t$ only.

$$\delta r = -\mathrm{r}^2 \delta \frac{1}{r} + \mathrm{r}^3 \left(\delta \frac{1}{r}\right)^2 - \mathrm{r}^4 \left(\delta \frac{1}{r}\right)^3 + \&c.$$

As $\left(\delta \frac{1}{r}\right)^2$ is already of the order m^3, it is evidently sufficient, when seeking terms in $\int \mathrm{d} R$ of the order m^5, to take

$$\mathrm{d} R = -\frac{\mathrm{r}\, \mathrm{d} R}{\mathrm{d} r} \mathrm{d}\,.\,\mathrm{r}\,\delta \frac{1}{r} + \frac{\mathrm{d} R}{\mathrm{d} \tau} \mathrm{d}\,\delta\lambda$$

$$= m^2 \left\{\frac{1}{2} + \frac{3}{2} \cos 2\tau\right\} \frac{171}{16} m^3 e \sin(2\tau - \xi)$$

$$- m^2 e \cos\xi\, (2 m^2 \sin 2\tau)$$

$$= \left\{\frac{171}{32} - 1\right\} m^5 e \sin(2\tau - \xi)$$

$$\mathrm{d} R = \left\{\frac{15}{8}^{*} m^4 + \frac{139}{32} m^5\right\} e \sin(2\tau - \xi)$$

$$\frac{\mathrm{a}}{\mu} \int \mathrm{d} R = \left\{-\frac{15}{8} m^4 - \frac{259}{32} m^5\right\} e \cos(2\tau - \xi)$$

---

* See p. 55.

Let $\mathrm{a}\,\delta\frac{1}{r} = \mathfrak{b}\,m + \mathfrak{c}\,m^2 + \mathfrak{d}\,m^3 + \mathfrak{e}\,m^4$

$$\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2 = \{2\,\mathfrak{b}\,\mathfrak{e} + 2\,\mathfrak{c}\,\mathfrak{d}\}\,m^5$$

$$-\frac{15}{4}\,m\,e\cos(2\,\tau-\xi)\left\{-\frac{179}{288}m^4+\frac{131}{18}m^4\cos 2\,\tau+\frac{7}{8}m^4\cos 4\,\tau\right\}$$

$$+\left\{\frac{m^2}{3}+2\,m^2\cos 2\,\tau+\frac{4}{3}m^2 e\cos\xi\right.$$

$$\left.+\frac{187}{16}m^2 e\cos(2\,\tau-\xi)+\frac{33}{8}m^2 e\cos(2\,\tau+\xi)\right\}$$

$$\left\{\frac{19}{6}m^3\cos 2\,\tau+\frac{33113}{1536}m^3 e\cos(2\,\tau-\xi)\right.$$

$$\left.+\frac{101}{16}m^3 e\cos(2\,\tau+\xi)+\frac{495}{128}m^3 e\cos(4\,\tau-\xi)\right\}$$

$$=\frac{19}{6}m^5+\frac{19}{18}m^5\cos 2\,\tau$$

$$+\left\{-\frac{2685}{1152}+\frac{105}{64}+\frac{33113}{4608}+\frac{38}{9}+\frac{495}{128}\right\}m^5 e\cos(2\,\tau-\xi)$$

$$=\frac{19}{6}m^5+\frac{19}{18}m^5\cos 2\,\tau+\frac{22403}{1536}m^5 e\cos(2\,\tau-\xi)$$

$\frac{3}{2}\,\frac{\mathrm{r}^4}{\mathrm{a}^2}\left(\delta\frac{1}{r}\right)^2$ contains the term

$$\left\{\frac{22403}{1024}-\frac{19}{6}\right\}m^5 e\cos(2\,\tau-\xi)$$

$$=\frac{57481}{3072}m^5 e\cos(2\,\tau-\xi)$$

It is sufficient to take

$$\mathrm{a}^3\left(\delta\frac{1}{r}\right)^3 = 3\,\mathfrak{b}\,\mathfrak{c}^2\,m^5$$

$$=\frac{45}{8}m\,e\cos(2\,\tau-\xi)\left\{\frac{19}{36}m^4+\frac{m^4}{3}\cos 2\tau+\frac{m}{2}\cos 4\,\tau\right\}$$

$$=\frac{315}{72}m^5 e\cos(2\,\tau-\xi)$$

Hence,

$\mathfrak{r}_3$ contains the terms

$$\left\{\frac{383}{18}+\frac{57481}{3072}-\frac{315}{36}\right\} m^5 e \cos(2\tau-\xi)$$

$$=\frac{287899}{9216} m^5 e \cos(2\tau-\xi)$$

Uniting to this, terms found before, (see p. 122,)

$$\mathfrak{r}_3=\frac{3}{2}m^2+\frac{91}{16}m^3+\frac{2657}{192}m^4+\frac{287899}{9216}m^5$$

Let $r_3=\frac{15}{8}m+\frac{187}{32}m^2+\frac{33113}{1536}m^3+A\,m^4$, then

$$\left\{\frac{15}{8}m+\frac{187}{32}m^2+\frac{33113}{1536}m^3+A\,m^4\right\}$$

$$\left\{-4m+\frac{11}{2}m^2+\frac{177}{16}m^3+\frac{2877}{64}m^4\right\} \quad \text{(See p. 122.)}$$

$$=\left\{1-4m+\frac{11}{2}m^2+\frac{177}{16}m^3\right\}$$

$$\left\{\frac{3}{2}m^2+\frac{91}{16}m^3+\frac{2657}{192}m^4+\frac{287899}{9216}m^5\right\}$$

$$-2\left\{\left\{2+2m+4m^2+\frac{13}{2}m^3\right\}\left\{\frac{9}{4}m^2+\frac{15}{16}m^3+\frac{235}{64}m^4+\frac{19787}{3072}m^5\right\}\right.$$

$$\left.-\frac{15}{8}m^4-\frac{259}{32}m^5\right\}$$

Equating coefficients,

$$\frac{43155}{16}+\frac{33099}{512}+\frac{364243}{3072}-4A$$

$$=\frac{287899}{9216}-\frac{2657}{48}+\frac{1001}{32}+\frac{531}{32}-\frac{19787}{768}-\frac{235}{16}-\frac{15}{2}-\frac{117}{4}+\frac{259}{16}$$

$$A=\frac{3406141}{4608}$$

$$\mathfrak{r}_4=\frac{3}{2}m^2+\frac{19}{4}m^3+\left\{\frac{131}{12}+\frac{33}{32}\right\}m^4. \quad \text{(See p. 123.)}$$

Let $r_4=\frac{33}{16}m^2+\frac{101}{16}m^3+A\,m^4$

$$\left\{\frac{33}{16}m^2+\frac{101}{16}m^3+A\,m^4\right\}\left\{8-12m-\frac{m^2}{2}\right\}$$

$$-\left\{9-12m-\frac{m^2}{2}\right\}\left\{\frac{3}{2}m^2+\frac{19}{4}m^3+\frac{1147}{96}m^4\right\}$$

$$-2\left\{2+\frac{2}{3}m+\frac{4}{9}m^2\right\}\left\{-\frac{3}{4}m^2-\frac{31}{32}{}^{*}m^3\right\}$$

Equating coefficients,

$$-\frac{33}{32}-\frac{303}{4}+8A=\frac{3441}{32}-57-\frac{3}{4}+\frac{31}{8}+\frac{2}{3}$$

$$A=\frac{6293}{384}$$

M. Plana has $A=\frac{5897}{384}$: these figures may be arrived at, as in former instances, by substituting $e\left(1-\frac{m^2}{2}\right)$ for e in my expression. The calculation of the terms in r_{132} and r_{133} multiplied by m^4 is given in p. 128.

The expression for the longitude is to be obtained, as before, from the equation

$$\frac{d\lambda}{dt}=\frac{h}{r^2}-\frac{1}{r^2}\int\frac{dR}{d\lambda}dt$$

$$=\sqrt{a\mu}\left\{1-\frac{m^2}{3}+\frac{11}{144}m^4\right\} \quad \text{(See p. 135.)}$$

$$\frac{d\lambda}{dt}=\{\mathfrak{a}\,\mathfrak{E}+\mathfrak{b}\,\mathfrak{D}+\mathfrak{c}\,\mathfrak{C}+\mathfrak{d}\,\mathfrak{B}+\mathfrak{e}\,\mathfrak{A}\}m^4$$

Neglecting e^3, $\mathfrak{a}=1+2e\cos\xi$, $\qquad \mathfrak{A}=1$

$\mathfrak{E}=\mathfrak{E}_{(1)}+\mathfrak{a}_{(1)}\mathfrak{E}$ $\qquad \mathfrak{b}\,\mathfrak{D}=\mathfrak{b}_1\mathfrak{D}$

$\mathfrak{C}=\mathfrak{c}\,\mathfrak{C}_{(1)}+\mathfrak{c}_{(1)}\mathfrak{C}$ $\qquad \mathfrak{d}\,\mathfrak{B}=0$ $\qquad \mathfrak{e}\,\mathfrak{A}=\mathfrak{e}_{(1)}$

Making m and e equal to unity,

$$\frac{d\lambda}{dt}=\mathfrak{E}_{(1)}{}^{\dagger}+\mathfrak{a}_{(1)}\mathfrak{E}+\mathfrak{b}_{(1)}\mathfrak{D}+\mathfrak{c}\,\mathfrak{C}_{(1)}+\mathfrak{c}_{(1)}\mathfrak{C}+\mathfrak{e}_{(1)}.$$

The last equation merely serves as a guide to show the combinations which enter into the formation of the terms now under investigation of the order $m^4 e$.

$$\frac{d\lambda}{dt}=-\frac{2649}{64}{}^{\ddagger}m^4e\cos\xi-\frac{141}{8}m^4e\cos(2\tau-\xi)+\frac{77}{72}m^4e\cos(2\tau+\xi)$$

---

[*] See p. 53. † This notation is explained in p. 186. ‡ See p. 60.

$$+\frac{399}{64}{}^{*}m^4 e\cos(4\tau-\xi)+\frac{9}{32}m^4 e\cos(4\tau+\xi)$$

$$+2e\cos\xi\left\{\frac{11}{144}m^4+\frac{m^4}{2}\cos 2\tau+\frac{9}{64}m^4\cos 4\tau\right\}$$

$$+\frac{15}{4}m e\cos(2\tau-\xi)\left\{\frac{3}{4}m^3\cos 2\tau\right\}$$

$$+\left\{\frac{m^2}{3}+2m^2\cos 2\tau\right\}\left\{-\frac{9}{2}m^2 e\cos(2\tau-\xi)+\frac{m^2}{2}e\cos(2\tau+\xi)\right\}$$

$$+\left\{\frac{5}{3}m^2 e\cos\xi+\frac{203}{16}m^2 e\cos(2\tau-\xi)+\frac{41}{8}m^2 e\cos(2\tau+\xi)\right\}$$

$$\left\{-\frac{m^2}{3}+\frac{3}{4}m^2\cos 2\tau\right\}$$

$$-\frac{1253}{144}m^4 e\cos\xi+\frac{1337369}{9216}m^4 e\cos(2\tau-\xi)+\frac{6293}{192}m^4 e\cos(2\tau+\xi)$$

$$+\frac{13725}{256}m^4 e\cos(4\tau-\xi)+\frac{805}{128}m^4 e\cos(4\tau+\xi)$$

$$+2e\cos\xi\left\{\frac{131}{18}m^4\cos 2\tau+\frac{7}{8}m^4\cos 4\tau\right\}$$

$$+\frac{4051}{288}{}^{*}m^4 e\cos\xi+\frac{251}{96}m^4 e\cos(2\tau-\xi)+\frac{65}{48}m^4 e\cos(2\tau+\xi)$$

$$+\frac{377}{32}{}^{*}m^4 e\cos(4\tau-\xi)+\frac{33}{16}m^4 e\cos(4\tau+\xi)$$

$$=\left\{-\frac{2649}{64}+\frac{11}{72}+\frac{45}{32}-\frac{9}{2}+\frac{1}{2}-\frac{5}{9}+\frac{609}{128}+\frac{123}{64}-\frac{1253}{144}+\frac{4051}{288}\right\}m^4 e\cos\xi$$

$$+\left\{-\frac{141}{8}+\frac{1}{2}-\frac{3}{2}+\frac{5}{8}+\frac{203}{48}+\frac{1337369}{9216}+\frac{131}{18}+\frac{251}{96}\right\}m^4 e\cos(2\tau-\xi)$$

$$+\left\{\frac{77}{72}+\frac{1}{2}+\frac{1}{6}+\frac{5}{8}-\frac{41}{24}+\frac{6293}{192}+\frac{131}{18}+\frac{65}{48}\right\}m^4 e\cos(2\tau+\xi)$$

$$+\left\{\frac{399}{64}+\frac{9}{64}+\frac{45}{32}-\frac{9}{2}+\frac{609}{128}+\frac{13725}{256}+\frac{7}{8}+\frac{377}{32}\right\}m^4 e\cos(4\tau-\xi)$$

$$+\left\{\frac{9}{32}+\frac{9}{64}+\frac{1}{2}+\frac{123}{64}+\frac{805}{128}+\frac{7}{8}+\frac{33}{16}\right\}m^4 e\cos(4\tau+\xi)$$

Uniting to these, terms found before, (see p. 196,)

$$\frac{d\lambda}{n\,dt}=\left\{2+m^2-\frac{165}{32}m^3-\frac{37259}{1152}m^4\right\}e\cos\xi$$

$$+\left\{\frac{15}{4}m+\frac{143}{16}m^2+\frac{28729}{768}m^3+\frac{1301625}{9216}m^4\right\}e\cos(2\tau-\xi)$$

---

* The terms in p. 55 and p. 61 require correction.

$$+ \left\{\frac{51}{8} m^2 + \frac{135}{8} m^3 + \frac{13989}{576} m^4\right\} e \cos(2\tau + \xi)$$

$$+ \left\{\frac{765}{64} m^3 + \frac{12373}{256} m^4\right\} e \cos(4\tau - \xi)$$

$$+ \frac{1545}{128} m^4 e \cos(4\tau + \xi)$$

Integrating,

$$\lambda = \left\{2 + \frac{5}{2} m^2 + \frac{285}{32} m^3 + \frac{41599}{1152} m^4\right\} e \sin \xi$$

[2]

$$+ \left\{\frac{15}{4} m + \frac{263}{16} m^2 + \frac{51817}{768} m^3 + \frac{3549833}{2304} m^4\right\} e \sin(2\tau - \xi)$$

[3]

$$+ \left\{\frac{17}{8} m^2 + \frac{169}{24} m^3 + \frac{2005}{192} m^4\right\} e \sin(2\tau + \xi)$$

[4]

$$+ \left\{\frac{255}{64} m^3 + \frac{16453}{768} m^4\right\} e \sin(4\tau - \xi)$$

[132]

$$+ \frac{309}{128} m^4 e \sin(4\tau + \xi)$$

[133]

Some of these terms cannot be expected to coincide with those given by M. Plana, on account of several discrepancies, which have already been noticed: as soon however as these discrepancies shall have been removed, it will be easy to introduce any modifications which may be rendered necessary in the expressions contained in the latter pages.

I now propose to obtain some of the terms in the reciprocal of the radius vector depending upon the fourth powers of the eccentricities, by methods precisely the same as those employed in p. 37 and p. 143.

$$a^2 \left(\delta \frac{1}{r}\right)^2 = \frac{15}{4} m e \cos(2\tau - \xi) \left\{ - \frac{75}{16} m e e_{,}^2 \cos(2\tau - \xi) + \frac{405}{64} m e^3 \cos(2\tau + \xi)\right.$$

$$+ \frac{35}{4} m e^3 e_{,} \cos(2\tau - \xi_{,}) - \frac{15}{4} m e^2 e_{,} \cos(2\tau + \xi_{,}) - \frac{105}{64} m e^3 \cos(2\tau - 3\xi)$$

$$- \frac{21}{4} m e^2 e_{,} \cos(2\xi + \xi_{,}) + \frac{21}{4} m e^2 e_{,} \cos(2\xi - \xi_{,}) - \frac{63}{32} m e e_{,}^2 \cos(\xi + 2\xi_{,})$$

$$+\frac{255}{32}m\,e\,e_{,}^{2}\cos(2\tau-\xi-2\xi_{,})+\frac{63}{32}m\,e\,e_{,}^{2}\cos(\xi-2\xi_{,})$$

$$-\frac{45}{32}m\,e\,e_{,}^{2}\cos(2\tau-\xi+2\xi_{,})\Big\}$$

$$+\Big\{\frac{15}{4}m\,e^{2}\cos 2\tau-\frac{21}{8}m\,e\,e_{,}\cos(\xi+\xi_{,})+\frac{35}{8}m\,e\,e_{,}\cos(2\tau-\xi-\xi_{,})$$

$$+\frac{21}{8}m\,e\,e_{,}\cos(\xi-\xi_{,})-\frac{15}{8}m\,e\,e_{,}\cos(2\tau-\xi+\xi_{,})\Big\}^{2}$$

$$=\frac{225}{32}m^{2}e^{4}+\underset{[0]}{\frac{91}{64}m^{2}e^{2}e_{,}^{2}}-\underset{[1]}{\frac{525}{32}m^{2}e^{2}e_{,}^{2}\cos 2\tau}$$

$$+\underset{[8]}{\left\{\frac{10575}{512}e^{2}-\frac{441}{64}e_{,}^{2}\right\}m^{2}e^{2}\cos 2\xi}$$

$$+\underset{[9]}{\frac{525}{32}m^{2}e^{2}e_{,}^{2}\cos(2\tau-2\xi)}+\underset{[11]}{\frac{75}{8}m^{2}e^{3}e_{,}\cos(\xi+\xi_{,})}$$

$$-\underset{[13]}{\frac{315}{16}m^{2}e^{3}e_{,}\cos(2\tau+\xi+\xi_{,})}+\underset{[14]}{\frac{75}{8}m^{2}e^{3}e_{,}\cos(\xi-\xi_{,})}$$

$$+\underset{[15]}{\frac{315}{32}m^{2}e^{3}e_{,}\cos(2\tau-\xi+\xi_{,})}+\underset{[16]}{\frac{315}{16}m^{2}e^{3}e_{,}\cos(2\tau+\xi-\xi_{,})}$$

$$-\underset{[17]}{\frac{357}{128}m^{2}e^{2}e_{,}^{2}\cos 2\xi_{,}}+\underset{[19]}{\frac{315}{256}m^{2}e^{2}e_{,}^{2}\cos(2\tau+2\xi_{,})}$$

$$-\underset{[42]}{\frac{315}{32}m^{2}e^{3}e_{,}\cos(2\tau-3\xi-\xi_{,})}+\underset{[45]}{\frac{315}{32}m^{2}e^{3}e_{,}\cos(2\tau-3\xi+\xi_{,})}$$

$$+\underset{[47]}{\frac{441}{128}m^{2}e^{2}e_{,}^{2}\cos(2\xi+2\xi_{,})}$$

$$-\underset{[48]}{\frac{3885}{256}m^{2}e^{2}e_{,}^{2}\cos(2\tau-2\xi-2\xi_{,})}+\underset{[50]}{\frac{441}{128}m^{2}e^{2}e_{,}^{2}\cos(2\xi-2\xi_{,})}$$

$$- \frac{315}{256} m^2 e^2 e_{,}^2 \cos (2\tau - 2\xi + 2\xi_{,}) + \frac{6975}{256} m^2 e^4 \cos 4\tau$$

[51] [131]

$$- \frac{825}{32} m^2 e^2 e_{,}^2 \cos (4\tau - 2\xi) + \frac{525}{16} m^2 e^3 e_{,} \cos (4\tau - \xi - \xi_{,})$$

[136] [138]

$$- \frac{225}{16} m^2 e^3 e_{,} \cos (4\tau - \xi + \xi_{,}) - \frac{1575}{512} m^2 e^4 \cos (4\tau - 4\xi).$$

[140]

$$+ \frac{6275}{256} m^2 e^2 e_{,}^2 \cos(4\tau - 2\xi - 2\xi_{,}) - \frac{225}{256} m^2 e^2 e_{,}^2 \cos(4\tau - 2\xi + 2\xi_{,})$$

$$\frac{3\,\mathrm{r}^4}{2\,\mathrm{a}^4} = \frac{3}{2} + \frac{15}{2} e^2 - 6e \cos \xi + 3 e^2 \cos 2\xi$$

For the terms in $\mathrm{a}^2 \left(\delta \frac{1}{r}\right)^2$ multiplied by $m^2 e^3$ &c. see p. 144; hence $\frac{3\,\mathrm{r}^4}{2\,\mathrm{a}^4} \left(\delta \frac{1}{r}\right)^2$ contains the terms

$$- 6e \cos \xi \left\{ \frac{225}{32} m^2 e^3 \cos \xi + \frac{75}{16} m^2 e^2 e_{,} \cos \xi_{,} + \frac{315}{64} m^2 e^2 e_{,} \cos (2\tau - \xi_{,}) \right.$$

$$- \frac{315}{64} m^2 e^2 e_{,} \cos (2\tau + \xi_{,}) - \frac{315}{64} m^2 e^2 e_{,} \cos (2\tau - 2\xi - \xi_{,})$$

$$+ \frac{315}{64} m^2 e^2 e_{,} \cos (2\tau - 2\xi + \xi_{,}) + \frac{525}{64} m^2 e^2 e_{,} \cos (4\tau - 2\xi - \xi_{,})$$

$$\left. - \frac{225}{64} m^2 e^2 e_{,} \cos (4\tau - 2\xi + \xi_{,}) \right\}$$

$$+ \left\{ \frac{15}{2} e^2 + 3 e^2 \cos 2\xi \right\} \left\{ \frac{225}{128} m^2 e^2 + \frac{225}{128} m^2 e^2 \cos (4\tau - 2\xi) \right\}$$

$$\frac{3\,\mathrm{r}^4}{2\,\mathrm{a}^4} \left(\delta \frac{1}{r}\right)^2 = \frac{1575}{64} m^2 e^2 e_{,}^2 \cos (2\tau - 2\xi) + \left\{ \frac{225}{16} - \frac{225}{16} \right\} m^2 e^3 e_{,} \cos (\xi + \xi_{,})$$

$$+ \frac{945}{64} m^2 e^3 e_{,} \cos (2\tau - \xi - \xi_{,}) + \left\{ \frac{225}{16} - \frac{225}{16} \right\} m^2 e^3 e_{,} \cos (\xi - \xi_{,})$$

$$+ \left\{ \frac{945}{64} - \frac{945}{64} \right\} m^2 e^3 e_{,} \cos (2\tau - \xi + \xi)$$

$$+ \left\{ - \frac{945}{64} + \frac{945}{64} \right\} m^2 e^3 e_{,} \cos (2\tau - 3\xi - \xi_{,})$$

$$+\left\{\frac{945}{64}-\frac{945}{64}\right\} m^2 e^3 e_{,} \cos(2\tau-3\xi+\xi_{,})$$

$$-\frac{1925}{1024} m^2 e^4 \cos(4\tau-4\xi)+\&c.$$

$$=\frac{1575}{128} m^2 e^2 e_{,}^2 \cos(2\tau-2\xi)+\frac{945}{64} m^2 e^3 e_{,} \cos(2\tau-\xi-\xi_{,})$$

[9] [12]

$$+\frac{945}{32} m^2 e^3 e_{,} \cos(2\tau-3\xi+\xi_{,})+\frac{675}{256} m^2 e^4 \cos(4\tau-4\xi)$$

[45]

$$\frac{\mathbf{r}^3}{\mathbf{a}^3}=1+3e^2\left(1+\frac{e^2}{8}\right)-3e\left(1+\frac{3}{8}e^2\right)\cos\xi$$

$$+\frac{5}{8}e^4\cos 2\xi+\frac{e^3}{8}\cos 3\xi+\frac{e^4}{8}\cos 4\xi$$

$$\mathfrak{r}_1=\left\{\frac{135}{128}+\frac{1215}{128}\right\} m e^4-\frac{225}{32} m e^2 e_{,}^2$$

$$=\frac{675}{64} m e^4-\frac{225}{32} m e^2 e_{,}^2$$

Let $r_1=\frac{15}{4} m e^2+A m e^4+B m^2 e_{,}^2$,

$$\left\{\frac{15}{4} m e^2+A m e^4+B m e^2 e_{,}^2\right\}\{3+12e^2\}$$

$$=\frac{675}{16} m e^4-\frac{225}{8} m e^2 e_{,}^2$$

$$A=-\frac{15}{16} \qquad B=-\frac{75}{8}$$

Let $r_9=-\frac{15}{4}m^2+A m^2 e^2+B m^2 e_{,}^2$, (See p. 124.)

$$-\left\{-\frac{15}{4}m^2+A m^2 e^2+B m^2 e_{,}^2\right\}=-2\left\{-\frac{15}{8}m^2+\frac{75}{16}m^2 e_{,}^2\right\}$$

$$A=0 \qquad B=\frac{75}{8}$$

$$\mathfrak{r}_{10}=\left\{\frac{1215}{128}-\frac{15}{128}\right\} m e^2=\frac{150}{16} m e^2 \qquad 15 r_{10}=16\mathfrak{r}_{10} \qquad r_{10}=10 m e^2$$

$$\mathfrak{r}_{11}=-\frac{9}{4}m^2-\frac{63}{8}m e^2+\left\{-\frac{27}{8}-\frac{27}{32}-\frac{2367}{64}\right\} m^2 e^2-\frac{81}{32}m^2 e_{,}^2$$

$$=-\frac{9}{4}m^2-\frac{63}{8}m e^2-\frac{2637}{64}m^2 e^2-\frac{81}{32}m^2 e_{,}^2$$

Let $r_{11} = -\frac{21}{8} - \frac{837}{64} m^2 + A\, m\, e^2 + B\, m\, e_{,}^2$

$$\left\{-\frac{21}{8} m - \frac{837}{64} m^2 + A\, m\, e^2 + B\, m\, e_{,}^2\right\} \left\{(1 + 3\, e^2)\,(1 + 2\, m) - 1\right\}$$

$$= \left\{1 + 2\, m\right\} \left\{-\frac{9}{4} m^2 - \frac{63}{8} m\, e^2 - \frac{2637}{64} m^2 e^2 - \frac{81}{32} m^2 e_{,}^2\right\}$$

$$- 4\left\{\frac{3}{4} m^2 - \frac{3}{32} m^2 e^2 + \frac{27}{32} m^2 e_{,}^2\right\}$$

$$-\frac{63}{4} - \frac{2511}{64} + 2\, A = -\frac{63}{4} - \frac{2637}{64} + \frac{3}{8} \qquad 2\, B = -\frac{81}{32} - \frac{27}{8}$$

$$A = -\frac{51}{64} \qquad\qquad B = -\frac{189}{64}$$

$$\mathfrak{r}_{12} = \frac{21}{4} m^2 + \frac{105}{8} m\, e^2 + \left\{\frac{2397}{32} + \frac{63}{32} - \frac{315}{16} + \frac{945}{64}\right\} m^2 e^2 - \frac{369}{32} m^2 e_{,}^2$$

$$= \frac{21}{4} m^2 + \frac{105}{8} m\, e^2 + \frac{4605}{64} m^2 e^2 - \frac{369}{32} m^2 e_{,}^2$$

Let $r_{12} = \frac{35}{8} m + \frac{1269}{64} m^2 + A\, m\, e^2 + B\, m\, e_{,}^2,$

$$\left\{\frac{35}{8} m + \frac{1269}{64} m^2 + A\, m\, e^2 + B\, m\, e_{,}^2\right\} \left\{(1 + 3\, e^2)\,(1 - 6\, m) - 1\right\}$$

$$= (1 - 6\, m) \left\{\frac{21}{4} m^2 + \frac{105}{8} m\, e^2 + \frac{4605}{64} m^2 e^2 - \frac{369}{32} m^2 e_{,}^2\right\}$$

$$- 4 \left\{\frac{63}{8} m^2 - \frac{273}{64} m^2 e^2 - \frac{1107}{64} m^2 e_{,}^2\right\}$$

$$-\frac{315}{4} + \frac{3807}{64} - 6\, A = -\frac{315}{4} + \frac{4605}{64} + \frac{273}{16}$$

$$-6\, B = -\frac{369}{32} + \frac{1107}{16}$$

$$A = -\frac{315}{64} \qquad\qquad B = -\frac{615}{64}$$

M. Plana has $A = \frac{35}{8}$.

$\mathfrak{r}_{13} = -\frac{45}{8} m\, e^2 \qquad r_{13} = -\frac{405}{64} m\, e^2$

M. Pl[illegible] $= -\frac{45}{64} m\, e^2$

$\mathfrak{r}$ [illegible] $+ \frac{3483}{64}\Big\}\, m^2 e^2 - \frac{81}{32} m^2 e_{,}^2$

$$= -\frac{9}{4}m^2 + \frac{63}{8}me^2 + \frac{3213}{64}m^2e^2 - \frac{81}{32}m^2e_{\prime}^2$$

Let $r_{14} = \frac{21}{8}m + \frac{1113}{64}m^2 + Ame^2 + Bme_{\prime}^2,$

$$\left\{\frac{21}{8}m + \frac{1113}{64}m^2 + Ame^2 + Bme_{\prime}^2\right\}\{(1+3e^2)(1-2m)-1\}$$

$$= \{1-2m\}\left\{-\frac{9}{4}m^2 + \frac{63}{8}me^2 + \frac{3213}{64}m^2e^2 - \frac{81}{32}m^2e_{\prime}^2\right\}$$

$$-4\left\{\frac{3}{4}m^2 - \frac{3}{32}m^2e^2 + \frac{27}{32}m^2e_{\prime}^2\right\}$$

$$-\frac{63}{4} + \frac{3339}{64} - 2A = -\frac{63}{4} + \frac{3213}{64} + \frac{3}{8}$$

$$-2B = -\frac{81}{32} - \frac{27}{8}$$

$$A = \frac{51}{64} \qquad B = -\frac{189}{64}$$

$$\mathfrak{r}_{15} = -\frac{3}{4}m^2 - \frac{45}{8}me^2 + \left\{-\frac{621}{32} - \frac{9}{32} + \frac{45}{16}\right\}m^2e^2 + \frac{3}{32}m^2e_{\prime}^2$$

$$= -\frac{3}{4}m^2 - \frac{45}{8}me^2 - \frac{135}{8}m^2e^2 + \frac{3}{32}m^2e_{\prime}^2$$

Let $r_{15} = -\frac{15}{8}m - \frac{97}{64}m^2 + Ame^2 + Bme_{\prime}^2,$

$$\left\{-\frac{15}{8}m - \frac{97}{64}m^2 + Ame^2 + Bme_{\prime}^2\right\}\{(1+3e^2)(1-2m)-1\}$$

$$= \{1-2m\}\left\{-\frac{3}{4}m^2 - \frac{45}{8}me^2 - \frac{135}{8}m^2e^2 + \frac{3}{32}m^2e^2\right\}$$

$$-4\left\{-\frac{9}{8}m^2 + \frac{39}{64}m^2e^2 + \frac{9}{4}m^2e_{\prime}^2\right\}$$

$$\frac{45}{4} - \frac{291}{64} - 2A = \frac{45}{4} - \frac{135}{8} - \frac{39}{16}$$

$$-2B = \frac{3}{32} - \frac{9}{16}$$

$$A = \frac{945}{128} \qquad B = \frac{15}{64}$$

M. Plana has $A = -\frac{15}{8}.$

$\mathfrak{r}_{16} = \frac{105}{8}me^2 \qquad 8r_{16} = 9\,\mathfrak{r}_{16} \qquad r_{16} = \frac{945}{64}me^2$

M. Plana has $r_{16} = \frac{105}{64}me^2$

$= 0$ $\qquad \mathfrak{r}_{18} = \frac{765}{64} e^2$ $\qquad r_{18} = \frac{255}{16} m\, e^2$

$= \frac{3}{2} r_{21} - \frac{1}{16} r_3 - \frac{1}{16} r_1 = - \frac{165}{64} m$

${}_{39} = 4\, \mathfrak{r}_{39}$ $\qquad r_{39} = - \frac{55}{16} m$

$= \frac{3}{2} r_{23} - \frac{1}{16} r_5 = - \frac{63}{8} m$

$r_{41} = 9\, \mathfrak{r}_{41}$ $\qquad r_{41} = - \frac{567}{64} m$

${}_2 = \frac{3}{2} r_{24} - \frac{1}{16} r_6 = \left\{ - \frac{315}{16} - \frac{7}{32} \right\} m^2 = - \frac{637}{32} m^2$

$m\, r_{42} = - \frac{637}{32} m^2 - \frac{4 \,.\, 49}{64} m^2$ $\qquad r_{42} = - \frac{245}{64} m$

${}_4 = \frac{3}{2} r_{26} - \frac{1}{16} r_5 = \frac{63}{8} m$

$r_{44} = 9\, \mathfrak{r}_{44}$ $\qquad r_{44} = \frac{567}{64} m$

${}_{45} = \frac{3}{2} r_{27} - \frac{1}{16} r_7 + \frac{945}{32} m^2 = \left\{ \frac{45}{16} + \frac{1}{32} + \frac{945}{32} \right\} m^2 = \frac{259}{8} m^2$

$m\, r_{45} = \frac{259}{8} m^2 + \frac{4 \,.\, 7}{64} m^2$ $\qquad r_{45} = \frac{625}{32} m$

M. Plana has $\frac{285}{64} m$.

${}_{47} = \frac{3}{2} r_{29} = - \frac{189}{64} m$ $\qquad 3\, r_{47} = 4\, \mathfrak{r}_{47}$ $\qquad r_{47} = - \frac{63}{16} m$

${}_{50} = \frac{3}{2} r_{32} = \frac{189}{64} m$ $\qquad 3\, r_{50} = 4\, \mathfrak{r}_{50}$ $\qquad r_{50} = \frac{63}{16} m$

${}_{53} = \frac{3}{2} r_{35} = - \frac{159}{32} m^2$

$m\, r_{53} = - \frac{159}{32} m^2 - \frac{4 \,.\, 53}{32} m$ $\qquad r_{53} = - \frac{371}{192} m$

M. Plana has $- \frac{53}{64} m$.

${}_{54} = \frac{3}{2} r_{36} = \frac{845}{32}$

$- 10\, m\, r_{54} = \frac{845}{32} - \frac{4 \,.\, 2535}{64}$ $\qquad r_{54} = \frac{345}{64} m$

${}_{56} = \frac{3}{2} r_{35} = - \frac{159}{32} m^2$

$$-6\,m\,r_{56} = -\frac{159}{32}\,m^2 - \frac{4\,.\,53}{32}\,m^2 \qquad\qquad r_{56} = \frac{371}{192}\,m$$

$$r_{57} = \frac{3}{2}\,r_{37} = \frac{m^2}{32}$$

$$2\,m\,r_{57} = \frac{m^2}{32} - \frac{4\,.\,3}{64}\,m^2 \qquad\qquad r_{57} = -\frac{5}{64}\,m$$

$$r_{59} = \frac{591}{32}\,m^2$$

I now proceed to find terms in the longitude multiplied by $m\,e^3\,e_{,}$, $m\,e^2\,e_{,}^2$, $m\,e\,e_{,}^3$ and $m\,e_{,}^4$, by means of the equation

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,t} = \frac{h}{r^2} - \frac{1}{r^2}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t$$

It is sufficient to take

$$\frac{\mathrm{d}\,.\,\delta\,\lambda}{\mathrm{d}\,t} = \frac{2\,h}{\mathrm{r}}\,\delta\,\frac{1}{\mathrm{r}} - \frac{1}{r^2}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t$$

$$-\frac{\mathrm{a}}{\mathrm{n}\,\mu}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t = -\frac{15}{8}\,m\,e^2\cos\left(2\,\tau - 2\,\xi\right)$$

$$-\frac{35}{8}\,m\,e^2\,e_{,}\cos\left(2\,\tau - 2\,\xi - \xi_{,}\right)$$

$$+\frac{15}{8}\,m\,e^2\,e_{,}\cos\left(2\,\tau - 2\,\xi + \xi_{,}\right)$$

Since

$$\frac{\mathrm{a}}{\mathrm{r}} = 1 + e\left(1 + \frac{e^2}{8}\right)\cos\xi + e^2\left(1 - \frac{e^2}{3}\right)\cos 2\,\xi$$

$$+\frac{9}{8}\,e^3\cos 3\,\xi + \frac{4}{3}\,e^4\cos 4\,\xi \qquad \text{(See p. 8.)}$$

$$\frac{\mathrm{d}\,\lambda}{\mathrm{n}\,\mathrm{d}\,t} = 2\,e\cos\xi\left\{-\frac{35}{8}\,m\,e^2\,e_{,}\cos\left(2\,\tau - 2\,\xi - \xi_{,}\right) + \frac{15}{8}\,m\,e^2\,e_{,}\cos\left(2\,\tau - 2\,\xi -\right.\right.$$

$$+\left\{\frac{e^2}{2} - \frac{5}{2}\,e^2\cos 2\,\xi\right\}\left\{-\frac{15}{8}\,m\,e^2\cos\left(2\,\tau - 2\,\xi\right)\right\}$$

$$\left\{-\frac{15}{8}\,m\,e^4 - \frac{75}{4}\,m\,e^2\,e_{,}^2\right\}\cos 2\,\tau + 20\,m\,e^4\cos\left(2\,\tau + 2\,\xi\right)$$

$$+\left\{-\frac{51}{32}\,m\,e^2 - \frac{189}{32}\,m\,e_{,}^2\right\}e\,e_{,}\cos\left(\xi + \xi_{,}\right)$$

$$+\left\{-\frac{315}{32}\,m\,e^2 - \frac{615}{32}\,m\,e_{,}^2\right\}e\,e_{,}\cos\left(2\,\tau - \xi - \xi_{,}\right)$$

$$-\frac{405}{32}\,m\,e^3\,e_{,}\cos\left(2\,\tau + \xi + \xi_{,}\right) + \left\{\frac{51}{32}\,m\,e^2 + \frac{189}{32}\,m\,e_{,}^2\right\}e\,e_{,}\cos\left(\xi -\right.$$

$$+\left\{\frac{945}{64}\,m\,e^2+\frac{15}{32}\,m\,e_{,}^2\right\}e\,e_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\frac{945}{32}\,m\,e^3e_{,}\cos(2\tau+\xi-\xi_{,})+\frac{255}{8}\,m\,e^2e_{,}^2\cos(2\tau-2\xi_{,})$$

$$+\frac{45}{8}\,m\,e^2e_{,}^2\cos(2\tau+2\xi_{,})-\frac{567}{32}\,m\,e^3e_{,}\cos(3\xi+\xi_{,})$$

$$-\frac{245}{32}\,m\,e^3e_{,}\cos(2\tau-3\xi-\xi_{,})+\frac{567}{32}\,m\,e^3e_{,}\cos(3\xi-\xi_{,})$$

$$+\frac{625}{16}\,m\,e^3e_{,}\cos(2\tau-3\xi+\xi_{,})-\frac{63}{8}\,m\,e^2e_{,}^2\cos(2\xi+2\xi_{,})$$

$$+\frac{63}{8}\,m\,e^2e_{,}^2\cos(2\xi-2\xi_{,})-\frac{371}{96}\,m\,e\,e_{,}^3\cos(\xi+3\xi_{,})$$

$$+\frac{845}{32}\,m\,e\,e_{,}^3\cos(2\tau-\xi-3\xi_{,})+\frac{371}{96}\,m\,e\,e_{,}^3\cos(\xi-3\xi_{,})$$

$$-\frac{5}{32}\,m\,e\,e_{,}^3\cos(2\tau-\xi+3\xi_{,})$$

$$2e\cos\xi\left\{-\frac{75}{16}\,m\,e\,e_{,}^2\cos(2\tau-\xi)+\frac{405}{64}\,m\,e^3\cos(2\tau+\xi)\right.$$

$$+\frac{35}{4}\,m\,e^2e_{,}\cos(2\tau-\xi_{,})-\frac{15}{4}\,m\,e^2e_{,}\cos(2\tau+\xi_{,})-\frac{105}{64}\,m\,e^3\cos(2\tau-3\xi)$$

$$-\frac{21}{4}\,m\,e^2e_{,}\cos(2\xi+\xi_{,})+\frac{21}{4}\,m\,e^2e_{,}\cos 2\xi-\xi_{,})$$

$$-\frac{63}{32}\,m\,e\,e_{,}^2\cos(\xi+2\xi_{,})+\frac{255}{32}\,m\,e\,e_{,}^2\cos(2\tau-\xi-2\xi_{,})$$

$$\left.+\frac{63}{32}\,m\,e\,e_{,}^2\cos(\xi-2\xi_{,})-\frac{45}{32}\,m\,e\,e_{,}^2\cos(2\tau-\xi+2\xi_{,})\right\}$$

$$\left\{2e^2\cos 2\xi-e^2\right\}\left\{\frac{15}{4}\,m\,e^2\cos 2\tau-\frac{21}{8}\,m\,e\,e_{,}\cos(\xi+\xi_{,})\right.$$

$$+\frac{35}{8}\,m\,e\,e_{,}\cos(2\tau-\xi-\xi_{,})+\frac{21}{8}\,m\,e\,e_{,}\cos(\xi-\xi_{,})$$

$$\left.-\frac{15}{8}\,m\,e\,e_{,}\cos(2\tau-\xi+\xi_{,})\right\}$$

$$\left\{\frac{9}{4}e^3\cos 3\xi-\frac{5\,e^3}{4}\cos\xi\right\}\left\{\frac{15}{8}\,m\,e\cos(2\tau-\xi)\right\}$$

$$=\left\{-\frac{15}{16}+\frac{75}{64}+\frac{405}{128}-\frac{15}{8}-\frac{75}{128}\right\}m\,e^4+\left\{-\frac{75}{8}-\frac{75}{32}\right\}m\,e^2e_{,}^2$$

$$=\frac{15}{16}\,m\,e^4-\frac{375}{32}\,m\,e^2e_{,}^2$$

$$\lambda_{10} = \left\{5 + \frac{405}{256} + \frac{15}{16} + \frac{135}{256}\right\} m\, e^2 = \frac{515}{64}\, m\, e^2$$

$$\lambda_{11} = \left\{-\frac{51}{32} - \frac{21}{4} + \frac{21}{8} + \frac{21}{8}\right\} m\, e^2 - \frac{189}{32}\, m\, e_{,}^2$$

$$= -\frac{51}{32} m\, e^2 - \frac{189}{32}\, m\, e_{,}^2$$

$$\lambda_{12} = \left\{-\frac{35}{8} - \frac{315}{32} + \frac{35}{4} - \frac{35}{8}\right\} m\, e^2 - \frac{615}{32}\, m\, e_{,}^2$$

$$= -\frac{315}{32}\, m\, e^2 - \frac{615}{32}\, m\, e_{,}^2$$

$$\lambda_{13} = \left\{-\frac{405}{32} - \frac{15}{4} - \frac{15}{8}\right\} m\, e^2 = -\frac{195}{32}\, m\, e^2$$

$$\lambda_{14} = \left\{-\frac{51}{32} + \frac{21}{4} - \frac{21}{8} - \frac{21}{8}\right\} m\, e^2 + \frac{189}{32}\, m\, e_{,}^2$$

$$= -\frac{51}{32} m\, e^2 + \frac{189}{32} m\, e_{,}^2$$

$$\lambda_{15} = \left\{\frac{945}{64} - \frac{15}{4} - \frac{15}{8}\right\} m\, e^2 + \frac{15}{32}\, m\, e_{,}^2$$

$$= \frac{545}{64}\, m\, e^2 + \frac{15}{32}\, m\, e_{,}^2$$

$$\lambda_{16} = \left\{\frac{945}{32} + \frac{35}{4} + \frac{35}{8}\right\} m\, e^2 = \frac{455}{32}\, m\, e^2$$

$$\lambda_{18} = \left\{\frac{255}{16} + \frac{255}{64}\right\} m\, e^2 = \frac{1275}{64}\, m\, e^2$$

$$\lambda_{19} = \left\{-\frac{45}{16} - \frac{45}{64}\right\} m\, e^2 = -\frac{225}{64}\, m\, e^2$$

$$\lambda_{39} = \left\{-\frac{75}{64} + \frac{105}{128} - \frac{135}{128}\right\} m = -\frac{45}{64}\, m$$

$$\lambda_{41} = \left\{-\frac{567}{32} - \frac{21}{4} - \frac{21}{8}\right\} \frac{m}{3} = -\frac{273}{32}\, m$$

$$\lambda_{44} = \left\{\frac{567}{32} + \frac{21}{4} + \frac{21}{8}\right\} \frac{m}{3} = \frac{273}{32}\, m$$

$$\lambda_{45} = \left\{\frac{105}{32} - \frac{15}{8} + \frac{15}{8}\right\} m = -\frac{105}{32}\, m$$

$$\lambda_{47} = \left\{-\frac{63}{8} - \frac{63}{32}\right\}\frac{m}{2} = -\frac{315}{64}m$$

$$\lambda_{50} = \left\{\frac{63}{8} + \frac{63}{32}\right\}\frac{m}{2} = \frac{315}{64}m$$

$$\lambda_{53} = -\frac{371}{96}m \qquad \lambda_{56} = \frac{371}{96}m \qquad \lambda_{57} = -\frac{5}{32}m$$

The expression for the reciprocal of the radius vector is required merely for the purpose of obtaining the expression for the longitude. Most of the terms in the latter expression, as far as the fifth order, which result when the latitude is neglected from that portion of the disturbing function which is equal to

$$-\frac{m_{,}r^2}{4\,r_{,}^3}\{1 + 3\cos(2\lambda - 2\lambda_{,})\}$$

have now been obtained.

There still remain a few terms, Arg. 5, Arg. 9, &c. where the divisor introduced by integration is a multiple of m, and a term in λ_{33}, which depends immediately upon the corresponding term in r_{33}.

I now proceed to find the term in λ_5, multiplied by m^4.

If δr, $\delta r_{,}$, $\delta\lambda$ and $\delta\lambda_{,}$ represent the terms in r, $r_{,}$, λ and $\lambda_{,}$ multiplied by $e_{,}$, $\left[\frac{dR}{dr}\right]$, $\left[\frac{dR}{d\lambda}\right]$, $\left[\frac{dR}{dr_{,}}\right]$, [between brackets,] the values of those quantities when δr, $\delta r_{,}$, $\delta\lambda$ and $\delta\lambda_{,}$ are neglected, the terms in R multiplied by $e_{,}$

$$= \left[\frac{dR}{dr}\right]\delta r + \left[\frac{dR}{d\lambda}\right]\delta\lambda + \left[\frac{dR}{d\lambda_{,}}\right]\delta\lambda_{,} + \left[\frac{dR}{dr_{,}}\right]\delta r_{,}$$

$$= -\left[r\frac{dR}{dr}\right][r]\,\delta\frac{1}{r} + \left[\frac{dR}{d\lambda}\right](\delta\lambda - \delta\lambda_{,}) + \left[r_{,}\frac{dR}{dr_{,}}\right]\left[\frac{1}{r_{,}}\right]\delta r_{,}$$

$$= -2[R]\,[r]\,\delta\frac{1}{r} + \left[\frac{dR}{d\lambda}\right](\delta\lambda - \delta\lambda_{,}) - 3[R]\left[\frac{1}{r_{,}}\right]\delta r_{,}$$

The expressions for $[R]$ and $\left[\frac{dR}{d\lambda}\right]$ are given in pages 138 and 141.

$$[r]\,\delta\frac{1}{r} = \left\{1 - \frac{m^2}{6} - m^2\cos 2\tau\right\}\left\{\left\{-\frac{3}{2}m^2 + \frac{449}{16}m^4\right\}e_{,}\cos\xi_{,}\right.$$

$$+\left\{\frac{7}{2}m^2+\frac{157}{8}m^3+\frac{3349}{48}m^4\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{-\frac{m^2}{2}-\frac{91}{24}m^3-\frac{1265}{144}m^4\right\}e_{,}\cos(2\tau+\xi_{,})$$

$$=\left\{-\frac{3}{2}m^2+\left\{\frac{449}{16}+\frac{1}{4}-\frac{7}{4}+\frac{1}{4}\right\}m^4\right\}e_{,}\cos\xi_{,}$$

$$+\left\{\frac{7}{2}m^2+\frac{157}{8}m^3+\left\{\frac{3349}{48}-\frac{7}{12}+\frac{3}{4}\right\}m^4\right\}e\cos(2\tau-\xi_{,})$$

$$+\left\{-\frac{m^2}{2}-\frac{91}{24}m^3+\left\{-\frac{1265}{144}+\frac{1}{12}+\frac{3}{4}\right\}m^4\right\}e_{,}\cos(2\tau+\xi_{,})$$

$$=\left\{-\frac{3}{2}m^2+\frac{429}{16}m^4\right\}e_{,}\cos\xi_{,}$$

$$+\left\{\frac{7}{2}m^2+\frac{157}{8}m^3+\frac{3357}{48}m^4\right\}e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{-\frac{m^2}{2}-\frac{91}{24}m^3-\frac{1145}{144}m^4\right\}e_{,}\cos(2\tau+\xi_{,})$$

With reference only to the argument 5 and to obtaining o the term in r_5 multiplied by m^5, it is sufficient to take

$$\frac{\mathrm{a}}{\mu}[R]=-\frac{m^2}{4}+\frac{179}{96}m^4+\frac{97}{16}m^5-\frac{3}{4}m^2\cos 2\tau \qquad \text{(See p. 138}$$

$$\frac{\mathrm{a}}{\mu}\left[\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right]=\frac{3}{2}m^2\sin 2\tau \qquad \delta r_{,}=-a_{,}e_{,}\cos\xi_{,}$$

$$\delta\lambda-\delta\lambda_{,}=\left\{\frac{77}{16}m^2+\frac{479}{16}m^3\right\}e_{,}\sin(2\tau-\xi_{,})$$

$$+\left\{-\frac{11}{16}m^2-\frac{257}{48}m^4\right\}e_{,}\sin(2\tau+\xi_{,})$$

$$\frac{\mathrm{a}R}{\mu}=\left\{\frac{m^2}{4}-\frac{179}{96}m^4-\frac{97}{16}m^5\right\}\left\{-3-3m^2\right\}e_{,}\cos\xi_{,}$$

$$+\frac{3}{4}m^2\cos 2\tau\left\{\left\{7m^2+\frac{157}{4}m^3\right\}e_{,}\cos(2\tau-\xi_{,})\right.$$

$$\left.+\left\{-m^2-\frac{91}{12}m^3\right\}e_{,}\cos(2\tau+\xi_{,})\right\}$$

$$+\frac{3}{2}m^2\sin 2\tau\left\{\left\{\frac{77}{16}m^2+\frac{479}{16}m^3\right\}e_{,}\sin(2\tau-\xi_{,})\right.$$

$$+\left\{-\frac{11}{16}m^2-\frac{257}{48}m^3\right\}e_{,}\sin(2\tau+\xi_{,})\Bigg\}$$

$$=\Bigg\{-\frac{3}{4}m^2+\left\{-\frac{3}{4}+\frac{179}{32}+\frac{21}{8}-\frac{3}{8}+\frac{231}{64}-\frac{33}{64}\right\}m^4$$

$$+\left\{\frac{291}{16}+\frac{471}{32}-\frac{91}{32}+\frac{1437}{64}-\frac{257}{64}\right\}m^5\Bigg\}e_{,}\cos\xi_{,}$$

$$=\left\{-\frac{3}{4}m^2+\frac{163}{16}m^4+\frac{97}{2}m^5\right\}e_{,}\cos\xi_{,}$$

$$dR=\frac{dR}{dt}dt-\frac{dR}{dr_{,}}dr_{,}-\frac{dR}{d\lambda_{,}}d\lambda$$

$$\int dR=R-\int\left\{\left(\frac{dR}{dr_{,}}\right)dr_{,}+\left(\frac{dR}{d\lambda_{,}}\right)d\lambda_{,}\right\}$$

$$\left(\frac{dR}{dr_{,}}\right)dr_{,}+\left(\frac{dR}{d\lambda_{,}}\right)d\lambda_{,}=\left[\frac{dR}{dr_{,}}\right]m\,\mathrm{n}\,e_{,}\sin\xi_{,}\,dt$$

$$+\left[\frac{dR}{d\lambda_{,}}\right]\{m+2m\,e_{,}\cos\xi_{,}\}\,\mathrm{n}\,dt$$

From the form of this expression it is obviously sufficient to take only the terms of Arg. 0 in $\left[\frac{dR}{dr_{,}}\right]$, and $\left[\frac{dR}{d\lambda_{,}}\right]$ may be neglected because this quantity contains no term Arg. 0, so that

$$\int dR=R+\left\{\frac{3}{4}m^2-\frac{179}{32}m^4-\frac{291}{16}m^5\right\}e_{,}\cos\xi_{,}$$

$$=\left\{\frac{147}{32}m^4+\frac{485}{16}m^5\right\}e_{,}\cos\xi_{,}$$

In this approximation $\mathfrak{r}_5$ may be taken $=0$, and finally if

$$r_5=-\frac{3}{2}m^2+\frac{449}{16}m^4+A\,m^5$$

$$(m^2-1)\left\{-\frac{3}{2}m^2+\frac{449}{16}m^4+A\,m^5\right\}$$

$$=\frac{3}{2}m^2-\frac{163}{8}m^4-97\,m^5-\frac{147}{16}m^4-\frac{485}{8}m^5$$

$$A=\frac{1261}{8}$$

$$r_5 = -\frac{3}{2}m^2 + \frac{449}{16}m^4 + \frac{1261}{8}m^5$$

I now propose to determine the term in λ_5 multiplied by m^4.

I have hitherto employed solely the equation

$$\frac{d\lambda}{dt} = \frac{1}{r^2}\left\{h - \int \frac{dR}{d\lambda}dt\right\}$$

to determine the inequalities of longitude; but this equation presents an inconvenience in certain cases where the divisor introduced by integration is a multiple of m, as in Arg. 5, Arg. 9, &c., because it follows from this circumstance, and from the double integration, that to find terms in λ_5, λ_9, &c., multiplied by any given power of m, as m^q, it is necessary (according to that method) first to obtain the terms in $\frac{dR}{d\lambda}$ of the same arguments multiplied by m^{q+2}.

The calculations which have hitherto been given, and which have been instituted with the view of confirming the accuracy of M. Plana's results, or of detecting some of those inaccuracies which may be supposed to exist in a work of such labour and extent, are only, however, examples of the methods of arriving at the lunar inequalities already explained in the former portion of this treatise. I shall now give examples of the determination of inequalities of longitude by a method which is peculiarly applicable to Arguments 5, 9, 77, &c., and which is not subject to the inconvenience I have just noticed. It will be seen, also, in treating these arguments, that reductions occur, which facilitate the process.

$$r'^2\frac{d\lambda'}{dt} = h - \int \frac{dR}{d\lambda'}dt \qquad \text{(See p. 3.)}$$

$$r\frac{d^2r}{dt^2} + \frac{dr^2}{dt^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int dR + r\frac{dR}{dr} = 0$$

$$\frac{dr^2}{dt^2} + \frac{r^2}{(1+s^2)}\frac{d\lambda^2}{dt^2} + \frac{r^2}{(1+s^2)^2}\frac{ds^2}{dt^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int dR = 0$$

$$r\frac{d^2r}{dt^2} - \frac{r^2}{(1+s^2)}\frac{d\lambda^2}{dt^2} - \frac{r^2}{(1+s^2)}\frac{ds^2}{dt^2} + \frac{\mu}{r} + \frac{r\,dR}{dr} = 0$$

$$r'^4\frac{d\lambda'^2}{dt^2} - 2hr'^2\left(\frac{d\lambda'}{dt}\right) + h^2 = \left\{\int \frac{dR}{d\lambda'}dt\right\}^2$$

$r^2 = r'^2(1+s^2)$ and eliminating $\frac{\mathrm{d}\lambda'^2}{\mathrm{d}t^2}$, I obtain the following equations:

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t} = \frac{h(1+s^2)}{2r^2} + \frac{\mu}{2hr} + \frac{r\,\mathrm{d}^2r}{2h\,\mathrm{d}t^2} - \frac{r^2}{2h(1+s^2)^2}\frac{\mathrm{d}s^2}{\mathrm{d}t^2}$$

$$+ \frac{r\,\mathrm{d}R}{2h\,\mathrm{d}r} - \frac{(1+s^2)}{2hr^2}\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\mathrm{d}t\right\}^2$$

or,

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t} = \frac{h(1+s^2)}{2r^2} + \frac{\mu}{hr} - \frac{\mu}{2ha} - \frac{\mathrm{d}r^2}{2h\,\mathrm{d}t^2} - \frac{r^2}{2h(1+s^2)^2}\frac{\mathrm{d}s^2}{\mathrm{d}t^2}$$

$$- \frac{1}{h}\int\mathrm{d}R - \frac{(1+s^2)}{2hr^2}\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\mathrm{d}t\right\}^2$$

Either of these equations may serve to determine the inequalities of longitude in the planetary or in the lunar theory, but the latter is rather the more advantageous.

$$s^2 = \frac{z^2}{r^2 - z^2} \qquad\qquad 1 + s^2 = \frac{r^2}{r^2 - z^2}$$

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t} = \frac{h}{2(r^2 - z^2)} + \frac{\mu}{hr} - \frac{\mu}{2ha} - \frac{\mathrm{d}r^2}{2h\,\mathrm{d}t^2} - \frac{r^2\left\{r\frac{\mathrm{d}z}{\mathrm{d}t} - z\frac{\mathrm{d}r}{\mathrm{d}t}\right\}^2}{2h(r^2 - z^2)}$$

$$- \frac{1}{h}\int\mathrm{d}R - \frac{1}{2h(r^2 - z^2)}\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t\right\}^2$$

Neglecting quantities depending upon γ^4,

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t} = \frac{h}{2r^2} + \frac{hs^2}{2r^2} + \frac{\mu}{hr} - \frac{\mu}{2ha} - \frac{\mathrm{d}r^2}{2h\,\mathrm{d}t^2}$$

$$- \frac{r^2}{2h}\frac{\mathrm{d}s^2}{\mathrm{d}t^2} - \frac{1}{h}\int\mathrm{d}R - \frac{(1+s^2)}{2hr^2}\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\mathrm{d}t\right\}^2$$

Neglecting quantities multiplied by γ^2,

$$-\frac{\mu}{ha} - \frac{\mathrm{d}r^2}{2h\mathrm{d}t^2} - \frac{1}{h}\int\mathrm{d}R - \frac{1}{2hr^2}\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t\right\}^2$$

$$= \frac{h}{2\,r^2} + \frac{\mu}{h\,r} - \frac{\mu}{2\,h\,a} + \frac{h}{r}\,\delta\frac{1}{r} + \frac{\mu}{h}\,\delta\frac{1}{r}$$

$$+ \frac{h}{2}\left(\delta\frac{1}{r}\right)^2 - \frac{d\,r^2}{2\,h\,d\,t^2} - \frac{1}{h}\int d\,R - \frac{1}{2\,h\,r^2}\left\{\int\frac{d\,R}{d\,\lambda}\,d\,t\right\}^2$$

Neglecting $\left(\delta\frac{1}{r}\right)^3$, this equation may be put in the form

$$\frac{d\,\lambda}{d\,t} = \frac{h}{2\,r^2} + \frac{\mu}{h\,r} - \frac{\mu}{2\,h\,a} + \frac{h}{r}\,\delta\frac{1}{r} + \frac{\mu}{h}\,\delta\frac{1}{r} + \frac{h}{2}\left(\delta\frac{1}{r}\right)^2$$

$$- \frac{d\,r^2}{2\,h\,d\,t^2} + \frac{2\,r\,d\,r^2}{h\;d\,t^2}\,\delta\frac{1}{r} + \frac{r^2\,d\,r}{h\,d\,t}\,\frac{d\,.\,\delta\frac{1}{r}}{d\,t} - \frac{5\,r^3\,d\,r}{2\,h\;d\,t}\,\frac{d\,.\left(\delta\frac{1}{r}\right)^2}{d\,t}$$

$$- \frac{5\,r^2\,d\,r^2}{h\;d\,t^2}\left(\delta\frac{1}{r}\right)^2 - \frac{r^4}{2\,h}\,\frac{\left(d\,.\,\delta\frac{1}{r}\right)^2}{d\,t^2} - \frac{1}{h}\int d\,R - \frac{1}{2\,h\,r^2}\left\{\int\frac{d\,R}{d\,\lambda}\,d\,t\right\}^2$$

I shall now endeavour, by means of the following equation, to determine the term in λ_5 multiplied by m^4.

$$\frac{d\,\lambda}{d\,t} = \left\{\frac{h}{r} + \frac{\mu}{h}\right\}\delta\frac{1}{r} + \frac{h}{2}\left(\delta\frac{1}{r}\right)^2 - \frac{a^4}{2\,h}\,\frac{\left(d\,.\,\delta\frac{1}{r}\right)^2}{d\,t^2}$$

$$- \frac{1}{h}\int d\,R - \frac{1}{2\,h\,r^2}\left\{\int\frac{d\,R}{d\,\lambda}\,d\,t\right\}^2$$

This equation is sufficient, because there is no term in $\frac{d\,r}{d\,t}$ independent of e.

It is sufficient to take

$$a^2\left(\delta\frac{1}{r}\right)^2 = \left\{\left\{m^2 + \frac{19}{6}\,m^3\right\}\left\{\frac{7}{2}\,m^2 + \frac{157}{8}\,m^3\right.\right.$$

$$\left.\left. - \frac{m^2}{2} - \frac{91}{24}\,m^3\right\} + \left(\frac{m^2}{3}\right)\left(-\frac{3\,m^2}{2}\right)\right\} e_{\prime}\cos\xi_{\prime}$$

$$= \left\{\frac{5}{2}\,m^4 + \frac{76}{3}\,m^5\right\} e_{\prime} \cos \xi_{\prime}$$

$$\mathrm{a}\,\delta\,\frac{1}{r} = \left(m^2 + \frac{19}{6}\,m^3\right) \cos 2\,\tau - \frac{3\,m^2}{2}\,e_{\prime} \cos \xi_{\prime}$$

$$+ \left\{\frac{7}{2}\,m^2 + \frac{157}{8}\,m^3\right\} e_{\prime} \cos (2\,\tau - \xi_{\prime})$$

$$+ \left\{-\frac{m^2}{2} - \frac{91}{24}\,m^3\right\} e_{\prime} \cos (2\,\tau + \xi_{\prime})$$

$$\frac{\mathrm{a}^2 \left(\mathrm{d}\,.\,\delta\frac{1}{r}\right)^2}{\mathrm{n}^2\,\mathrm{d}\,t^2} = \{2 - 2\,m\} \left(m^2 + \frac{19}{6}\,m^3\right)$$

$$\left\{\left(\frac{7}{2}\,m^2 + \frac{157}{8}\,m^3\right)(2 - 3\,m)\right.$$

$$\left. + \left(-\frac{m^2}{2} - \frac{91}{24}\,m^3\right)(2 - m)\right\} e_{\prime} \cos \xi_{\prime}$$

$$= \left\{12\,m^4 + \frac{208}{3}\,m^5\right\} e_{\prime} \cos \xi_{\prime}$$

$$\left\{\int \frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t\right\}^2 = \left\{\frac{3}{4}\,m^2 + \frac{3}{4}\,m^3\right\} \left\{\frac{21}{8}\,m^2 + \frac{99}{16}\,m^3\right.$$

$$\left. - \frac{3}{8}\,m^2 - \frac{39}{16}\,m^3\right\} e_{\prime} \cos \xi_{\prime}$$

$$= \left\{\frac{27}{16}\,m^4 + \frac{9}{2}\,m^5\right\} e_{\prime} \cos \xi_{\prime}$$

$$h = \sqrt{\mathrm{a}\,\mu}\left(1 - \frac{m^2}{3}\right) \qquad \frac{1}{h}\int \mathrm{d}\,R = \frac{147}{32}\,m^4 + \frac{485}{16}\,m^5$$

$$\frac{\mathrm{d}\,\lambda}{\mathrm{n}\,\mathrm{d}\,t} = \left(1 - \frac{m^2}{3}\right)\left(-\frac{3}{2}\,m^2 + \frac{449}{16}\,m^4 + \frac{1261}{8}\,m^5\right)$$

$$+ \left(1 + \frac{m^2}{3}\right)\left(-\frac{3}{2}\,m^2 + \frac{449}{16}\,m^4 + \frac{1261}{8}\,m^5\right)$$

$$+ \frac{5}{4}\,m^4 + \frac{38}{3}\,m^5 - 6\,m^4 - \frac{104}{3}\,m^5 - \frac{147}{32}\,m^4 - \frac{485}{16}\,m^5$$

$$- \frac{27}{32}\,m^4 - \frac{9}{4}\,m^5$$

$$\lambda_5 = \left(-\frac{3}{2} - \frac{3}{2}\right) m + \left(\frac{449}{16} + \frac{449}{16} + \frac{5}{4} - 6 - \frac{147}{32} - \frac{27}{32}\right) m^3$$

$$+\left(\frac{1261}{8}+\frac{1261}{8}+\frac{38}{3}-\frac{104}{3}-\frac{485}{16}-\frac{9}{4}\right)m^4$$

$$= -3m+\frac{735}{16}m^3+\frac{4171}{16}m^4$$

I next proceed to determine some of the terms in λ_9; but in order to procure that multiplied by m^3 it is necessary previously to procure the term in r_9 multiplied by m^4.

$$\int \mathrm{d}R = R - \int\left\{\left(\frac{\mathrm{d}R}{\mathrm{d}r_{,}}\right)\mathrm{d}r_{,} + \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda_{,}}\right)\mathrm{d}\lambda_{,}\right\}$$

Since $\mathrm{d}r_{,}$ is multiplied by $e_{,}$, and since

$$\frac{\mathrm{d}R}{\mathrm{d}\lambda_{,}} = -\frac{\mathrm{d}R}{\mathrm{d}\lambda},$$

in order to obtain the term in $\int \mathrm{d}R$, Arg. 9., independent of $e_{,}$, it is sufficient to take $\mathrm{d}\lambda_{,} = m\,\mathrm{n}\,\mathrm{d}t$,

$$\int \mathrm{d}R = R + m\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{n}\,\mathrm{d}t$$

$$\frac{\mathrm{a}}{\mu}\,\delta\,\frac{\mathrm{d}R}{\mathrm{d}\lambda} = -3m^2\sin 2\tau\left\{\frac{5}{4}m^2e^2\cos 2\xi+\frac{225}{64}m^2e^2\cos(4\tau-2\xi)\right\}$$

$$+9m^2e\sin(2\tau-\xi)\left(\frac{m^2}{2}e\cos\xi\right)-\frac{15}{2}m^2e^2\sin(2\tau-2\xi)\left(\frac{m^2}{6}\right)$$

$$+3m^2\cos 2\tau\left\{\frac{43}{16}m^2e^2\sin 2\xi+\frac{1125}{256}m^2e^2\sin(4\tau-2\xi)\right\}$$

$$-9m^2e\cos(2\tau-\xi)\left(\frac{5}{2}m^2e\sin\xi\right)$$

$$+\frac{9}{2}m^2\sin 2\tau\left\{\frac{225}{128}m^2e^2\cos(4\tau-2\xi)\right\}$$

$$-6m^2\cos 2\tau\left\{\frac{225}{64}m^2e^2\sin(4\tau-2\xi)\right\}$$

$$-3m^2\sin 2\tau\left\{-\frac{225}{32}m^2e^2\cos(4\tau-2\xi)\right\}$$

$$=\left\{-\frac{15}{8}-\frac{675}{128}+\frac{9}{4}-\frac{5}{4}-\frac{129}{32}+\frac{3375}{512}+\frac{45}{4}\right.$$

$$\left.+\frac{2025}{512}-\frac{675}{64}-\frac{675}{64}\right\}m^4e^2\sin(2\tau-2\xi)$$

Hence it appears that

$\frac{\mathrm{d} R}{\mathrm{d} \lambda}$ contains the term

$$\left\{\frac{15}{4} m^2 - \frac{1213}{128} m^4\right\} e^2 \sin(2\tau - 2\xi)$$

$$\frac{\mathrm{a}}{\mathrm{n}\mu} \int \frac{\mathrm{d} R}{\mathrm{d} \lambda} \mathrm{d} t = \left\{\frac{15}{4} m - \frac{1213}{128} m^3\right\} \frac{1}{2} \left\{1 + \frac{3}{4} m + \frac{243}{32} m^2\right\} e^2 \cos(2\tau - 2\xi)$$

$$= \left\{\frac{15}{8} m + \frac{45}{32} m^2 + \frac{19}{2} m^3\right\} e^3 \cos(2\tau - 2\xi)$$

$$\frac{\mathrm{a} R}{\mu} = \left\{-\frac{15}{8} m^2 - \frac{45}{32} m^3 - \frac{8203}{512} m^4\right\} e^2 \cos(2\tau - 2\xi)$$

(See p. 166.)

$$\text{Hence } \frac{\mathrm{a}}{\mu} \int \mathrm{d} R = -\frac{3339}{512} m^4 e^2 \cos(2\tau - 2\xi)$$

$$r_9 = \frac{45}{16} m + \frac{561}{32} m^2$$

$$\text{Let } r_9 = -\frac{15}{4} m^2 - \frac{225}{16} m^3 + A m^4$$

$$(4m^2 - 1)\left\{-\frac{15}{4} m^2 - \frac{225}{16} m^3 + A m^4\right\} = (4m^2 - 6m^3)\left\{\frac{45}{16} m + \frac{561}{32} m^2\right\}$$

$$+ \frac{15}{4} m^2 + \frac{45}{16} m^3 + \frac{8203}{256} m^4 + \frac{3339}{256} m^4$$

$$-15 - A = \frac{561}{8} - \frac{135}{8} + \frac{8203}{256} + \frac{3339}{256}$$

$$A = -\frac{14507}{128}$$

$$r_9 = -\frac{15}{4} m^2 - \frac{225}{16} m^3 - \frac{14507}{128} m^4$$

I now revert to the equation

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{h}{2r^2} + \frac{\mu}{hr} - \frac{\mathrm{d}r^2}{2h\,\mathrm{d}t^2} - \frac{\mu}{ha} - \frac{1}{h}\int \mathrm{d} R - \frac{r^2}{2h}\left\{\int \frac{\mathrm{d} R}{\mathrm{d} \lambda} \mathrm{d} t\right\}^2$$

When the fourth powers of the eccentricities are neglected, Arg. 9 results only from the combinations of Arg. 2 with Arg. 3, 8 with 1, and 136 with 1; and it will be found sufficient, in order to obtain the term of Arg. 9 multiplied by m^4, to take the terms multiplied by m^2 in Arg. 1

. . . . m^3 . . 2

. . . . m^4 . . . 3

. . . . m^2 . . 8

. . . . m^2 . . 136

$\frac{r^2}{2h}\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t\right\}^2$ gives no term multiplied by m^4 of Arg. 9.

$$r = \mathrm{r} - \mathrm{r}^2\,\delta\frac{1}{r} + \mathrm{r}^3\left(\delta\frac{1}{r}\right)^2 - \mathrm{r}^4\left(\delta\frac{1}{r}\right)^3 + \&\mathrm{c}.$$

$$\frac{\mathrm{r}^3}{\mathrm{a}}\left(\delta\frac{1}{r}\right)^2 = \{1 - 3e\cos\xi\}\left\{\frac{m^4}{3}\cos 2\tau + \frac{15}{8}m^3 e\cos\xi\right.$$

$$\left. + \left\{\frac{5}{8}m^3 + \frac{251}{96}m^4\right\} e\cos(2\tau - \xi) + \frac{225}{128}m^2 e^2\cos(4\tau - 2\xi)\right\}$$

$$= \frac{15}{8}m^3 e\cos\xi + \left\{\frac{5}{8}m^3 + \frac{203}{96}m^4\right\} e\cos(2\tau - \xi)$$

$$+ \frac{225}{128}m^2 e^2\cos(4\tau - 2\xi)$$

$$\mathrm{r}^2 = -2e\cos\xi - \frac{e^2}{2}\cos 2\xi$$

It will be found sufficient, in order to procure $\frac{\mathrm{d}r}{\mathrm{d}t}$, to take

$$\frac{r}{\mathrm{a}} = -r_1\cos 2\tau - \left\{1 + r_2 - 2r_0 - \frac{15}{8}m^3\right\} e\cos\xi$$

$$-\left\{r_3 - r_1 - \frac{5}{8}m^3 - \frac{203}{96}m^4\right\} e\cos(2\tau - \xi)$$

$$-\left\{\frac{1}{2} + r_8 - r_2 - \frac{1}{2}r_0\right\} e^2\cos 2\xi$$

$$-\left\{r_{136} - \frac{225}{128}m^2\right\} e^2\cos(4\tau - 2\xi)$$

$$\frac{h}{2r^2} = \frac{h}{2\mathrm{r}^2} + \frac{h}{\mathrm{r}}\,\delta\frac{1}{r} + \frac{h}{2}\left(\delta\frac{1}{\mathrm{r}}\right)^2$$

$$\frac{\mathrm{a}}{\mathrm{r}} = 1 + e\cos\xi + e^2\cos 2\xi \qquad h = \sqrt{\mathrm{a}\mu}\left(1 - \frac{m^2}{3}\right)$$

$$\frac{\mathrm{d}r^2}{2h\,\mathrm{d}t^2} = \left\{\left\{r_3 - r_1 - \frac{5}{8}m^3 - \frac{203}{96}m^4\right\}\{1 + r_2 - 2r_0\}\frac{\mathrm{c}}{2}\{2 - 2m - \mathrm{c}\}\left(1 + \frac{m^2}{3}\right)\right.$$

$$+ r_1 \left\{ \frac{1}{2} + r_3 - r_2 - \frac{1}{2} r_0 \right\} \text{c} \left\{ 2 - 2m \right\} \left\{ 1 + \frac{m^2}{3} \right\}$$

$$\left. + 4 r_1 \left\{ r_{136} - \frac{225}{128} m^2 \right\} \right\} \text{n}\, e^2 \cos (2\tau - 2\xi)$$

$$= \left\{ \left\{ r_3 - r_1 - \frac{5}{8} m^3 - \frac{203}{96} m^4 \right\} \left\{ \frac{1}{2} - m + \frac{m^2}{3} + \frac{m^3}{12} \right\} \right.$$

$$\left. + r_1 \left\{ 1 - m + \frac{17}{12} m^2 \right\} + \frac{225}{32} m^2 r_1 \right\} \text{n}\, e^2 \cos (2\tau - 2\xi)$$

$$= \left\{ r_3 \left\{ \frac{1}{2} - m + \frac{m^2}{3} + \frac{m^3}{12} \right\} + r_1 \left\{ \frac{1}{2} + \frac{779}{96} m^2 \right\} \right.$$

$$\left. - \frac{5}{16} m^3 - \frac{83}{192} m^4 \right\} \text{n}\, e^2 \cos (2\tau - 2\xi)$$

$$\frac{h}{r} \delta \frac{1}{r} + \frac{\mu}{h} \delta \frac{1}{r} = \left\{ 2 r_9 + \frac{1}{2} \left(1 - \frac{m^2}{3} \right) r_3 \right.$$

$$\left. + \frac{1}{2} \left(1 - \frac{m^2}{3} \right) r_1 \right\} \text{n}\, e^2 \cos (2\tau - 2\xi)$$

$$\frac{d\lambda}{\text{n}\, dt} = \left\{ 2 r_9 + r_3 \left\{ m - \frac{m^2}{2} - \frac{1}{12} m^3 \right\} - \frac{795}{96} m^4 + \frac{5}{8} \overset{*}{m}{}^3 + \frac{501}{128} m^4 \right.$$

$$\left. + \frac{5}{16} m^3 + \frac{83}{192} m^4 + \frac{3339}{512} m^4 \right\} e^2 \cos (2\tau - 2\xi)$$

$$\lambda_9 = - \left\{ - \frac{15}{2} m - \frac{225}{8} m^2 - \frac{14507}{64} m^3 \right.$$

$$+ \left\{ \frac{15}{8} m + \frac{187}{32} m^2 + \frac{33113}{1536} m^3 \right\} \left\{ 1 - \frac{m}{2} - \frac{1}{12} m^2 \right\}$$

$$- \frac{795}{96} m^3 + \frac{5}{8} m^2 + \frac{501}{128} m^3 + \frac{5}{16} m^2$$

$$\left. + \frac{83}{192} m^3 + \frac{3339}{512} m^3 \right\} \frac{1}{2} \left\{ 1 + \frac{3}{4} m + \frac{243}{32} m^2 \right\}$$

$$= \left\{ \frac{15}{4} - \frac{15}{16} \right\} m + \left\{ \frac{225}{16} - \frac{187}{64} + \frac{15}{32} - \frac{5}{16} - \frac{5}{32} + \frac{135}{64} \right\} m^2$$

$$+ \left\{ \frac{14507}{128} - \frac{28385}{3072} + \frac{795}{192} - \frac{501}{256} - \frac{83}{384} - \frac{3339}{1024} + \frac{2229}{256} + \frac{10935}{512} \right\} m^3$$

$$= \frac{45}{16} m + \frac{53}{4} m^2 + \frac{48007}{384} m^3$$

In this investigation certain quantities have been left indetermi-

---

[*] For this term and the following, see p. 155 and p. 171.

nate, in order to show more clearly the nature of the reducti which ensue, and without which λ_9 would contain a term indep dent of m. Without these reductions the term in r_3 multiplied m^4 would have been required.

I shall now endeavour to calculate the term in λ_{17} multiplied m^3, for which purpose it is necessary first to procure the term r_{17} multiplied by m^4.

$$\int d\,R = R - \int \left\{ \frac{d\,R}{d\,r_,} d\,r_, + \frac{d\,R}{d\,\lambda_,} d\,\lambda_, \right\}$$

$$\frac{r_, d\,R}{d\,r_,} = -3\,R \qquad \frac{d\,R}{d\,\lambda_,} = -\frac{d\,R}{d\,\lambda}$$

$\frac{d\,R}{d\,\lambda}$ contains no term Arg. 0 or 5 multiplied by m^4, and may the fore be neglected, and it is sufficient to take

$$\frac{d\,\mathrm{r}_,}{\mathrm{a}_,} = \{m\,e_, \sin \xi_, + m\,e_,^2 \sin 2\,\xi_,\}\,\mathrm{n}\,d\,t$$

$$\frac{d\,r_,}{r_,} = \left\{ m\,e_, \sin \xi_, + \frac{3}{2}\,m\,e_,^2 \sin 2\,\xi_, \right\} \mathrm{n}\,d\,t$$

$$\frac{\mathrm{a}}{\mu} \int d\,R = \left\{ -\frac{9}{8} m^2 + \frac{1419}{64} m^4 + \frac{9}{16} m^2 - \frac{537}{128} m^4 \right.$$

$$\left. + \frac{9}{16} m^2 - \frac{489}{64} m^4 \right\} e_,^2 \cos 2\,\xi_,$$

$$= \frac{1323}{128} m^4\,e_,^2 \cos 2\,\xi_,$$

Let $r_{17} = -\frac{9}{4} m^2 + A\,m^4$

$$(4\,m^2 - 1) \left(-\frac{9}{4} m^2 + A\,m^4 \right) = \frac{9}{4} m^2 - \frac{1419}{32} m^4 - \frac{1323}{64} m^4$$

$$-9 - A = -\frac{4161}{64}$$

$$A = \frac{3585}{64}$$

In order to have $\frac{d\,r^2}{d\,t^2}$, Arg. 17, it is sufficient to take

$$\frac{r}{\mathrm{a}} = -m^2 \cos 2\,\tau - \frac{7}{2}\,m^3\,e_, \cos (2\,\tau - \xi_,)$$

$$+ \frac{m^2}{2} e_, \cos (2 \tau + \xi_,) - \frac{17}{2} m^2 e_, \cos (2 \tau - 2 \xi_,)$$

$$\cdot \frac{a}{n \mu} \int \frac{d R}{d \lambda} d t = \frac{3}{4} m^2 \cos 2 \tau + \frac{21}{8} m^2 e_, \cos (2 \tau - \xi_,)$$

$$- \frac{3}{8} m^2 e_, \cos (2 \tau + \xi_,) + \frac{51}{8} m^2 e_,^2 \cos (2 \tau - 2 \xi_,)$$

$$\frac{d r^2}{1^2 d t^2} = 27 m^4 e_,^2 \cos 2 \xi_,$$

$$\left\{ \int \frac{d R}{d \lambda} d t \right\}^2 = \frac{243}{64} m^4 e_,^2 \cos 2 \xi_,$$

$$_{17} = - \frac{9}{4} m + \left\{ \frac{3585}{64} + \frac{57}{32} - \frac{27}{4} - \frac{1323}{256} - \frac{243}{256} \right\} m^3$$

$$= - \frac{9}{4} m + \frac{5751}{128} m^3$$

Calculation of terms in the latitude :

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} + \frac{d R}{d z} = 0$$

Considering R as a function of r', the projection of the radius ector r upon the ecliptic, λ' the longitude reckoned upon the ame plane, and s the tangent of the latitude,

$$\frac{d R}{d z} = \left(\frac{d R}{d s} \right) \frac{d s}{d z} \qquad \frac{d s}{d z} = \frac{1}{r'}$$

Hence, through an expression for R, given in p. 25,

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} + \frac{m_, z}{r_,^3} + \frac{3 m_, z r' \cos (\lambda' - \lambda)}{r_,^4} = 0$$

I shall now endeavour to arrive at an approximate expression for the coordinate z of the moon, by the integration of the equation

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} + \frac{m_, z}{r_,^3} + \frac{3 m_, z r' \cos (\lambda' - \lambda)}{r_,^4} = 0$$

The last quantity gives only terms multiplied by the small quantity $\frac{a}{a_,}$; these I omit for the present, so that the equation assumes he following very simple form,

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} + \frac{m_{\prime} z}{r_{\prime}^3} = 0$$

I make $\frac{a^2 z}{r^3} + \frac{m_{\prime} a^2 z}{\mu r_{\prime}^3} = z + \mathbb{Z}$

and if $z = a E \gamma z_n \sin (i n t + q)$

z_n being the coefficient corresponding to the n^{th} argument in t expression for z and E, a function of the eccentricities of t form $e^i e_{\prime}^k \gamma^l$

$$\{i^2 - 1\} z_n = \mathbb{Z}_n$$

$$\frac{a^2 z}{r^3} = z \left\{ \frac{a^2}{r^3} + \frac{3 a^2}{r^3} \left(\delta \frac{1}{r}\right) + \frac{3 a^2}{r} \left(\delta \frac{1}{r}\right)^2 + a^2 \left(\delta \frac{1}{r}\right)^3 \right\}$$

$\frac{a^3}{r^3} = 1 + \frac{3}{2} e^2 + 3 e \cos \xi + \frac{9}{2} e^2 \cos 2 \xi$ (See p. 8.)

$$\frac{a^2}{r^2} = 1 + \frac{e^2}{2} + 2 e \cos \xi + \frac{5}{2} e^2 \cos 2 \xi$$

$\frac{3 a^3}{r^2} \delta \frac{1}{r}$ contains the terms

$$6 e \cos \xi \left\{ \frac{m^2}{6} + \left\{ m^2 + \frac{19}{6} m^3 \right\} \cos 2 \tau + \frac{2}{3} m^2 e \cos \xi \right.$$

$$+ \left\{ \frac{15}{8} m + \frac{187}{32} m^2 \right\} e \cos (2 \tau - \xi) + \frac{33}{16} m^2 e \cos (2 \tau + \xi)$$

$$\left. - \frac{3}{2} m^2 e_{\prime} \cos \xi_{\prime} + \frac{7}{2} m^2 e_{\prime} \cos (2 \tau - \xi_{\prime}) - \frac{m^2}{2} e_{\prime} \cos (2 \tau + \xi_{\prime}) \right\}$$

$$+ \left\{ \frac{3}{2} e^2 + \frac{15}{2} e^2 \cos 2 \xi \right\} \left\{ \frac{m^2}{6} + m^2 \cos 2 \tau \right\}$$

$3 a^2 \left(\delta \frac{1}{r}\right)^2$ contains the following terms, (See p. 55.)

$$\frac{19}{12} m^4 + m^4 \cos 2 \tau + \frac{45}{8} m^3 e \cos \xi + \frac{15}{8} m^3 e \cos (2 \tau - \xi)$$

$$+ \frac{3}{2} m^4 \cos 4 \tau + \frac{45}{8} m^3 e \cos (4 \tau - \xi)$$

Hence

$$\frac{a^3}{r^3} + \frac{m_{,}a^3}{\mu\, r_{,}^3} = 1 + \frac{3}{2}e^2 + \left\{\frac{1}{2}+1\right\} m^2 + \left\{-\frac{179}{96}+\frac{19}{12}\right\} m^4$$

$$+ \left\{2 + \frac{1}{4} + \frac{675}{128}\right\} m^2 e^2 + \left\{\frac{3}{4}+\frac{3}{4}\right\} m^2 e_{,}^2$$

$$+ \left\{3\, m^2 + \frac{19}{2} m^3 + \left\{\frac{131}{6}+1\right\} m^4 + \left\{\frac{45}{4}+\frac{45}{8}\right\} m\, e^2\right.$$

$$\left. + \left\{\frac{567}{16}+\frac{561}{32}+\frac{99}{16}+\frac{3}{2}\right\} m^2 e^2 - \frac{15}{2} m^2 e_{,}^2\right\} \cos 2\tau$$

$$+ \left\{3 + \{2+1\} m^2 + \frac{45}{8} m^3\right\} e \cos \xi$$

$$+ \left\{\frac{45}{8} m + \left\{\frac{561}{32}+3\right\} m^2 + \left\{\frac{33113}{512}+\frac{19}{2}+\frac{15}{8}\right\} m^3\right\} e \cos (2\tau - \quad)$$

$$+ \left\{\left\{\frac{99}{16}+3\right\} m^2 + \left\{\frac{303}{16}+\frac{19}{2}\right\} m^3\right\} e \cos (2\tau + \xi)$$

$$+ \left\{3 - \frac{9}{2}\right\} m^2 e \cos \xi_{,} + \left\{\frac{21}{2} m^2 + \frac{471}{8} m^3\right\} e_{,} \cos (2\tau - \xi_{,})$$

$$+ \left\{-\frac{3}{2} m^2 - \frac{91}{8} m^3\right\} e_{,} \cos (2\tau + \xi_{,})$$

$$+ \left\{\frac{9}{2} + \left\{5 + 2 + \frac{5}{4}\right\} m^2 + \left\{-\frac{495}{64}+\frac{45}{16}+\frac{1485}{128}\right\} m^3\right\} e^2 \cos 2\xi$$

$$+ \left\{\frac{45}{8} m + \left\{-\frac{45}{4}+\frac{561}{32}+\frac{15}{4}\right\} m^2\right\} e^2 \cos (2\tau - 2\xi)$$

$$+ \left\{\frac{21}{2}+\frac{99}{16}+\frac{15}{4}\right\} m^2 e^2 \cos (2\tau + 2\xi)$$

$$+ \left\{-\frac{63}{8} m + \left\{-\frac{2511}{64}-\frac{9}{2}\right\} m^2\right\} e\, e_{,} \cos (\xi + \xi_{,})$$

$$+ \left\{\frac{105}{8} m + \left\{\frac{3807}{64}+\frac{21}{2}\right\} m^2\right\} e\, e_{,} \cos (2\tau - \xi - \xi_{,})$$

$$+ \left\{-\frac{99}{32}-\frac{3}{2}\right\} m^2 e\, e_{,} \cos (2\tau + \xi + \xi_{,})$$

$$+ \left\{\frac{63}{8} m + \left\{\frac{3339}{64}-\frac{9}{2}\right\} m^2\right\} e\, e_{,} \cos (\xi - \xi_{,})$$

$$+ \left\{-\frac{45}{8} m + \left\{-\frac{291}{64}-\frac{3}{2}\right\} m^2\right\} e\, e_{,} \cos (2\tau - \xi + \xi_{,})$$

$$+ \left\{\frac{693}{32}+\frac{21}{2}\right\} m^2 e\, e_{,} \cos (2\tau + \xi - \xi_{,}) + \left\{-\frac{27}{4}+\frac{9}{2}\right\} m^2 e_{,}^2 \cos 2\xi_{,}$$

$$+ \frac{51}{2} m^2 e_{\prime}^2 \cos(2\tau - 2\xi_{\prime}) - \frac{9}{4} m^3 e_{\prime}^2 \cos(2\tau + 2\xi_{\prime}) + \left\{\frac{21}{8} + \frac{3}{2}\right\} m^4 \cos 4\tau$$

$$+ \left\{\frac{1485}{128} + \frac{45}{8}\right\} m^3 e \cos(4\tau - \xi) + \frac{675}{128} m^2 e^2 \cos(4\tau - 2\xi)$$

$$= 1 + \frac{3}{2} e^2 + \frac{3}{2} m^2 - \frac{9}{32} m^4 + \frac{963}{128} m^2 e^2 + \frac{3}{2} m^2 e_{\prime}^2$$

$$+ \left\{ 3 m^2 + \frac{19}{2} m^3 + \frac{137}{6} m^4 + \frac{135}{8} m e^2 \right.$$

$$\left. + \frac{1941}{32} m^2 e^2 - \frac{15}{2} m^2 e_{\prime}^2 \right\} \cos 2\tau$$

$$+ \left\{ 3 + 3 m^2 + \frac{45}{8} m^3 \right\} e \cos \xi$$

$$+ \left\{ \frac{45}{8} m + \frac{657}{32} m^2 + \frac{38937}{512} m^3 \right\} e \cos(2\tau - \xi)$$

$$+ \left\{ \frac{147}{16} m^2 + \frac{455}{16} m^3 \right\} e \cos(2\tau + \xi)$$

$$- \frac{3}{2} m^2 e_{\prime} \cos \xi_{\prime} + \left\{ \frac{21}{2} m^2 + \frac{471}{8} m^3 \right\} e_{\prime} \cos(2\tau - \xi_{\prime})$$

$$+ \left\{ - \frac{3}{2} m^2 - \frac{91}{8} m^3 \right\} e_{\prime} \cos(2\tau + \xi_{\prime})$$

$$+ \left\{ \frac{9}{2} + \frac{33}{4} m^2 + \frac{855}{128} m^3 \right\} e^2 \cos 2\xi$$

$$+ \left\{ \frac{45}{8} m + \frac{321}{32} m^2 \right\} e^2 \cos(2\tau - 2\xi)$$

$$+ \frac{327}{16} m^2 e^2 \cos(2\tau + 2\xi) + \left\{ - \frac{63}{8} m - \frac{2799}{64} m^2 \right\} e e_{\prime} \cos(\xi + \xi_{\prime}$$

$$+ \left\{ \frac{105}{8} m + \frac{4479}{64} m^2 \right\} e e_{\prime} \cos(2\tau - \xi - \xi_{\prime})$$

$$- \frac{147}{32} m^2 e e_{\prime} \cos(2\tau + \xi + \xi_{\prime})$$

$$+ \left\{ \frac{63}{8} m + \frac{3051}{64} m^2 \right\} e e_{\prime} \cos(\xi - \xi_{\prime})$$

$$+ \left\{ - \frac{45}{8} m - \frac{387}{64} m^2 \right\} e e_{\prime} \cos(2\tau - \xi + \xi_{\prime})$$

$$+\frac{1029}{32}m^2 e\, e_{,} \cos(2\tau+\xi-\xi_{,})-\frac{9}{4}m^2 e_{,}^2\cos 2\xi_{,}$$

$$+\frac{51}{2}m^2 e_{,}^2\cos(2\tau-2\xi_{,})-\frac{9}{4}m^3 e_{,}^2\cos(2\tau+2\xi_{,})+\frac{33}{8}m^4\cos 4\tau$$

$$+\frac{2925}{128}m^3 e\cos(4\tau-\xi)+\frac{675}{128}m^2 e^2\cos(4\tau-2\xi)$$

I first take

$$\mathbb{Z}=\gamma\sin\eta\left\{\frac{3}{2}m^2+3m^2\cos 2\tau\right\}$$

$$=\frac{3}{2}m^2\gamma\sin\eta-\frac{3}{2}m^2\gamma\sin(2\tau-\eta)+\frac{3}{2}m^2\gamma\sin(2\tau+\eta)$$

If $z=\mathrm{a}\gamma z_{147}\sin(2\tau-\eta)+\mathrm{a}\gamma z_{148}\sin(2\tau+\eta)$

$$\{(2-2m-\mathrm{g})^2-1\}z_{147}=-\frac{3}{2}m^2 \qquad z_{147}=\frac{3}{8}m$$

$$\{(2-2m+\mathrm{g})^2-1\}z_{148}=\frac{3}{2}m^2 \qquad z_{148}=\frac{3}{16}m^2$$

I next take

$$\mathbb{Z}=\gamma\sin\eta\left\{\frac{19}{2}m^3\cos 2\tau\right\}$$

$$+\frac{3}{8}m\gamma\sin(2\tau-\eta)\left\{\frac{3}{2}m^2+3m^2\cos 2\tau\right\}$$

$$=-\frac{9}{16}m^3\gamma\sin\eta-\frac{67}{16}m^3\gamma\sin(2\tau-\eta)$$

$$+\frac{19}{4}m^3\gamma\sin(2\tau+\eta)+\frac{9}{16}m^3\gamma\sin(4\tau-\eta)$$

$$\eta=\mathrm{g}\mathrm{n}t+\varepsilon-\vartheta$$

$$\mathrm{g}^2-1=\frac{3m^2}{2}-\frac{9}{16}m^3$$

$$\mathrm{g}=1+\frac{3}{4}m^2-\frac{9}{32}m^3$$

If $z_{147}=\frac{3}{8}m+Am^2$

$$\left\{-4m+\frac{5}{2}m^2\right\}\left\{\frac{3}{8}m+Am^2\right\}=-\frac{3}{2}m^2-\frac{67}{16}m^3$$

$$\frac{15}{16} - 4A = -\frac{67}{16} \qquad\qquad A = \frac{41}{32}$$

If $z_{148} = \frac{3}{16} m^2 + A m^3$

$$\left\{8 - 12m\right\}\left\{\frac{3}{16} m^2 + A m^3\right\} = \frac{3}{2} m^2 + \frac{19}{4} m^3$$

$$-\frac{9}{4} + 8A = \frac{19}{4} \qquad\qquad A = \frac{7}{8}$$

If z contain the term $A \mathrm{a} \gamma \sin(4\tau - \eta)$,

$$8A = \frac{9}{16} m^3 \qquad\qquad A = \frac{9}{128} m^3$$

In order to find the term in g multiplied by m^4 and that in z_{147} multiplied by m^3, I now take

$$\mathbf{Z} = \gamma \sin \eta \left\{-\frac{9}{32} m^4 + \frac{137}{6} m^4 \cos 2\tau\right\}$$

$$+ \frac{3}{8} m \gamma \sin(2\tau - \eta) \left\{\frac{19}{2} m^3 \cos 2\tau\right\}$$

$$+ \left\{\frac{41}{32} m^2 \sin(2\tau - \eta) + \frac{3}{16} m^2 \sin(2\tau + \eta)\right\}$$

$$\left\{\frac{3}{2} m^2 + 3 m^2 \cos 2\tau\right\}$$

$$= \left\{-\frac{9}{32} - \frac{57}{32} + \frac{9}{32} - \frac{123}{64}\right\} m^4 \gamma \sin \eta$$

$$+ \left\{-\frac{137}{12} + \frac{123}{64}\right\} m^4 \gamma \sin(2\tau - \eta)$$

$$= -\frac{711}{192} m^4 \gamma \sin \eta - \frac{1823}{192} m^4 \gamma \sin(2\tau - \eta)$$

I now find

$$g^2 - 1 = \frac{3m^2}{2} - \frac{9}{16} m^3 - \frac{711}{192} m^4$$

$$g = 1 + \frac{3}{4} m^2 - \frac{9}{32} m^3 - \frac{273}{128} m^4$$ See p. 46 for M. Plana's expression.

If $z_{147} = \frac{3}{8} m + \frac{41}{32} m^2 + A m^3$

$$\left\{-4m+\frac{5}{2}m^2+\frac{57}{16}m^3\right\}\left\{\frac{3}{8}m+\frac{41}{32}m^2+A\,m^3\right\}=-{}^{*}\frac{1823}{192}m^3$$

$$\frac{171}{128}+\frac{205}{64}-4A=-\frac{1823}{192}\qquad A=\frac{5389}{1536}$$

$$z=\mathbf{a}\gamma\sin\eta+\left\{\frac{3}{8}m+\frac{41}{32}m^2+\frac{5389}{1536}m^3\right\}\mathbf{a}\gamma\sin(2\tau-\eta)$$

$$+\left\{\frac{3}{16}m^2+\frac{7}{8}m^3\right\}\mathbf{a}\gamma\sin(2\tau+\eta)+\frac{9}{128}m^3\mathbf{a}\gamma\sin(4\tau-\eta)$$

I next take

$$\frac{\mathbf{a}^3}{\mathbf{r}^3}=1+3e\cos\xi\qquad z=\mathbf{a}\gamma\sin\eta$$

$$\mathbf{Z}=\frac{3}{2}\gamma^2\{\sin(\xi+\eta)-\sin(\xi-\eta)\}$$

$$\{(\mathrm{c}-\mathrm{g})^2-1\}z_{149}=-\frac{3}{2}\qquad z_{149}=\frac{3}{2}$$

$$\{(\mathrm{c}+\mathrm{g})^2-1\}z_{150}=\frac{3}{2}\qquad z_{150}=\frac{1}{2}$$

In order to procure a further approximation I take

$$\mathbf{Z}=\gamma\sin\eta\left\{3m^2e\cos\xi+\left\{\frac{45}{8}m+\frac{657}{32}m^2\right\}e\cos(2\tau-\xi)\right.$$

$$+\frac{147}{16}m^2e\cos(2\tau+\xi)-\frac{3}{2}m^2e_{\prime}\cos\xi_{\prime}$$

$$\left.+\frac{21}{2}m^2e_{\prime}\cos(2\tau-\xi_{\prime})-\frac{3}{2}m^2e_{\prime}\cos(2\tau+\xi_{\prime})\right\}$$

$$+\frac{3}{8}m\gamma\sin(2\tau-\eta)\left\{3e\cos\xi+\frac{45}{8}me\cos(2\tau-\xi)\right\}$$

$$+\left\{\frac{41}{32}m^2\gamma e\sin(2\tau-\eta)+\frac{3}{16}m^2\gamma e\sin(2\tau+\eta)\right\}3e\cos\xi$$

$$+\left\{\frac{3}{2}\gamma e\sin(\xi-\eta)+\frac{1}{2}\gamma e\sin(\xi+\eta)\right\}\left\{\frac{3}{2}m^2+3m^2\cos2\tau\right\}$$

$$=\left\{-\frac{3}{2}+\frac{135}{128}+\frac{9}{4}\right\}m^2\gamma e\sin(\xi-\eta)+\left\{\frac{3}{2}+\frac{3}{4}\right\}m^2\gamma e\sin(\xi+\eta)$$

---

* As the terms multiplied by m and m^2 can have no influence upon the value of A, they are omitted.

$$+\left\{-\frac{9}{4}m+\left\{-\frac{657}{64}+\frac{123}{64}-\frac{3}{4}\right\}m^2\right\}\gamma e\sin(2\tau-\xi-\eta)$$

$$+\left\{\frac{45}{16}m+\left\{\frac{657}{64}+\frac{9}{32}-\frac{9}{4}\right\}m^2\right\}\gamma e\sin(2\tau-\xi+\eta)$$

$$+\left\{\frac{9}{16}m+\left\{-\frac{147}{32}+\frac{123}{64}+\frac{9}{4}\right\}m^2\right\}\gamma e\sin(2\tau+\xi-\eta)$$

$$+\left\{\frac{147}{32}+\frac{9}{32}+\frac{3}{4}\right\}m^2\gamma e\sin(2\tau+\xi+\eta)$$

$$+\frac{3}{4}m^2\gamma e_{,}\sin(\xi_{,}-\eta)-\frac{3}{4}m^2\gamma e_{,}\sin(\xi_{,}+\eta)$$

$$-\frac{21}{4}m^2\gamma e_{,}\sin(2\tau-\xi_{,}-\eta)+\frac{21}{4}m^2\gamma e_{,}\sin(2\tau-\xi_{,}+\eta)$$

$$+\frac{3}{4}m^2\gamma e_{,}\sin(2\tau+\xi_{,}-\eta)-\frac{3}{4}m^2\gamma e_{,}\sin(2\tau+\xi_{,}+\eta)$$

$$=\frac{231}{128}m^2\gamma e\sin(\xi-\eta)+\frac{9}{4}m^2\gamma e\sin(\xi+\eta)$$

[149] [150]

$$+\left\{-\frac{9}{4}m+\frac{291}{32}m^2\right\}\gamma e\sin(2\tau-\xi-\eta)$$

[151]

$$+\left\{\frac{45}{16}m+\frac{531}{64}m^2\right\}\gamma e\sin(2\tau-\xi+\eta)$$

[152]

$$+\left\{\frac{9}{16}m-\frac{27}{64}m^2\right\}\gamma e\sin(2\tau+\xi-\eta)$$

[153]

$$+\frac{45}{8}m^2\gamma e\sin(2\tau+\xi+\eta)+\frac{3}{4}m^2\gamma e_{,}\sin(\xi_{,}-\eta)$$

[154] [155]

$$-\frac{3}{4}m^2\gamma e_{,}\sin(\xi_{,}+\eta)-\frac{21}{4}m^2\gamma e_{,}\sin(2\tau-\xi_{,}-\eta)$$

[156] [157]

$$+\frac{21}{4}m^2\gamma e_{,}\sin(2\tau-\xi_{,}+\eta)+\frac{3}{4}m^2\gamma e_{,}\sin(2\tau+\xi_{,}-$$

[158] [159]

$$-\frac{3}{4}m^2\gamma e_{,}\sin(2\tau+\xi_{,}+\eta)$$

[160]

rom this development and by the integration of the differential ation it is easy to deduce the following terms in z,

$$-\frac{231}{128}m^2 a\gamma e\sin(\xi-\eta)+\frac{3}{4}m^2 a\gamma e\sin(\xi+\eta)$$

$$+\left\{\frac{9}{4}m+\frac{291}{32}m^2\right\}a\gamma e\sin(2\tau-\xi-\eta)$$

$$+\left\{\frac{15}{16}m+\frac{337}{64}m^2\right\}a\gamma e\sin(2\tau-\xi+\eta)$$

$$+\left\{\frac{3}{16}m+\frac{23}{64}m^2\right\}a\gamma e\sin(2\tau+\xi-\eta)$$

$$+\frac{3}{8}m^2 a\gamma e\sin(2\tau+\xi+\eta)-\frac{3}{8}m\,a\gamma e_{,}\sin(\xi_{,}-\eta)$$

$$-\frac{3}{8}m\,a\,e_{,}\gamma\sin(\xi_{,}+\eta)+\frac{7}{8}m\,a\,e_{,}\gamma\sin(2\tau-\xi_{,}-\eta)$$

$$+\frac{21}{32}m^2 a\gamma e_{,}\sin(2\tau-\xi_{,}+\eta)-\frac{3}{8}m\,a\gamma e_{,}\sin(2\tau+\xi_{,}-\eta)$$

$$-\frac{3}{32}m^2 a\gamma e_{,}\sin(2\tau+\xi_{,}+\eta)$$

In order to obtain the terms in z multiplied by m^3, Arguments 5, 156, 157 and 159, I now take

$$=\gamma\sin\eta\left\{\frac{471}{8}m^3 e_{,}\cos(2\tau-\xi_{,})-\frac{91}{8}m^3 e_{,}\cos(2\tau+\xi_{,})\right\}$$

$$+\frac{3}{8}m\gamma\sin(2\tau-\eta)\left\{-\frac{3}{2}m^2 e_{,}\cos\xi_{,}+\frac{21}{2}m^2 e_{,}\cos(2\tau-\xi_{,})\right.$$

$$\left.-\frac{3}{2}m^2 e_{,}\cos(2\tau+\xi_{,})\right\}$$

$$+\left\{-\frac{3}{8}m^2\gamma e_{,}\sin(\xi_{,}-\eta)-\frac{3}{8}m\gamma e_{,}\sin(\xi_{,}+\eta)+\frac{7}{8}m\gamma e_{,}\sin(2\tau-\xi_{,}-\eta)\right.$$

$$\left.-\frac{3}{8}m\gamma e_{,}\sin(2\tau+\xi_{,}-\eta)\right\}\left\{\frac{3}{2}m^2+3m^2\cos 2\tau\right\}$$

$$=\left\{\frac{63}{32}-\frac{9}{16}-\frac{9}{16}\right\}m^3\gamma e_{,}\sin(\xi_{,}-\eta)$$

$$+\left\{\frac{9}{32}-\frac{9}{16}-\frac{21}{16}\right\} m^3 \gamma e_, \sin(\xi_, + \eta)$$

$$+\left\{-\frac{471}{16}-\frac{9}{32}+\frac{9}{16}+\frac{21}{16}\right\} m^3 \gamma e_, \sin(2\tau - \xi_, - \eta)$$

$$+\left\{\frac{91}{16}-\frac{9}{32}-\frac{9}{16}-\frac{9}{16}\right\} m^3 \gamma e_, \sin(2\tau + \xi_, - \eta)$$

$$= \underset{[155]}{\frac{27}{32} m^3 \gamma e_, \sin(\xi_, - \eta)} - \underset{[156]}{\frac{51}{32} m^3 \gamma e_, \sin(\xi_, + \eta)}$$

$$- \underset{[157]}{\frac{891}{32} m^3 \gamma e_, \sin(2\tau - \xi_, - \eta)} + \underset{[159]}{\frac{137}{32} m^3 \gamma e_, \sin(2\tau + \xi_, -}$$

If $z_{155} = -\frac{3}{8} m + A m^2$

$$\left\{-2m + \frac{5}{2} m^2\right\}\left\{-\frac{3}{8} m + A m^2\right\} = \frac{3}{4} m^2 + \frac{27}{32} m^3$$

$$-\frac{15}{16} - 2A = \frac{27}{32} \qquad A = -\frac{57}{64}$$

If $z_{156} = -\frac{3}{8} m + A m^2$

$$\left\{2m + \frac{5}{2} m^2\right\}\left\{-\frac{3}{8} m + A m^2\right\} = -\frac{3}{4} m^2 - \frac{51}{32} m^3$$

$$-\frac{15}{16} + 2A = -\frac{51}{32} \qquad A = -\frac{21}{64}$$

If $z_{157} = \frac{7}{8} m + A m^2$

$$\left\{-6m + \frac{15}{2} m^2\right\}\left\{\frac{7}{8} m + A m^2\right\} = -\frac{21}{4} m^2 - \frac{891}{32} m^3$$

$$\frac{105}{16} - 6A = -\frac{891}{32} \qquad A = \frac{367}{64}$$

I next take

$$\frac{\mathbf{a}^3}{r^3} = 1 + \frac{3}{2} e^2 + 3 e \cos \xi + \frac{9}{2} e^2 \cos 2\xi$$

$$z = \mathrm{a}\,\gamma \sin \eta + \frac{3}{2}\,\mathrm{a}\,\gamma\, e \sin (\xi - \eta) + \frac{\mathrm{a}\,\gamma\, e}{2} \sin (\xi + \eta_{,})$$

$$\mathbb{Z} = \left\{\frac{3}{2} - \frac{9}{4} + \frac{3}{4}\right\} e\,\gamma \sin \eta + \left\{\frac{9}{4} - \frac{9}{4}\right\} e^2 \gamma \sin (2\,\xi - \eta)$$

$$+ \left\{\frac{3}{4} + \frac{9}{4}\right\} e^2 \gamma \sin (2\,\xi + \eta)$$

$$= 3\, e^2 \gamma \sin (2\,\xi + \eta) \qquad\qquad z_{162} = \frac{3}{8}$$

In order to procure the term in z_{161} multiplied by m^3, it is necessary first to procure that in z_{149} multiplied by m^3; for this purpose it is sufficient to take

$$\frac{\mathrm{a}^3}{r^3} + \frac{m_{,}\,\mathrm{a}^3}{\mu\, r_{,}^3} = 1 + \frac{3\, m^2}{2} + 3\, m^2 \cos 2\,\tau$$

$$+ \left\{3 + 3\, m^2 + \frac{45}{8}\, m^3\right\} e \cos \xi$$

$$+ \left\{\frac{45}{8}\, m + \frac{657}{32}\, m^2\right\} e \cos (2\,\tau \quad \xi)$$

$$z = \mathrm{a}\,\gamma \sin \eta + \left\{\frac{3}{8}\, m + \frac{41}{32}\, m^2\right\} \mathrm{a}\,\gamma \sin (2\,\tau - \eta) + \frac{3}{2}\, a\, e \sin (\xi_{,} - \eta)$$

$$+ \frac{15}{16}\, m\, \mathrm{a}\,\gamma\, e \sin (2\,\tau - \xi + \eta) + \frac{3}{16}\, m\, \mathrm{a}\,\gamma\, e \sin (2\,\tau + \xi - \eta)$$

$$\mathbb{Z}_{149} = -\frac{3}{2} + \frac{231}{128}\, m^2 + \left\{-\frac{45}{16} + \frac{1971}{512} + \frac{1845}{512} - \frac{45}{32} + \frac{9}{32}\right\} m^3$$

$$= -\frac{3}{2} + \frac{231}{128}\, m^2 + \frac{225}{64}\, m^3$$

$$z_{149} = \frac{3}{2} - \frac{231}{128}\, m^2 - \frac{225}{64}\, m^3$$

I next take

$$\frac{\mathrm{a}^3}{\mathrm{r}^3} + \frac{m_{,}\,\mathrm{a}^3}{\mu\, r_{,}^3} = 1 + \frac{3}{2}\, m^2 + 3\, m^2 \cos 2\,\tau$$

$$+ \left\{3 + 3\, m^2 + \frac{45}{8}\, m^3\right\} e \cos \xi$$

$$+ \left\{\frac{45}{8}\, m + \frac{657}{32}\, m^2\right\} e \cos (2\,\tau - \xi)$$

$$+\frac{147}{16}m^2 e\cos(2\tau+\xi)$$

$$+\left\{\frac{9}{2}+\frac{33}{4}m^2+\frac{855}{128}m^3\right\}e^2\cos 2\xi$$

$$+\left\{\frac{45}{8}m+\frac{321}{32}m^2\right\}e^2\cos(2\tau-2\xi)$$

$$z=\mathrm{a}\gamma\sin\eta+\left\{\frac{3}{8}m+\frac{41}{32}m^2\right\}\mathrm{a}\gamma\sin(2\tau-\eta)$$

$$+\left\{\frac{3}{2}-\frac{231}{128}m^2-\frac{225}{64}m^3\right\}\mathrm{a}\gamma e\sin(\xi-\eta)$$

$$+\left\{\frac{15}{16}m+\frac{337}{64}m^2\right\}\mathrm{a}\gamma e\sin(2\tau-\xi+\eta)$$

$$+\left\{\frac{3}{16}m+\frac{23}{64}m^2\right\}\mathrm{a}\gamma e\sin(2\tau+\xi-\eta)$$

$$+z_{164}\mathrm{a}\gamma e^2\sin(2\tau-2\xi+\eta)$$

$$+\frac{9}{64}m\,\mathrm{a}\gamma e^2\sin(2\tau+2\xi-\eta)$$

$$\mathbb{Z}_{161}=\frac{3}{2}m^2 z_{161}+\left\{-\frac{693}{256}+\frac{9}{4}-\frac{33}{8}+\frac{135}{256}+\frac{135}{128}\right\}m^2-\frac{3}{2}m^2 z_{164}$$

$$+\left\{\frac{27}{128}-\frac{675}{128}+\frac{135}{32}+\frac{1035}{1024}+\frac{1971}{1024}-\frac{2205}{512}-\frac{855}{256}+\frac{963}{512}+\frac{1845}{512}\right\}m^3$$

$$\left\{-\frac{9}{2}m^2-\frac{441}{16}m^3\right\}z_{161}=\frac{3}{2}m^2 z_{161}-3m^2-\frac{135}{128}m^3-\frac{9}{128}m^3$$

Let $z_{161}=A+Bm$ $\qquad$ $z_{164}=\frac{45}{64}m$. (See p. 240.

$$-\frac{9}{2}A=\frac{3}{2}A-3 \qquad A=\frac{1}{2}$$

$$-\frac{441}{32}-\frac{9}{2}B=\frac{3}{2}B-\frac{135}{128}-\frac{9}{128} \qquad B=-\frac{135}{64}$$

In order to have the term in z_{161} multiplied by m^2 by this n thod, it would be necessary to procure previously the terms in and in r_8 multiplied by m^4. M. Plana notices the difficulty of taining this term *.

---

* De tous les coefficiens de la latitude le plus difficile à développer est celui qui partient à l'argument $2gv-cv$. (vol. ii. p. 269.)

In order to arrive at z_{183}, I take

$$\mathbb{Z} = -\frac{9}{4} m^2 e_{,} \cos 2\,\xi_{,} \left\{ \frac{3}{8}\, m\,\gamma \sin (2\,\tau - \eta) \right\}$$

$$-\frac{9}{4} m^3 e_{,}^2 \cos (2\,\tau + 2\,\xi_{,}) \left\{ \gamma \sin \eta \right\}$$

$$+ \left\{ \frac{3\,m^2}{2} + 3\,m^2 \cos 2\,\tau - \frac{3\,m^2}{2} e_{,} \cos \xi_{,} - \frac{3}{2} m^2 e_{,} \cos (2\,\tau + \xi_{,}) \right\}$$

$$\left\{ -\frac{3}{8} m\,\gamma\,e_{,} \sin (\xi_{,} - \eta) - \frac{3}{8} m\,\gamma\,e_{,} \sin (2\,\tau + \xi_{,} - \eta) \right.$$

$$\left. -\frac{9}{32} m\,\gamma\,e_{,}^2 \sin (2\,\xi_{,} - \eta) + z_{183}\,\gamma\,e_{,}^2 \sin (2\,\tau + 2\,\xi_{,} - \eta) \right\}$$

$$= \left\{ \left\{ -\frac{27}{64} + \frac{9}{8} + \frac{9}{32} + \frac{9}{32} - \frac{27}{64} \right\} m^3 + \frac{3}{2} m^2 z_{183} \right\} \gamma\,e_{,}^2 \sin (2\,\tau + 2\,\xi_{,} - \eta)$$

$$= \left\{ \frac{27}{32} m^3 + \frac{3}{2} m^2 z_{183} \right\} \gamma\,e_{,}^2 \sin (2\,\tau + 2\,\xi_{,} - \eta)$$

$$-\frac{3}{2} m^2 z_{183} = \frac{27}{32} m^3 + \frac{3}{2} m^2 z_{183} \qquad z_{183} = -\frac{9}{32} m$$

In order to arrive at the remaining terms in z multiplied by $m\,e^2$, $m\,e\,e_{,}$, and $m\,e_{,}^2$, a further development of $\mathbb{Z}$ is necessary.

$\mathbb{Z}$ contains the terms

$$\frac{3}{2} e^2 \left\{ \left\{ \frac{3}{8} m + \frac{41}{32} m^2 \right\} \sin (2\,\tau - \eta) + \frac{3}{16} m^2 \sin (2\,\tau + \eta) \right\}$$

$$+ 3\,e \cos \xi \left\{ -\frac{231}{128} m^2\,\gamma\,e \sin (\xi - \eta) + \frac{3}{4} m^2\,\gamma\,e \sin (\xi + \eta) \right.$$

$$+ \left\{ \frac{9}{4} m + \frac{291}{32} m^2 \right\} \gamma\,e \sin (2\,\tau - \xi - \eta) + \left\{ \frac{15}{16} m + \frac{337}{64} m^2 \right\} \gamma\,e \sin (2\,\tau - \xi + \eta)$$

$$+ \left\{ \frac{3}{16} m + \frac{23}{64} m^2 \right\} \gamma\,e \sin (2\,\tau + \xi - \eta) + \frac{3}{8} m^2\,\gamma\,e \sin (2\,\tau + \xi + \eta)$$

$$-\frac{3}{8} m\,e_{,} \sin (\xi_{,} - \eta) - \frac{3}{8} m\,e_{,} \sin (\xi_{,} + \eta)$$

$$+ \frac{7}{8} m\,e_{,} \sin (2\,\tau - \xi_{,} - \eta) - \frac{3}{8} m\,e_{,} \sin (2\,\tau + \xi_{,} - \eta)$$

$$+ \left\{ \left\{ \frac{45}{8} m + \frac{657}{32} m^2 \right\} e \cos (2\,\tau - \xi) + \frac{147}{16} m^2 e \cos (2\,\tau + \xi) \right\}$$

$$\left\{ \frac{3}{2} \gamma\,e \sin (\xi - \eta) + \frac{1}{2} \gamma\,c \sin (\xi + \eta) + \frac{9}{4} m\,\gamma\,e \sin (2\,\tau - \xi - \eta) \right.$$

$$+ \frac{15}{16} m \gamma e \sin (2\tau - \xi + \eta) + \frac{3}{16} m \gamma e \sin (2\tau + \xi - \eta) \Big\}$$

$$+ \left\{ \frac{45}{8} m e^2 \cos (2\tau - 2\xi) - \frac{9}{4} m^2 e_{,}^2 \cos 2\xi_{,} \right\} \left\{ \frac{3}{8} m \gamma \sin (2\tau - \eta) \right\}$$

$$+ \Bigg\{ \frac{963}{128} m^2 e^2 + \left\{ 3 m^2 + \frac{135}{8} m e^2 + \frac{1941}{32} m^2 e^2 - \frac{15}{2} m^2 e_{,}^2 \right\} \cos 2\tau$$

$$+ \frac{33}{4} m^2 e^2 \cos 2\xi + \frac{321}{32} m^2 e^2 \cos (2\tau - 2\xi) \Bigg\} \gamma \left(1 - \frac{e^2}{2}\right) \sin \eta$$

$$+ \left\{ \frac{3 m^2}{2} + 3 m^2 \cos 2\tau \right\} \left\{ \frac{1}{2} \gamma e^2 \sin (2\xi - \eta) + \frac{3}{8} \gamma e^2 \sin (2\xi + \eta) \right\}$$

$$\mathbb{Z} = \Bigg\{ \left\{ \frac{9}{16} + \frac{27}{8} + \frac{9}{32} - \frac{135}{16} + \frac{135}{32} \right\} m e^2$$

$$+ \left\{ \frac{123}{64} - \frac{1941}{64} + \frac{873}{64} + \frac{69}{128} + \frac{1971}{128} - \frac{147}{64} + \frac{3}{4} \right\} m^2 e^2 + \frac{15}{4} m^2 e_{,}^2 \Bigg\} \sin (2\tau - \eta)$$

[147]

$$+ \left\{ \frac{45}{32} + \frac{45}{32} + \frac{135}{16} \right\} m e^2 \gamma \sin (2\tau + \eta)$$

$$+ \Bigg\{ \left\{ \frac{27}{8} + \frac{27}{32} - \frac{45}{16} - \frac{45}{32} \right\} m$$

$$+ \left\{ \frac{873}{64} - \frac{657}{128} + \frac{369}{128} - \frac{321}{64} - \frac{9}{16} \right\} m^2 \Bigg\} \gamma e^2 \sin (2\tau - 2\xi - \eta)$$

[163]

$$+ \Bigg\{ \left\{ \frac{45}{32} + \frac{45}{16} - \frac{135}{32} \right\} m$$

$$+ \left\{ \frac{1011}{128} - \frac{1971}{128} + \frac{27}{64} + \frac{321}{64} - \frac{3}{4} \right\} m^2 \Bigg\} \gamma e^2 \sin (2\tau - 2\xi + \eta)$$

$$+ \left\{ \frac{9}{32} + \frac{27}{32} \right\} m \gamma e^2 \sin (2\tau + 2\xi - \eta)$$

$$+ \left\{ \frac{63}{16} - \frac{9}{16} \right\} m \gamma e e_{,} \sin (\xi + \xi_{,} - \eta) + \left\{ - \frac{63}{16} - \frac{9}{16} \right\} m \gamma e e_{,} \sin (\xi + \xi_{,} + \eta)$$

$$+ \left\{ - \frac{105}{16} + \frac{21}{16} \right\} m \gamma e e_{,} \sin (\xi - \xi_{,} - \eta) + \frac{105}{16} m \gamma e e_{,} \sin (2\tau - \xi - \xi_{,} + \eta)$$

$$- \frac{9}{16} m \gamma e e_{,} \sin (2\tau + \xi + \xi_{,} - \eta) + \left\{ - \frac{63}{16} + \frac{9}{16} \right\} m \gamma e e_{,} \sin (\xi - \xi_{,} - \eta)$$

$$+ \left\{ \frac{63}{16} + \frac{9}{16} \right\} m \gamma c e_{,} \sin (\xi - \xi_{,} + \eta) + \left\{ \frac{45}{16} - \frac{9}{16} \right\} m \gamma e e_{,} \sin (2\tau - \xi + \xi_{,} - \eta)$$

$$- \frac{45}{16} m \gamma e e_{,} \sin (2\tau - \xi + \xi_{,} + \eta) + \frac{21}{16} m \gamma e e_{,} \sin (2\tau + \xi - \xi_{,} - \eta)$$

$$+ \frac{9}{8} m^2 \gamma e_{,}^2 \sin (2\xi_{,} - \eta) - \frac{9}{8} m^2 \gamma e_{,}^2 \sin (2\xi_{,} + \eta)$$

$$-\frac{51}{4} m^2 \gamma e_{\prime}^2 \sin(2\tau - 2\xi_{\prime} - \eta) + \frac{51}{4} m^2 \gamma e_{\prime}^2 \sin(2\tau - 2\xi_{\prime} + \eta)$$

$$_{147} = -\frac{3}{8} m^2 e^2 \quad \mathbb{Z}_{163} = \frac{93}{16} m^2 \quad z_{163} = \frac{93}{64} m \quad \mathbb{Z}_{164} = -\frac{45}{16} m^2 \quad z_{164} = \frac{45}{64} m$$

From the last development containing terms in $\mathbb{Z}$ of the order the squares of the eccentricities, it is easy to deduce other terms z of the same order, and uniting these to terms previously ob-ned, the following expression results for z.

$$= \mathrm{a}\gamma\left(1 - \frac{e^2}{2}\right) \sin\eta + \left\{\frac{3}{8} m + \frac{41}{32} m^2 + \frac{5389}{1536} m^3\right.$$

[146]

$$\left. + \frac{3}{32} m e^2 + \frac{15}{16} m e_{\prime}^2\right\} \mathrm{a}\gamma \sin(2\tau - \eta)$$

[147]

$$+ \left\{\frac{3}{16} m^2 + \frac{7}{8} m^3 + \frac{45}{32} m e^2\right\} \mathrm{a}\gamma \sin(2\tau + \eta)$$

[148]

$$+ \left\{\frac{3}{2} - \frac{231}{128} m^2\right\} \mathrm{a}\gamma e \sin(\xi - \eta)$$

[149]

$$+ \left\{\frac{1}{2} + \frac{3}{4} m^2\right\} \mathrm{a}\gamma e \sin(\xi + \eta)$$

[150]

$$+ \left\{\frac{9}{4} m + \frac{291}{32} m^2\right\} \mathrm{a}\gamma e \sin(2\tau - \xi - \eta)$$

[151]

$$+ \left\{\frac{15}{16} m + \frac{337}{64} m^2\right\} \mathrm{a}\gamma e \sin(2\tau - \xi + \eta)$$

[152]

$$+ \left\{\frac{3}{16} m + \frac{23}{64} m^2\right\} \mathrm{a}\gamma e \sin(2\tau + \xi - \eta)$$

[153]

$$+ \frac{3}{8} m^2 \mathrm{a}\gamma e \sin(2\tau + \xi + \eta)$$

[154]

$$+ \left\{-\frac{3}{8} m - \frac{57}{32} m^2\right\} \mathrm{a}\gamma e_{\prime} \sin(\xi_{\prime} - \eta)$$

[155]

$$+ \left\{ - \frac{3}{8}\, m - \frac{21}{64} m^2 \right\} \mathrm{a}\, \gamma\, e_{\prime} \sin (\xi_{\prime} + \eta)$$

[156]

$$+ \left\{ \frac{7}{8}\, m + \frac{367}{64} m^2 \right\} \mathrm{a}\, \gamma\, e_{\prime} \sin (2\,\tau - \xi_{\prime} - \eta)$$

[157]

$$+ \frac{21}{32} m^2\, \mathrm{a}\, \gamma\, e_{\prime} \sin (2\,\tau - \xi_{\prime} + \eta)$$

[158]

$$+ \left\{ - \frac{3}{8}\, m - \frac{131}{64}\, m^2 \right\} \mathrm{a}\, \gamma\, e_{\prime} \sin (2\,\tau + \xi_{\prime} - \eta)$$

[159]

$$- \frac{3}{32} m^2\, \mathrm{a}\, \gamma\, e_{\prime} \sin (2\,\tau + \xi_{\prime} + \eta)$$

[160]

$$+ \left\{ \frac{1}{2} - \frac{135}{64}\, m \right\} \mathrm{a}\, \gamma\, e^2 \sin (2\,\xi - \eta)$$

[161]

$$+ \frac{3}{8}\, \mathrm{a}\, \gamma\, e^2 \sin (2\,\xi + \eta)$$

[162]

$$+ \frac{93}{64} m\, \mathrm{a}\, \gamma\, e^2 \sin (2\,\tau - 2\,\xi - \eta)$$

[163]

$$+ \frac{45}{64} m\, \mathrm{a}\, \gamma\, e^2 \sin (2\,\tau - 2\,\xi + \eta)$$

[164]

$$+ \frac{9}{64}\, \mathrm{a}\, m\, e^2 \sin (2\,\tau + 2\,\xi - \eta)$$

[165]

$$- \frac{15}{4} m\, \mathrm{a}\, \gamma\, e\, e_{\prime} \sin (\xi + \xi_{\prime} - \eta) - \frac{3}{2}\, m\, \mathrm{a}\, \gamma\, e\, e_{\prime} \sin (\xi + \xi_{\prime} + \eta)$$

[167] [168]

$$+ \frac{21}{4} m\, \mathrm{a}\, \gamma\, e\, e_{\prime} \sin (2\,\tau - \xi - \xi_{\prime} - \eta)$$

[169]

$$+\frac{35}{16}m\,\mathrm{a}\,\gamma\,e\,e_{\prime}\sin(2\tau-\xi-\xi_{\prime}+\eta)$$

[170]

$$-\frac{3}{16}m\,\mathrm{a}\,\gamma\,e\,e_{\prime}\sin(2\tau+\xi+\xi_{\prime}-\eta)+\frac{27}{8}m\,\mathrm{a}\,\gamma\,e\,e_{\prime}\sin(\xi-\xi_{\prime}-\eta)$$

[171] [173]

$$+\frac{3}{2}m\,\mathrm{a}\,\gamma\,e\,e_{\prime}\sin(\xi-\xi_{\prime}+\eta)-\frac{9}{4}m\,\mathrm{a}\,\gamma\,e\,e_{\prime}\sin(2\tau-\xi+\xi_{\prime}-\eta)$$

[174] [175]

$$-\frac{15}{16}m\,\mathrm{a}\,\gamma\,e\,e_{\prime}\sin(2\tau-\xi+\xi_{\prime}+\eta)+\frac{7}{16}m\,\mathrm{a}\,\gamma\,e\,e_{\prime}\sin(2\tau+\xi-\xi_{\prime}-\eta)$$

[176] [177]

$$-\frac{9}{32}m\,\mathrm{a}\,\gamma\,e_{\prime}^{2}\sin(2\,\xi_{\prime}-\eta)-\frac{9}{32}m\,\mathrm{a}\,\gamma\,e_{\prime}^{2}\sin(2\,\xi_{\prime}+\eta)$$

[179] [180]

$$+\frac{51}{32}m\,\mathrm{a}\,\gamma\,e_{\prime}^{2}\sin(2\tau-2\,\xi_{\prime}-\eta)-\frac{9}{32}m\,\mathrm{a}\,\gamma\,e_{\prime}^{2}\sin(2\tau+2\,\xi_{\prime}-\eta)$$

[181] [183]

$$+\frac{9}{128}m^{3}\,\mathrm{a}\,\gamma\sin(4\tau-\eta)$$

The latitude of the moon is equal to the arc of which the sine $=\frac{z}{r}$, and therefore the moon's

$$\text{Latitude}=\frac{z}{r}+\frac{z^{3}}{2\,.\,3\,r^{3}}+\frac{3\,z^{5}}{2\,.\,4\,.\,5\,r^{5}}+\&c.$$

Hence, neglecting γ^{3}, the moon's latitude $=\frac{z}{r}$.

In addition to $\frac{z}{\mathrm{a}}$, the latitude contains the following terms:

$$e\cos\xi\left\{\gamma\sin\eta+\left\{\frac{3}{8}m+\frac{41}{32}m^{2}\right\}\gamma\sin(2\tau\quad\eta)\right.$$

$$+\frac{3}{16}m^{2}\gamma\sin(2\tau+\eta)+\frac{3}{2}\gamma\,e\sin(\xi-\eta)+\frac{1}{2}\gamma\,e\sin(\xi+\eta)$$

$$+\frac{9}{4}m\,\gamma\,e\sin(2\tau-\xi-\eta)+\frac{15}{16}m\,\gamma\,e\sin(2\tau-\xi+\eta)$$

$$+\frac{3}{16}m\,\gamma\,e\sin(2\tau+\xi-\eta)-\frac{3}{8}m\,\gamma\,e_{\prime}\sin(\xi_{\prime}-\eta)$$

$$- \frac{3}{8} m \gamma e_{,} \sin (\xi_{,} + \eta) + \frac{7}{8} m \gamma e_{,} \sin (2 \tau - \xi_{,} - \eta)$$

$$\left. - \frac{3}{8} m \gamma e_{,} \sin (2 \tau + \xi_{,} - \eta) \right\}$$

$$+ \left\{ \frac{m^2}{6} + m^2 \cos 2 \tau + \frac{15}{8} m e \cos (2 \tau - \xi) + e^2 \cos 2 \xi \right\}$$

$$\left\{ \gamma \sin \eta + \frac{3}{8} m \gamma \sin (2 \tau - \eta) + \frac{3}{2} e \sin (\xi - \eta) + \frac{1}{2} e \sin (\xi + \eta) \right\}$$

$$+ \left\{ \left\{ \frac{15}{4} m e^2 + \frac{19}{6} m^3 \right\} \cos 2 \tau + \frac{2}{3} m^2 e \cos \xi + \frac{187}{32} m^3 e \cos (2 \tau - \xi) \right.$$

$$+ \frac{33}{16} m^2 e \cos (2 \tau + \xi) - \frac{3}{2} m^2 e_{,} \cos \xi_{,} + \frac{7}{2} m^2 e_{,} \cos (2 \tau - \xi_{,})$$

$$- \frac{m^2}{2} e_{,} \cos (2 \tau + \xi_{,}) - \frac{21}{8} m e e_{,} \cos (\xi + \xi_{,})$$

$$+ \frac{35}{8} m e e_{,} (2 \tau - \xi - \xi_{,}) + \frac{21}{8} m e e_{,} \cos (\xi - \xi_{,})$$

$$\left. - \frac{15}{8} m e e_{,} \cos (2 \tau - \xi + \xi_{,}) \right\} \gamma \sin \eta$$

Hence the latitude of the moon

$$= \left\{ 1 + \frac{m^2}{6} + \left\{ - \frac{1}{2} - \frac{3}{4} + \frac{1}{4} \right\} e^2 - \frac{3}{16} m^3 \right\} \gamma \sin \eta$$

$$+ \left\{ \frac{3}{8} m + \left\{ \frac{41}{32} - \frac{1}{2} \right\} m^2 + \left\{ \frac{5389}{1536} + \frac{1}{16} - \frac{19}{12} \right\} m^3 \right.$$

$$\left. + \left\{ \frac{3}{32} + \frac{9}{8} + \frac{3}{32} + \frac{45}{32} - \frac{15}{8} \right\} m e^2 - \frac{15}{16} m e_{,}^2 \right\} \gamma \sin (2 \tau - \eta)$$

$$+ \left\{ \left\{ \frac{3}{16} + \frac{1}{2} \right\} m^2 + \left\{ \frac{7}{8} + \frac{19}{12} \right\} m^3 \right.$$

$$\left. + \left\{ \frac{45}{32} + \frac{15}{32} + \frac{15}{32} + \frac{15}{8} \right\} m e^2 \right\} \gamma \sin (2 \tau + \eta)$$

$$+ \left\{ \frac{3}{2} - \frac{1}{2} + \left\{ - \frac{231}{128} + \frac{1}{4} + \frac{45}{128} - \frac{1}{3} \right\} m^2 \right\} \gamma e \sin (\xi - \eta)$$

$$+ \left\{ \frac{1}{2} + \frac{1}{2} + \left\{ \frac{3}{4} + \frac{1}{12} + \frac{1}{3} \right\} m^2 \right\} \gamma e \sin (\xi + \eta)$$

$$+ \left\{ \left\{ \frac{9}{4} + \frac{3}{16} - \frac{15}{16} \right\} m + \left\{ \frac{291}{32} + \frac{41}{64} - \frac{1}{4} - \frac{187}{64} \right\} m^2 \right\} \gamma e \sin (2 \tau - \xi - \eta)$$

$$+ \left\{ \left\{ \frac{15}{16} + \frac{15}{16} \right\} m + \left\{ \frac{337}{64} + \frac{3}{32} - \frac{3}{4} + \frac{187}{64} \right\} m^2 \right\} \gamma e \sin (2 \tau - \xi + \eta)$$

$$+\left\{\left\{\frac{3}{16}+\frac{3}{16}\right\}m+\left\{\frac{23}{64}+\frac{41}{64}+\frac{3}{4}-\frac{33}{32}\right\}m^2\right\}\gamma e\sin(2\tau+\xi-\eta)$$

$$+\left\{\frac{3}{8}+\frac{3}{32}+\frac{1}{4}+\frac{33}{32}\right\}m^2\gamma e\sin(2\tau+\xi+\eta)$$

$$+\left\{-\frac{3}{8}m+\left\{-\frac{57}{64}+\frac{3}{4}\right\}m^2\right\}\gamma e_{,}\sin(\xi_{,}-\eta)$$

$$+\left\{-\frac{3}{8}m+\left\{-\frac{21}{64}-\frac{3}{4}\right\}m^2\right\}\gamma e_{,}\sin(\xi_{,}+\eta)$$

$$+\left\{\frac{7}{8}m+\left\{\frac{367}{64}-\frac{7}{4}\right\}m^2\right\}\gamma e_{,}\sin(2\tau-\xi_{,}-\eta)$$

$$+\left\{\frac{21}{32}+\frac{7}{4}\right\}m^2\gamma e_{,}\sin(2\tau-\xi_{,}+\eta)$$

$$+\left\{-\frac{3}{8}m+\left\{-\frac{131}{64}+\frac{1}{4}\right\}m^2\right\}\gamma e_{,}\sin(2\tau+\xi_{,}-\eta)$$

$$+\left\{-\frac{3}{32}-\frac{1}{4}\right\}m^2 e_{,}\sin(2\tau+\xi_{,}+\eta)$$

$$+\left\{\left\{\frac{1}{2}+\frac{3}{4}-\frac{1}{2}\right\}-\frac{135}{64}m\right\}\gamma e^2\sin(2\xi-\eta)$$

$$+\left\{\frac{3}{8}+\frac{1}{4}+\frac{1}{2}\right\}\gamma e^2\cos(2\xi+\eta)$$

$$+\left\{\frac{93}{64}+\frac{9}{8}-\frac{15}{32}+\frac{3}{16}\right\}m\gamma e^2\sin(2\tau-2\xi-\eta)$$

$$+\left\{\frac{45}{64}+\frac{15}{32}-\frac{45}{32}\right\}m\gamma e^2\sin(2\tau-2\xi+\eta)$$

$$+\left\{\frac{9}{64}+\frac{3}{32}+\frac{3}{16}\right\}m\gamma e^2\sin(2\tau+2\xi-\eta)$$

$$+\left\{-\frac{15}{4}-\frac{3}{16}+\frac{21}{16}\right\}m\gamma e e_{,}\sin(\xi+\xi_{,}-\eta)$$

$$+\left\{-\frac{3}{2}-\frac{3}{16}-\frac{21}{16}\right\}m\gamma e e_{,}\sin(\xi+\xi_{,}+\eta)$$

$$+\left\{\frac{21}{4}+\frac{7}{16}-\frac{35}{16}\right\}m\gamma e e_{,}\sin(2\tau-\xi-\xi_{,}-\eta)$$

$$+\left\{\frac{35}{16}+\frac{35}{16}\right\}m\gamma e e_{,}\sin(2\tau-\xi-\xi_{,}+\eta)$$

$$+\left\{-\frac{3}{16}-\frac{3}{16}\right\}m\gamma e e_{,}\sin(2\tau+\xi+\xi_{,}-\eta)$$

$$+\left\{\frac{27}{8}+\frac{3}{16}+\frac{21}{16}\right\}m\gamma e e_{,}\sin(\xi-\xi_{,}-\eta)$$

$$+\left\{\frac{3}{2}+\frac{3}{16}+\frac{21}{16}\right\} m\,\gamma\, e\, e_{,} \sin(\xi-\xi_{,}+\eta)$$

$$+\left\{-\frac{9}{4}-\frac{3}{16}+\frac{15}{16}\right\} m\,\gamma\, e\, e_{,} \sin(2\tau-\xi+\xi_{,}-\eta)$$

$$+\left\{-\frac{15}{16}-\frac{15}{16}\right\} m\,\gamma\, e\, e_{,} \sin(2\tau-\xi+\xi_{,}+\eta)$$

$$+\left\{\frac{7}{16}+\frac{7}{16}\right\} m\,\gamma\, e\, e_{,} \sin(2\tau+\xi-\xi_{,}-\eta)$$

$$-\frac{9}{32} m\,\gamma\, e_{,}^2 \sin(2\xi_{,}-\eta)-\frac{9}{32} m\,\gamma\, e_{,}^2 \sin(2\xi_{,}+\eta)$$

$$+\frac{51}{32} m\,\gamma\, e_{,}^2 \sin(2\tau-2\xi_{,}-\eta)-\frac{9}{32} m\,\gamma\, e_{,}^2 \sin(2\tau+2\xi_{,}-\eta)$$

$$+\left\{\frac{9}{128}+\frac{3}{16}\right\} m^3\gamma \sin(4\tau-\eta)+\left\{\frac{45}{128}+\frac{45}{128}\right\} m^2\gamma\, e \sin(4\tau-\xi-\eta)$$

Hence arises the following approximate expression for the latitude of the moon:

$$=\left\{1-e^2+\frac{m^2}{6}-\frac{3}{16}m^3\right\}\gamma \sin \eta$$

[146]

$$+\left\{\frac{3}{8}m+\frac{25}{32}m^2+\frac{3053}{1536}m^3+\frac{27}{32}m\,e^2-\frac{15}{16}m\,e_{,}^2\right\}\gamma \sin(2\tau-\eta)$$

[147]

$$+\left\{\frac{11}{16}m^2+\frac{59}{24}m^3+\frac{135}{32}m\,e^2\right\}\gamma \sin(2\tau+\eta)$$

[148]

$$+\left\{1-\frac{295}{192}m^2\right\}\gamma\, e \sin(\xi-\eta)+\left\{1+\frac{7}{6}m^2\right\}\gamma\, e \sin(\xi+\eta)$$

[149 [150]

$$+\left\{\frac{3}{2}m+\frac{105}{16}m^2\right\}\gamma\, e \sin(2\tau-\xi-\eta)$$

[151]

$$+\left\{\frac{15}{8}m+\frac{241}{32}m^2\right\}\gamma\, e \sin(2\tau-\xi+\eta)$$

[152]

$$+\left\{\frac{3}{8}m+\frac{23}{32}m^2\right\}\gamma\, e \sin(2\tau+\xi-\eta)$$

[153]

$$+\frac{7}{4}m^2\gamma e\sin(2\tau+\xi+\eta)+\left\{-\frac{3}{8}m-\frac{9}{64}m^2\right\}\gamma e_{,}\sin(\xi_{,}-\eta)$$

[154] [155]

$$+\left\{-\frac{3}{8}m-\frac{69}{64}m^2\right\}\gamma e_{,}\sin(\xi_{,}+\eta)$$

[156]

$$+\left\{\frac{7}{8}m+\frac{255}{64}m^2\right\}\gamma e_{,}\sin(2\tau-\xi_{,}-\eta)$$

[157]

$$+\frac{77}{32}m^2\gamma e_{,}\sin(2\tau-\xi_{,}+\eta)$$

[158]

$$+\left\{-\frac{3}{8}m-\frac{115}{64}m^2\right\}\gamma e_{,}\sin(2\tau+\xi_{,}-\eta)$$

[159]

$$-\frac{11}{32}m^2\gamma e_{,}\sin(2\tau+\xi_{,}+\eta)+\left\{\frac{3}{4}-\frac{135}{64}m\right\}\gamma e^2\sin(2\xi-\eta)$$

[160] [161]

$$+\frac{9}{8}\gamma e^2\sin(2\xi+\eta)+\frac{147}{64}m\gamma e^2\sin(2\tau-2\xi-\eta)$$

[162] [163)

$$-\frac{15}{64}m\gamma e^2\sin(2\tau-2\xi+\eta)+\frac{27}{64}m\gamma e^2\sin(2\tau+2\xi-\eta)$$

[164] [165]

$$-\frac{9}{4}m\gamma e e_{,}\sin(\xi+\xi_{,}-\eta)-3m\gamma e e_{,}\sin(\xi+\xi_{,}+\eta)$$

[167] [168]

$$+\frac{7}{2}m\gamma e e_{,}\sin(2\tau-\xi-\xi_{,}-\eta)$$

[169]

$$+\frac{35}{8}m\gamma e e_{,}\sin(2\tau-\xi-\xi_{,}+\eta)-\frac{3}{8}m\gamma e e_{,}\sin(2\tau+\xi+\xi_{,}-\eta)$$

[170] [171]

$$+\frac{9}{4}m\gamma e e_{,}\sin(\xi-\xi_{,}-\eta)+3m\gamma e e_{,}\sin(\xi-\xi_{,}+\eta)$$

[173] [174]

$$-\underset{[175]}{\frac{3}{2} m \gamma e e_{,} \sin(2\tau - \xi + \xi_{,} - \eta)} - \underset{[176]}{\frac{15}{8} m \gamma e e_{,} \sin(2\tau - \xi + \xi_{,} + \eta)}$$

$$+\underset{[177]}{\frac{7}{8} m \gamma e e_{,} \sin(2\tau + \xi - \xi_{,} - \eta)} - \underset{[179]}{\frac{9}{32} m \gamma e_{,}^{2} \sin(2\xi_{,} - \eta)}$$

$$-\underset{[180]}{\frac{9}{32} m \gamma e_{,}^{2} \sin(2\xi_{,} + \eta)} + \underset{[181]}{\frac{51}{32} m \gamma e_{,}^{2} \sin(2\tau - 2\xi_{,} - \eta)}$$

$$-\underset{[183]}{\frac{9}{32} m \gamma e_{,}^{2} \sin(2\tau + 2\xi_{,} - \eta)} + \frac{33}{128} m^{3} \gamma \sin(4\tau - \eta)$$

$$+\frac{45}{64} m^{2} \gamma e \sin(4\tau - \xi - \eta)$$

As the quantities which I have denoted by e and γ are not identical with the quantities denoted by the same letters in M. Plana's work, many of the calculations given in this Treatise lead to results which are consistent with those of M. Plana, although they are not identical with them.

Thus, M. Plana has the terms

$$\left\{1 - \frac{141}{64} m^{2}\right\} e \gamma \sin(\xi - \eta) + \left\{1 + \frac{m^{2}}{2}\right\} e \gamma \sin(\xi + \eta),$$

while I have given the terms, (see p. 246.)

$$\left\{1 - \frac{295}{192} m^{2}\right\} e \gamma \sin(\xi - \eta) + \left\{1 + \frac{7}{6} m^{2}\right\} e \gamma \sin(\xi + \eta)$$

But these results are not really inconsistent with each other, for the coefficient of $\sin \eta$ in M. Plana's expression for the latitude has not the term $\frac{m^{2}}{6}$; hence $\gamma\left(1 - \frac{m^{2}}{6}\right)$ must be substituted for γ; and, as in former instances, $e\left(1 - \frac{m^{2}}{6}\right)$ must be substituted for e in my expression, in order to institute a just comparison. These substitutions suffice to obtain M. Plana's results. I have ascertained that the only terms in my approximate expression for the latitude inconsistent with those given by M. Plana are the following:

$$-\frac{3}{8} m \gamma e e_{,} \sin(2\tau + \xi + \xi_{,} - \eta) + \frac{33}{128} m^{3} \gamma \sin(4\tau - \eta)$$

Calculation of terms in the reciprocal of the radius vector multiplied by γ^2.

From the term $-\frac{5}{8}\gamma^2 e \cos(\xi - 2\eta)$ (see p. 40), in the reciprocal of the radius vector it is easy to deduce the term $\frac{5}{4}\gamma^2 e \sin(\xi - 2\eta)$ in the longitude. I now take

$$\frac{\mathrm{a}\,\delta R}{\mu} = m^2 \left\{ \frac{1}{2} + \frac{3}{2}\cos 2\tau + \frac{3}{2} e_{\prime} \cos \xi_{\prime} + \frac{21}{4} e_{\prime} \cos(2\tau - \xi_{\prime}) \right.$$

$$\left. - \frac{3}{4} e_{\prime} \cos(2\tau + \xi_{\prime}) \right\} \left\{ -\frac{5}{8}\gamma e^2 \cos(\xi - 2\eta) \right\}$$

$$+ m^2 \left\{ \frac{3}{2} \sin 2\tau + \frac{21}{4} e_{\prime} \sin(2\tau - \xi_{\prime}) \right.$$

$$\left. - \frac{3}{4} e_{\prime} \sin(2\tau + \xi_{\prime}) \right\} \left\{ \frac{5}{4}\gamma e^2 \sin(\xi - 2\eta) \right\}$$

Hence $\frac{\mathrm{a} R}{\mu}$ contains the terms

$$\left\{ \frac{9}{8} - \frac{5}{16} \right\} m^2 \gamma^2 e \cos(\xi - 2\eta) + \left\{ -\frac{15}{32} + \frac{15}{16} \right\} m^2 \gamma^2 e \cos(2\tau - \xi + 2\eta)$$

$$+ \left\{ \frac{3}{8} - \frac{15}{32} - \frac{15}{16} \right\} m^2 \gamma^2 e \cos(2\tau + \xi - 2\eta) + \left\{ \frac{27}{16} - \frac{15}{32} \right\} m^2 \gamma^2 e e_{\prime} \cos(\xi + \xi_{\prime} - 2\eta)$$

$$+ \left\{ -\frac{105}{64} + \frac{105}{32} \right\} m^2 \gamma^2 e e_{\prime} \cos(2\tau - \xi - \xi_{\prime} + 2\eta)$$

$$+ \left\{ -\frac{3}{16} + \frac{15}{64} + \frac{15}{32} \right\} m^2 \gamma^2 e e_{\prime} \cos(2\tau + \xi + \xi_{\prime} - 2\eta)$$

$$+ \left\{ \frac{27}{16} - \frac{15}{32} \right\} m^2 \gamma^2 e e_{\prime} \cos(\xi - \xi_{\prime} - 2\eta) + \left\{ \frac{15}{64} - \frac{15}{32} \right\} m^2 \gamma^2 e e_{\prime} \cos(2\tau - \xi + \xi_{\prime} + 2\eta)$$

$$+ \left\{ \frac{21}{16} - \frac{105}{64} - \frac{105}{32} \right\} m^2 \gamma^2 e e_{\prime} \cos(2\tau + \xi - \xi_{\prime} - 2\eta)$$

$$= \underset{[65]}{\frac{13}{16} m^2 \gamma^2 e \cos(\xi - 2\eta)} + \underset{[68]}{\frac{15}{32} m^2 \gamma^2 e \cos(2\tau - \xi + 2\eta)}$$

$$- \underset{[69]}{\frac{33}{32} m^2 \gamma^2 e \cos(2\tau + \xi - 2\eta)} + \underset{[83]}{\frac{39}{32} m^2 \gamma^2 e e_{\prime} \cos(\xi + \xi_{\prime} - 2\eta)}.$$

$$8\,r_{68} = -\frac{9\,.\,15}{16}m^2 - \frac{4\,.\,15}{32}m^2 \qquad r_{68} = -\frac{165}{128}m^2$$

$$-4\,m\,r_{69} = -\frac{33}{16}m^2 + \frac{4\,.\,33}{32}m^2 \qquad r_{69} = -\frac{33}{64}m \qquad r_{70} = 0$$

$$3\,r_{71} = \frac{4\,.\,9}{16}m^2 \qquad r_{71} = \frac{3}{4}m^2 \quad 3\,r_{72} = \frac{4\,.\,9}{16}m^2 \qquad r_{72} = \frac{3}{4}m$$

$$r_{73} = -\frac{21}{8}m^2 \qquad \text{M. Plana has } \frac{7}{8}m - \frac{7}{8}m^2$$

$$r_{74} = 0 \qquad r_{75} = \frac{3}{8}m^2 \qquad r_{76} = 0$$

$$\mathfrak{r}_{83} = \frac{3}{2}r_{71} + \frac{45}{32}m^2 = \frac{81}{32}m^2 \qquad \mathfrak{r}_{85} = \frac{3}{2}r_{73} = -\frac{63}{16}m^2$$

$$\mathfrak{r}_{87} = \frac{3}{2}r_{75} + \frac{15}{32}m^2 = \frac{33}{32}m^2 \qquad \mathfrak{r}_{89} = \frac{3}{2}r_{72} + \frac{45}{32}m^2 = \frac{81}{32}m^2$$

$$\mathfrak{r}_{91} = \frac{3}{2}r_{75} = \frac{9}{16}m^2 \qquad \mathfrak{r}_{93} = \frac{3}{2}r_{73} - \frac{105}{32}m^2 = -\frac{231}{32}$$

$$-2\,m\,r_{83} = \frac{81}{32}m^2 - \frac{4\,.\,39}{32}m^2 \qquad r_{83} = \frac{75}{64}m$$

$$6\,m\,r_{85} = -\frac{63}{16}m^2 - \frac{4\,.\,21}{16}m^2 \qquad r_{85} = -\frac{49}{32}m$$

$$-2\,m\,r_{87} = \frac{33}{32}m^2 - \frac{4\,.\,33}{64}m^2 \qquad r_{87} = \frac{33}{64}m$$

$$2\,m\,r_{89} = \frac{81}{32}m^2 - \frac{4\,.\,39}{32}m^2 \qquad r_{89} = -\frac{75}{64}m$$

$$2\,m\,r_{91} = \frac{9}{16}m^2 + \frac{4\,.\,3}{16}m^2 \qquad r_{91} = \frac{21}{32}$$

$$-6\,m\,r_{93} = -\frac{231}{32}m^2 + \frac{4\,.\,231}{64}m^2 \qquad r_{93} = -\frac{77}{64}m$$

M. Plana has $r_{85} = -\frac{35}{32}m$ and $r_{93} = -\frac{49}{64}m$

In order to find the term in r_{65} multiplied by m, I take

$$\frac{\mathrm{a}\,\delta R}{\mu} = m^2\left\{\frac{1}{2} + \frac{3}{2}\cos 2\tau + \frac{3}{4}\gamma^2\cos(2\tau - 2\eta)\right\}\left\{\frac{15}{8}m\,e\cos(2\tau - \xi)\right.$$

$$+ r_{65}\, e\, \gamma^2 \cos(\xi - 2\eta) - \frac{33}{64} m\, e\, \gamma^2 \cos(2\tau + \xi - 2\eta) \Big\}$$

$$+ m^2 \left\{ \frac{3}{2} \sin 2\tau - \frac{9}{2} e \sin(2\tau - \xi) + \frac{3}{4} \gamma^2 \sin(2\tau - 2\eta) \right\}$$

$$\Big\{ \frac{15}{4} m\, e \sin(2\tau - \xi) + \frac{9}{16} m\, \gamma^2 \sin(2\tau - 2\eta)$$

$$- \frac{15}{16} m\, e\, \gamma^2 \sin(2\tau - \xi + 2\eta) - \frac{33}{32} m\, e\, \gamma^2 \sin(2\tau + \xi - 2\eta) \Big\}$$

$$+ m^2 \left\{ \frac{3}{4} \cos 2\tau - \frac{9}{4} e \cos(2\tau - \xi) \right\}$$

$$\left\{ \frac{3}{8} m\, \gamma^2 \cos(2\tau - 2\eta) - \frac{15}{8} m\, e\, \gamma^2 \cos(2\tau - \xi + 2\eta) \right\},$$

$$= \Big\{ \frac{m^2}{2} r_{65} + \Big\{ -\frac{99}{256} + \frac{45}{64} - \frac{45}{64} - \frac{99}{128} - \frac{81}{64}$$

$$+ \frac{45}{32} - \frac{45}{64} - \frac{27}{64} \Big\} m^3 \Big\} e\, \gamma^2 \cos(\xi - 2\eta)$$

$$= \left\{ \frac{m^2}{2} r_{65} - \frac{549}{256} m^3 \right\} e\, \gamma^2 \cos(\xi - 2\eta)$$

$$a^2 \left(\delta \frac{1}{r} \right)^2 = \left\{ \frac{m^2}{3} + 2 m^2 \cos 2\tau + \frac{15}{4} m\, e \cos(2\tau - \xi) \right\}$$

$$\left\{ r_{65}\, e\, \gamma^2 \cos(\xi - 2\eta) - \frac{3}{4} m^2 \gamma^2 \cos(2\tau - 2\eta) - \frac{33}{64} m\, e\, \gamma^2 \cos(2\tau + \xi - 2\eta) \right\}$$

$$= \left\{ \frac{m^2}{3} r_{65} + \left\{ -\frac{33}{64} - \frac{45}{32} \right\} m^3 \right\} e\, \gamma^2 \cos(\xi - 2\eta)$$

Let $r_{65} = -\frac{5}{8} + A\, m$ (see p. 40), then

$$\left\{ \frac{9}{2} + \frac{207}{16} m \right\} \left\{ -\frac{5}{8} + A\, m \right\} = \frac{3}{4} - \frac{9}{8} m - \frac{5}{16}$$

$$+ \frac{A}{2} m - \frac{369}{128} m + \frac{5}{4} - 2 A\, m - \frac{9}{2} + \frac{549}{64} m$$

$$- \frac{1035}{128} + \frac{9}{2} A = - \frac{9}{8} + \frac{A}{2} - \frac{369}{128} - 2 A + \frac{549}{64}$$

$$A = \frac{135}{64}$$

I shall next endeavour to obtain terms in $a\, \delta \frac{1}{r}$ multiplied by $m^3 \gamma^2$. For this purpose I take

$$\delta \lambda' = \frac{9}{16} \overset{*}{m} \gamma^2 \sin (2\tau - 2\eta)$$

$$2 s \delta s = -\frac{3}{8} m \gamma^2 \cos 2\tau + \frac{3}{8} m \gamma^2 \cos (2\tau - 2\eta)$$

$$\frac{a \delta R}{\mu} = \frac{3}{2} m^2 \sin 2\tau \left\{ \frac{9}{16} m \gamma^2 \sin (2\tau - 2\eta) \right\}$$

$$+ m^2 \left\{ \frac{3}{4}^{\dagger} + \frac{3}{4} \cos 2\tau \right\} \left\{ -\frac{3}{8} m \gamma^2 \cos 2\tau + \frac{3}{8} m \gamma^2 \cos (2\tau - 2\eta) \right\}$$

$$= -\frac{9}{32} m^3 \gamma^2 \cos 2\tau + \frac{9}{16} m^3 \gamma^2 \cos 2\eta + \frac{9}{32} m^3 \gamma^2 \cos (2\tau - 2\eta)$$

$$- \frac{9}{64} m^3 \gamma^2 \cos 4\tau - \frac{9}{32} m^3 \gamma^2 \cos (4\tau - 2\eta)$$

Let $r_1 = -\frac{m^2 \gamma^2}{2} + A m^3 \gamma^2$

$$(3 - 8m) \left(-\frac{m^2}{2} + A m^3 \right) = -(4 - 2m) \left\{ \frac{3}{8} m^2 + \frac{9}{32} m^3 \right\}$$

$$4 - 3A = -\frac{3}{4} + \frac{9}{8} \qquad\qquad A = -\frac{29}{24}$$

Let $r_{62} = \frac{m^2}{2} + A m^3$

$$3 \left\{ \frac{m^2}{2} + A m^3 \right\} = \frac{3}{2} m^2 - \frac{9}{4} m^3 \qquad\qquad A = -\frac{3}{4}$$

$$- r_{63} = \frac{3}{4} m^2 - \frac{9}{8} m^3 + \frac{9}{16} m^3 \qquad\qquad r_3 = -\frac{3}{4} m^2 + \frac{9}{16} m^3$$

$$15\, r_{131} = \frac{9}{16} m^3 \gamma^2 \qquad r_{131} = \frac{3}{80} m^3 \gamma^2 \qquad 3\, r_{144} = \frac{9}{8} m^3 \qquad r_{144} = \frac{3}{8} m^3$$

Calculation of terms in the longitude multiplied by γ^2.

I shall now endeavour to arrive at some of the terms in the longitude multiplied by γ^2, by means of the equation

$$\frac{d\lambda'}{dt} = \frac{(1 + s^2)}{r^2} \left\{ h - \int \frac{dR}{d\lambda'} dt \right\} .$$

For this purpose I begin by the development of those terms in

---

* This term will be obtained hereafter.

† See the expression for $\frac{dR}{ds}$ given at p. 28.

$\frac{\mathrm{d}R}{\mathrm{d}\lambda}$ which arise from the terms $-\frac{5}{8}e\gamma^2\cos(\xi-2\eta)$ and $\frac{5}{4}e\gamma^2\sin(\xi-2\eta)$ in $\mathrm{a}\,\delta\frac{1}{r}$ and $\delta\lambda$.

$$\frac{\mathrm{a}}{\mu}\,\delta\,\frac{\mathrm{d}R}{\mathrm{d}\lambda}=\left\{-3\,m^2\sin 2\tau-\frac{21}{2}m^2e_{,}\sin(2\tau-\xi_{,})\right.$$

$$\left.+\frac{3}{2}m^2e_{,}\sin(2\tau+\xi_{,})\right\}\left\{-\frac{5}{8}e\gamma^2\cos(\xi-2\eta)\right\}$$

$$+\left\{3\,m^2\cos 2\tau+\frac{21}{2}m^2e_{,}\cos(2\tau-\xi_{,})\right.$$

$$\left.-\frac{3}{2}m^2e_{,}\cos(2\tau+\xi_{,})\right\}\left\{\frac{5}{4}e\gamma^2\sin(\xi-2\eta)\right\}$$

Hence

$$\frac{\mathrm{a}}{\mu}\frac{\mathrm{d}R}{\mathrm{d}\lambda}=\left\{-\frac{3}{4}+\frac{15}{16}+\frac{15}{8}\right\}m^2e\gamma^2\sin(2\tau+\xi-2\eta)$$

$$+\left\{\frac{15}{16}-\frac{15}{8}\right\}m^2e\gamma^2\sin(2\tau-\xi+2\eta)$$

$$+\left\{\frac{105}{32}-\frac{105}{16}\right\}m^2e\,e_{,}\gamma^2\sin(2\tau-\xi-\xi_{,}+2\eta)$$

$$+\left\{\frac{3}{8}-\frac{15}{32}-\frac{15}{16}\right\}m^2e\,e_{,}\gamma^2\sin(2\tau+\xi+\xi_{,}-2\eta)$$

$$+\left\{-\frac{15}{32}+\frac{15}{16}\right\}m^2e\,e_{,}\gamma^2\sin(2\tau-\xi+\xi_{,}+2\eta)$$

$$+\left\{-\frac{21}{8}+\frac{105}{32}+\frac{105}{16}\right\}m^2e\,e_{,}\gamma^2\sin(2\tau+\xi-\xi_{,}-2\eta)$$

$$=\frac{33}{16}m^2e\gamma^2\sin(2\tau+\xi-2\eta)-\frac{15}{8}m^2e\gamma^2\sin(2\tau-\xi+2\eta)$$

[67] [68]

$$-\frac{105}{32}m^2e\,e_{,}\gamma^2\sin(2\tau-\xi-\xi_{,}+2\eta)$$

[86]

$$-\frac{33}{32}m^2e\,e_{,}\gamma^2\sin(2\tau-\xi-\xi_{,}+2\eta)$$

[87]

$$+\frac{15}{32}m^2e\,e_{,}\gamma^2\sin(2\tau-\xi+\xi_{,}+2\eta)$$

[92]

$$+\frac{231}{32}m^2 e e_{,}\gamma^2 \sin(2\tau+\xi-\xi_{,}-2\eta)$$

[93]

I next proceed to the development of s^2. (See p. 246 for tl value of s.)

$$s^2=\frac{\gamma^2}{2}+\left\{\frac{1}{2}+\frac{1}{2}\right\}e^2\gamma^2+\left\{\frac{1}{6}+\frac{9}{128}\right\}m^2\gamma^2+\left\{-\frac{3}{16}+\frac{75}{256}\right\}m^3\gamma^2$$

$$+\left\{-\frac{3}{8}m+\left\{-\frac{25}{32}+\frac{11}{16}\right\}m^2+\left\{-\frac{3053}{1536}+\frac{59}{24}-\frac{1}{16}\right\}m^3\right.$$

$$\left.+\left\{-\frac{27}{32}+\frac{135}{32}+\frac{3}{8}-\frac{15}{8}+\frac{3}{8}-\frac{3}{2}\right\}m e^2+\frac{15}{16}m e_{,}^2\right\}\gamma^2\cos 2\tau$$

$$+\left\{-1+1+\left\{\frac{295}{192}+\frac{7}{6}-\frac{1}{6}+\frac{1}{6}+\frac{9}{16}+\frac{9}{64}\right\}m^2\right\}\gamma^2 e\cos\xi$$

$$+\left\{\left\{-\frac{3}{2}+\frac{15}{8}+\frac{3}{8}\right\}m+\left\{-\frac{105}{16}+\frac{241}{32}+\frac{25}{32}+\frac{11}{16}\right\}m^2\right\}\gamma^2 e\cos(2\tau-$$

$$+\left\{\left\{-\frac{3}{8}-\frac{3}{8}\right\}m+\left\{-\frac{23}{32}+\frac{7}{4}-\frac{25}{32}-\frac{11}{16}\right\}m^2\right\}\gamma^2 e\cos(2\tau+\xi)$$

$$+\left\{\left\{\frac{3}{8}-\frac{3}{8}\right\}m+\left\{\frac{9}{64}-\frac{69}{64}+\frac{21}{64}-\frac{9}{64}\right\}m^2\right\}\gamma^2 e_{,}\cos\xi_{,}$$

$$+\left\{-\frac{7}{8}m+\left\{-\frac{255}{64}+\frac{77}{32}-\frac{9}{64}\right\}m^2\right\}\gamma^2 e_{,}\cos(2\tau-\xi_{,})$$

$$+\left\{\frac{3}{8}m+\left\{\frac{115}{64}-\frac{11}{32}+\frac{9}{64}\right\}m^2\right\}e_{,}\cos(2\tau+\xi_{,})$$

$$+\left\{-\frac{3}{4}+\frac{9}{8}-1+\frac{135}{64}m\right\}\gamma^2 e^2\cos 2\xi$$

$$+\left\{-\frac{147}{64}-\frac{15}{64}+\frac{9}{32}+\frac{3}{2}+\frac{15}{8}\right\}m\gamma^2 e^2\cos(2\tau-2\xi)$$

$$+\left\{-\frac{27}{64}-\frac{27}{64}-\frac{3}{8}\right\}m\gamma^2 e^2\cos(2\tau+2\xi)$$

$$+\left\{\frac{3}{8}+\frac{3}{8}+\frac{9}{4}-3\right\}m\gamma^2 e e_{,}\cos(\xi+\xi_{,})$$

$$+\left\{\frac{7}{8}-\frac{7}{2}+\frac{35}{8}\right\}m\gamma^2 e e_{,}\cos(2\tau-\xi-\xi_{,})$$

$$+\left\{\frac{3}{8}+\frac{3}{8}\right\}m\gamma^2 e e_{,}\cos(2\tau+\xi+\xi_{,})$$

$$+\left\{-\frac{3}{8}-\frac{3}{8}-\frac{9}{4}+3\right\} m\,\gamma^2 e\,e_{,}\cos(\xi-\xi_{,})$$

$$+\left\{-\frac{3}{8}+\frac{3}{2}-\frac{15}{8}\right\} m\,\gamma^3 e\,e_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\left\{-\frac{7}{8}-\frac{7}{8}\right\} m\,\gamma^2 e\,e_{,}\cos(2\tau+\xi-\xi_{,})$$

$$+\left\{\frac{9}{32}-\frac{9}{32}\right\} m\,\gamma^2 e_{,}^2\cos 2\,\xi_{,}-\frac{51}{32}m\,\gamma^2 e_{,}^2\cos(2\tau-2\,\xi_{,})$$

$$+\frac{9}{32}m\,\gamma^2 e_{,}^2\cos(2\tau+2\,\xi_{,})$$

$$+\left\{-\frac{1}{2}+e^2-\frac{m^2}{6}+\left\{-\frac{3}{16}+\frac{33}{128}\right\}m^3\right\}\gamma^2\cos 2\,\eta$$

$$+\left\{\frac{3}{8}m+\frac{25}{32}m^2+\left\{\frac{3053}{1536}+\frac{1}{16}\right\}m^3\right.$$

$$\left.+\left\{\frac{27}{32}-\frac{3}{8}-\frac{3}{2}+\frac{3}{8}\right\}m\,e^2-\frac{15}{16}m\,e_{,}^2\right\}\gamma^2\cos(2\tau-2\,\eta)$$

$$+\left\{-\frac{11}{16}m^2-\frac{59}{24}m^3+\left\{-\frac{135}{32}-\frac{15}{8}\right\}m\,e^2\right\}\gamma^2\cos(2\tau+2\,\eta)$$

$$+\left\{1+\left\{-\frac{295}{192}+\frac{1}{6}+\frac{45}{64}\right\}m^2\right\}\gamma^2 e\cos(\xi-2\,\eta)$$

$$+\left\{-1+\left\{-\frac{7}{6}-\frac{1}{6}\right\}m^2\right\}\gamma^2 e\cos(\xi+2\,\eta)$$

$$+\left\{\left\{\frac{3}{2}+\frac{3}{8}\right\}m+\left\{\frac{105}{16}+\frac{25}{32}\right\}m^2\right\}\gamma^2 e\cos(2\tau-\xi-2\,\eta)$$

$$+\left\{-\frac{15}{8}m+\left\{-\frac{241}{32}+\frac{11}{16}\right\}m^2\right\}\gamma^2 e\cos(2\tau-\xi+2\,\eta)$$

$$+\left\{\left\{\frac{3}{8}-\frac{3}{8}\right\}m+\left\{\frac{23}{32}-\frac{25}{32}\right\}m^2\right\}\gamma^2 e\cos(2\tau+\xi-2\,\eta)$$

$$+\left\{-\frac{7}{4}-\frac{11}{16}\right\}m^2\gamma^2 e\cos(2\tau+\xi+2\,\eta)$$

$$+\left\{-\frac{3}{8}m-\frac{9}{64}m^2\right\}\gamma^2 e_{,}\cos(\xi_{,}-2\,\eta)+\left\{\frac{3}{8}m+\frac{69}{64}m^2\right\}\gamma^2 e_{,}\cos(\xi_{,}+2\,\eta)$$

$$+\left\{\frac{7}{8}m+\left\{\frac{255}{64}-\frac{9}{64}\right\}m^2\right\}\gamma^2 e_{,}\cos(2\tau-\xi_{,}-2\,\eta)$$

$$-\frac{77}{32}m^2\gamma^2 c_{,}\cos(2\tau-\xi_{,}+2\,\eta)$$

$$+\left\{-\frac{3}{8}m+\left\{-\frac{115}{64}+\frac{9}{64}\right\}m^2\right\}\gamma^2 e_{,}\cos(2\tau+\xi_{,}\quad 2\eta)$$

$$+\frac{11}{32}m^2\gamma^2 e_{,}\cos(2\tau+\xi_{,}+2\eta)$$

$$+\left\{s_{161}-\frac{1}{2}+\left\{\frac{1}{8}-\frac{45}{512}+\frac{295}{192}+\frac{45}{64}\right\}m^2\right\}\gamma^2 c^2\cos(2\xi-2\eta)$$

$$+\left\{-\frac{9}{8}-\frac{1}{2}\right\}\gamma^2 e^2\cos(2\xi+2\eta)$$

$$+\left\{\frac{147}{64}+\frac{27}{64}+\frac{3}{2}\right\}m\gamma^2 e^2\cos(2\tau-2\xi-2\eta)$$

$$+\left\{\frac{15}{64}+\frac{15}{8}\right\}m\gamma^2 e^2\cos(2\tau-2\xi+2\eta)$$

$$+\left\{\frac{27}{64}-\frac{9}{32}-\frac{3}{8}\right\}m\gamma^2 e^2\cos(2\tau+2\xi-2\eta)$$

$$+\left\{\frac{3}{8}-\frac{9}{4}\right\}m\gamma^2 e e_{,}\cos(\xi+\xi_{,}-2\eta)+\left\{\frac{3}{8}+3\right\}m\gamma^2 e e_{,}\cos(\xi+\xi_{,}+2$$

$$+\left\{\frac{7}{8}+\frac{7}{2}\right\}m\gamma^2 e e_{,}\cos(2\tau-\xi-\xi_{,}-2\eta)-\frac{35}{8}m\gamma^2 e e_{,}\cos(2\tau-\xi-\xi_{,}+2$$

$$+\left\{\frac{3}{8}-\frac{3}{8}\right\}m\gamma^2 e e_{,}\cos(2\tau+\xi+\xi_{,}-2\eta)$$

$$+\left\{-\frac{3}{8}+\frac{9}{4}\right\}m\gamma^2 e e_{,}\cos(\xi-\xi_{,}-2\eta)$$

$$+\left\{-\frac{3}{8}-3\right\}m\gamma^2 e e_{,}\cos(\xi-\xi_{,}+2\eta)$$

$$+\left\{-\frac{3}{8}-\frac{3}{2}\right\}m e e_{,}\cos(2\tau-\xi+\xi_{,}-2\eta)+\frac{15}{8}m e e_{,}\cos(2\tau-\xi+\xi_{,}+2$$

$$+\left\{-\frac{7}{8}+\frac{7}{8}\right\}m\gamma^2 e e_{,}\cos(2\tau+\xi-\xi_{,}-2\eta)-\frac{9}{32}m\gamma^2 e_{,}^2\cos(2\xi_{,}-2\eta)$$

$$+\frac{9}{32}m\gamma^2 e_{,}^2\cos(2\xi_{,}+2\eta)+\frac{51}{32}m\gamma^2 e_{,}^2\cos(2\tau-2\xi_{,}-2\eta)$$

$$-\frac{9}{32}m\gamma^2 e_{,}^2\cos(2\tau+2\xi_{,}-2\eta)$$

$$+\left\{-\frac{33}{128}-\frac{33}{128}\right\}m^3\gamma^2\cos 4\tau+\left\{-\frac{45}{64}-\frac{45}{64}\right\}m^2\gamma^2 e\cos(4\tau-\xi)$$

$$+\left\{-\frac{9}{128}m^2+\left\{\frac{33}{128}-\frac{75}{256}\right\}m^3\right\}\gamma^2\cos(4\tau-2\eta)$$

$$\left\{\frac{45}{64}-\frac{9}{16}\right\}m^2\gamma^2 e\cos 4\tau-\xi-2\eta)-\frac{9}{64}m^2\gamma^2 e\cos(4\tau+\xi-2\eta)$$

$$-\frac{21}{64}m^2\gamma^2 e_{,}\cos(4\tau-\xi_{,}-2\eta)+\frac{9}{64}m^2\gamma^2 e_{,}\cos(4\tau+\xi_{,}-2\eta)$$

$$= \underset{[0]}{\frac{\gamma^2}{2} + e^2\gamma^2 + \frac{91}{384}m^2\gamma^2 + \frac{27}{256}m^3\gamma^2}$$

$$+ \left\{-\frac{3}{8}m - \frac{3}{32}m^2 + \frac{209}{512}m^3 + \frac{3}{4}m e^2 + \frac{15}{16}m e_{,}^2\right\}\underset{[1]}{\gamma^2\cos 2\tau}$$

$$+ \underset{[2]}{\frac{109}{32}m^2\gamma^2 e\cos\xi} + \left\{\frac{3}{4}m + \frac{39}{16}m^2\right\}\underset{[3]}{\gamma^2 e\cos(2\tau-\xi)}$$

$$+ \left\{-\frac{3}{4}m + \frac{9}{8}m^2\right\}\underset{[4]}{\gamma^2 e\cos(2\tau+\xi)} - \underset{[5]}{\frac{3}{4}m^2\gamma^2 e_{,}\cos\xi_{,}}$$

$$+ \left\{-\frac{7}{8}m - \frac{55}{32}m^2\right\}\underset{[6]}{\gamma^2 e_{,}\cos(2\tau-\xi_{,})}$$

$$+ \left\{\frac{3}{8}m + \frac{51}{32}m^2\right\}\underset{[7]}{\gamma^2 e_{,}\cos(2\tau+\xi_{,})}$$

$$+ \left\{-\frac{5}{8} + \frac{135}{64}m\right\}\underset{[8]}{\gamma^2 e^2\cos 2\xi} + \underset{[9]}{\frac{9}{8}m\gamma^2 e^2\cos(2\tau-2\xi)}$$

$$- \underset{[10]}{\frac{39}{32}m\gamma^2 e^2\cos(2\tau+2\xi)} + \underset{[12]}{\frac{7}{4}m\gamma^2 e e_{,}\cos(2\tau-\xi-\xi_{,})}$$

$$+ \underset{[13]}{\frac{3}{4}m\gamma^3 e e_{,}\cos(2\tau+\xi+\xi_{,})}$$

$$- \underset{[15]}{\frac{3}{4}\gamma^2 e e_{,}\cos(2\tau-\xi+\xi_{,})} - \underset{[16]}{\frac{7}{4}m\gamma^2 e e_{,}\cos(2\tau+\xi-\xi_{,})}$$

$$- \underset{[18]}{\frac{51}{32}m\gamma^2 e_{,}^2\cos(2\tau-2\xi_{,})} + \underset{[19]}{\frac{9}{32}m\gamma^2 e_{,}^2\cos(2\tau+2\xi_{,})}$$

$$+ \left\{-\frac{1}{2} + e^2 - \frac{m^2}{6} + \frac{9}{128}m^3\right\}\underset{[62]}{\gamma^2\cos 2\eta}$$

$$+\left\{\frac{3}{8}m+\frac{25}{32}m^2+\frac{3149}{1536}m^3-\frac{21}{32}m\,e^2-\frac{15}{16}m\,e_{,}^2\right\}\gamma^2\cos(2\tau-2\eta)$$

[63]

$$+\left\{-\frac{11}{16}m^2-\frac{59}{24}m^3-\frac{195}{32}m\,e^2\right\}\gamma^2\cos(2\tau+2\eta)$$

[64]

$$+\left\{1-\frac{2}{3}m\right\}\gamma^2 e\cos(\xi-2\eta)$$

[65]

$$+\left\{-1-\frac{4}{3}m^2\right\}\gamma^2 e\cos(\xi+2\eta)$$

[66]

$$+\left\{\frac{15}{8}m+\frac{235}{32}m^2\right\}\gamma^2 e\cos(2\tau-\xi-2\eta)$$

[67]

$$+\left\{-\frac{15}{8}m-\frac{219}{32}m^2\right\}\gamma^2 e\cos(2\tau-\xi+2\eta)$$

[68]

$$-\frac{m^2}{16}\gamma^2 e\cos(2\tau+\xi-2\eta)-\frac{39}{16}m^2\gamma^2 e\cos(2\tau+\xi+2\eta)$$

[69] [70]

$$+\left\{-\frac{3}{8}m-\frac{9}{64}m^2\right\}\gamma^2 e_{,}\cos(\xi_{,}-2\eta)$$

[71]

$$+\left\{\frac{3}{8}m+\frac{69}{64}m^2\right\}\gamma^2 e_{,}\cos(\xi_{,}+2\eta)$$

[72]

$$+\left\{\frac{7}{8}m+\frac{123}{32}m^2\right\}\gamma^2 e_{,}\cos(2\tau-\xi_{,}-2\eta).$$

[73]

$$-\frac{77}{32}m^2\gamma^2 e_{,}\cos(2\tau-\xi_{,}+2\eta)$$

[74]

$$+\left\{-\frac{3}{8}m-\frac{53}{32}m^2\right\}\gamma^2 e_{,}\cos(2\tau+\xi_{,}-2\eta)$$

[75]

$$+\underset{[76]}{\frac{11}{32} m^2 \gamma^2 e_{\prime} \cos(2\tau + \xi_{\prime} + 2\eta)}$$

$$+\underset{[77]}{\left\{ s_{161} - \frac{1}{2} + \frac{3497}{1536} m^2 \right\} \gamma^2 e^2 \cos(2\xi - 2\eta)}$$

$$-\underset{[78]}{\frac{13}{8} \gamma^2 e^2 \cos(2\xi + 2\eta)}$$

$$+\underset{[79]}{\frac{135}{32} m \gamma^2 e^2 \cos(2\tau - 2\xi - 2\eta)} + \underset{[80]}{\frac{135}{64} m \gamma^2 e^2 \cos(2\tau - 2\xi + 2\eta)}$$

$$-\underset{[81]}{\frac{15}{64} m \gamma^2 e^2 \cos(2\tau + 2\xi - 2\eta)} - \underset{[83]}{\frac{15}{8} m \gamma^2 e e_{\prime} \cos(\xi + \xi_{\prime} - 2\eta)}$$

$$+\underset{[84]}{\frac{27}{8} m \gamma^2 e e_{\prime} \cos(\xi + \xi_{\prime} + 2\eta)} + \underset{[85]}{\frac{35}{8} m \gamma^2 e e_{\prime} \cos(2\tau - \xi - \xi_{\prime} - 2\eta)}$$

$$-\underset{[86]}{\frac{35}{8} m \gamma^2 e e_{\prime} \cos(2\tau - \xi - \xi_{\prime} + 2\eta)} + \underset{[89]}{\frac{15}{8} m \gamma^2 e e_{\prime} \cos(\xi - \xi_{\prime} - 2\eta)}$$

$$-\underset{[90]}{\frac{27}{8} m \gamma^2 e e_{\prime} \cos(\xi - \xi_{\prime} + 2\eta)} - \underset{[91]}{\frac{15}{8} m e e_{\prime} \cos(2\tau - \xi + \xi_{\prime} - 2\eta)}$$

$$+\underset{[92]}{\frac{15}{8} m e e_{\prime} \cos(2\tau - \xi + \xi_{\prime} + 2\eta)} - \underset{[95]}{\frac{9}{32} m \gamma^2 e_{\prime}^2 \cos(2\xi_{\prime} - 2\eta)}$$

$$+\underset{[96]}{\frac{9}{32} m \gamma^2 e_{\prime}^2 \cos(2\xi_{\prime} + 2\eta)} + \underset{[97]}{\frac{51}{32} m \gamma^2 e_{\prime}^2 \cos(2\tau - 2\xi_{\prime} - 2\eta)}$$

$$-\underset{[99]}{\frac{9}{32} m \gamma^2 e_{\prime}^2 \cos(2\tau + 2\xi_{\prime} - 2\eta)} - \underset{[131]}{\frac{33}{64} m^3 \gamma^2 \cos 4\tau}$$

$$-\underset{[132]}{\frac{45}{32} m^2 \gamma^2 e \cos(4\tau - \xi)}$$

$$+\left\{-\frac{9}{128}m^2-\frac{9}{256}m^3\right\}\gamma^2\cos(4\tau-2\eta)$$

[144]

$$+\frac{9}{64}m^2\gamma^2 e\cos(4\tau-\xi-2\eta)-\frac{9}{64}m^2\gamma^2 e\cos(4\tau+\xi-2\eta)$$

$$-\frac{21}{64}m^2\gamma^2 e_{\prime}\cos(4\tau-\xi_{\prime}-2\eta)+\frac{9}{64}m^2\gamma^2 e_{\prime}\cos(4\tau+\xi_{\prime}-2\eta)$$

$$\frac{d\lambda'}{dt}=\frac{(1+s^2)}{r^2}\left\{h-\int\frac{dR}{d\lambda'}dt\right\}$$

If $a\,\delta\frac{1}{r}$ and $\delta\left\{h-\int\frac{dR}{d\lambda'}dt\right\}$ are restricted to mean the terms in $\frac{a}{r}$ and in $h-\int\frac{dR}{d\lambda'}dt$ which are multiplied by γ^2, and if $\left[\frac{a}{r}\right]$ and $\left[\frac{d\lambda'}{n\,dt}\right]$ represent the terms in those quantities independent of γ^2,

$$\frac{d\lambda'}{dt}=\left[\frac{d\lambda'}{dt}\right]s^2+\left[\frac{1}{r^2}\right]\delta\left\{h-\int\frac{dR}{d\lambda'}dt\right\}+\left[\frac{2h}{r}\right]\delta\frac{1}{r}$$

$$\frac{d\lambda'}{n\,dt}=1+\left\{\frac{11}{4}m^2+\frac{85}{12}m^3+\frac{75}{8}m\,e^2\right\}\cos 2\tau$$

$$+\{2+m^2\}e\cos\xi+\left\{\frac{15}{4}m+\frac{143}{16}m^2\right\}e\cos(2\tau-\xi)$$

$$+\frac{51}{8}m^2 e\cos(2\tau+\xi)-3\,m^2 e_{\prime}\cos\xi_{\prime}$$

$$+\frac{77}{8}m^2 e_{\prime}\cos(2\tau-\xi_{\prime})-\frac{11}{8}m^2 e_{\prime}\cos(2\tau+\xi_{\prime})$$

$$+\frac{5}{2}e^2\cos 2\xi-\frac{21}{4}m\,e\,e_{\prime}\cos(\xi+\xi_{\prime})+\frac{35}{4}m\,e\,e_{\prime}\cos(2\tau-\xi-\xi_{\prime})$$

$$+\frac{21}{4}m\,e\,e_{\prime}\cos(\xi-\xi_{\prime})-\frac{15}{4}m\,e\,e_{\prime}\cos(2\tau-\xi+\xi_{\prime})$$

See pp. 132, 142, 192, 204, &c.

I take

$$\frac{a^2}{r^2}=1+\frac{e^2}{2}+\frac{m^2}{3}+2\,m^2\cos 2\tau+2\,e\cos\xi$$

$$+\frac{15}{4}\,m\,e\cos(2\tau-\xi)+\frac{5}{2}e^2\cos 2\xi+\&c.$$

$$\delta\left\{h-\int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\mathrm{d}t\right\}=-\frac{\gamma^2}{2}-\frac{3}{8}m^2\gamma^2\cos 2\tau$$

$$+\frac{9}{4}m^2\gamma^2 e\cos(2\tau-\xi)$$

$$-\frac{1}{4}m^2\gamma^2 e\cos(2\tau+\xi)-\frac{21}{16}m^2\gamma^2 e_{,}\cos(2\tau-\xi_{,})$$

$$+\frac{3}{16}m^2\gamma^2 e_{,}\cos(2\tau+\xi_{,})$$

$$+\frac{15}{16}m\,\gamma^2 e^2\cos(2\tau-2\xi)-\frac{3}{8}m\,\gamma^2\cos(2\tau-2\eta)$$

$$+\frac{3}{4}m^2\gamma^2 e\cos(2\tau-\xi-2\eta)$$

$$+\frac{33}{16}m^2 e\,\gamma^2\cos(2\tau+\xi-2\eta)-\frac{5}{8}m^2 e\,\gamma^2\cos(2\tau-\xi+2\eta)$$

$$-\frac{7}{8}m\,e_{,}\gamma^2\cos(2\tau-\xi_{,}-2\eta)+\frac{3}{8}m\,\gamma^2 e_{,}\cos(2\tau+\xi_{,}-2\eta)$$

The following expression for $a\,\delta\frac{1}{r}$ will serve as a recapitulation terms multiplied by γ^2 obtained already in this treatise:

$$\frac{1}{r}=\left\{-\frac{m^2}{2}-\frac{29}{24}m^3\right\}\gamma^2\cos 2\tau-\frac{15}{16}m\,\gamma^2 e\cos(2\tau-\xi)$$

$$+\frac{9}{4}m^2\gamma^2 e_{,}\cos\xi_{,}-\frac{7}{4}m^2\gamma^2 e_{,}\cos(2\tau-\xi_{,})$$

$$+\frac{m^2\gamma^2}{4}e_{,}\cos(2\tau+\xi_{,})+\frac{63}{16}m\,\gamma^2 e\,e_{,}\cos(\xi+\xi_{,})$$

$$-\frac{35}{16}m\,\gamma^2 e\,e_{,}\cos(2\tau-\xi-\xi_{,})-\frac{63}{16}m\,\gamma^2 e\,e_{,}\cos(\xi-\xi_{,})$$

$$+\frac{15}{16}m\,\gamma^2 e\,e_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\left\{\frac{m^2}{2}-\frac{3}{4}m^3\right\}\gamma^2\cos 2\eta$$

$$+\left\{-\frac{3}{4}m^2+\frac{9}{16}m^3\right\}\gamma^2\cos(2\tau-2\eta)$$

$$+\left\{-\frac{5}{8}+\frac{135}{64}m\right\}\gamma^2 e\cos(\xi-2\eta)$$

$$-\frac{21}{32}m\gamma^2 e\cos(2\tau-\xi-2\eta)$$

$$-\frac{33}{64}m\gamma^2 e\cos(2\tau+\xi-2\eta)+\frac{3}{4}m^2\gamma^2 e_{,}\cos(\xi_{,}-2\eta)$$

$$+\frac{3}{4}m^2\gamma^2 e_{,}\cos(\xi_{,}+2\eta)$$

$$-\frac{21}{8}m^2\gamma^2 e_{,}\cos(2\tau-\xi_{,}-2\eta)+\frac{3}{8}m^2\gamma^2 e_{,}\cos(2\tau+\xi_{,}-2\eta$$

$$+\frac{75}{64}m\gamma^2 e e_{,}\cos(\xi+\xi_{,}-2\eta)-\frac{49}{32}m e e_{,}\gamma^2\cos(2\tau-\xi-\xi_{,}-$$

$$+\frac{33}{64}m\gamma^2 e e_{,}\cos(2\tau+\xi+\xi_{,}-2\eta)-\frac{75}{64}m e e_{,}\gamma^3\cos(\xi-\xi_{,}-2$$

$$+\frac{21}{32}m e e_{,}\gamma^2\cos(2\tau-\xi+\xi_{,}-2\eta)$$

$$-\frac{77}{64}m e e_{,}\gamma^2\cos(2\tau+\xi-\xi_{,}-2\eta)$$

$\left[\frac{d\lambda}{n\,dt}\right]$ s^2 contains the terms

$$2\gamma^2 e\cos\xi\left\{\frac{1}{2}+\frac{91}{384}m^2+\left\{-\frac{3}{8}m-\frac{3}{32}m^2\right\}\cos 2\tau\right.$$

$$+\frac{3}{4}m e\cos(2\tau-\xi)-\frac{3}{4}m e\cos(2\tau+\xi)$$

$$-\frac{7}{8}m e_{,}\cos(2\tau-\xi_{,})+\frac{3}{8}m e_{,}\cos(2\tau+\xi_{,})$$

$$+\left\{-\frac{1}{2}-\frac{m^2}{6}\right\}\cos 2\eta+\left\{\frac{3}{8}m+\frac{25}{32}m^2\right\}\cos(2\tau-2\eta)$$

$$-\frac{11}{16}m^2\cos(2\tau+2\eta)+e\cos(\xi-2\eta)-e\cos(\xi+2\eta)$$

$$+\frac{15}{8}m e\cos(2\tau-\xi-2\eta)-\frac{15}{8}m e\cos(2\tau-\xi+2\eta)$$

$$-\frac{3}{8}m e_{,}\cos(\xi_{,}-2\eta)+\frac{3}{8}m e_{,}\cos(\xi_{,}+2\eta)$$

$$+\frac{7}{8}m e_{,}\cos(2\tau-\xi_{,}-2n)-\frac{3}{8}m e_{,}\cos(2\tau+\xi_{,}-2n)\Big\}$$

$$+\gamma^2\left\{\frac{11}{4}m^2\cos 2\tau+\frac{15}{4}m e\cos(2\tau-\xi)\right\}\Big\{\frac{1}{2}-\frac{3}{8}m\cos 2\tau$$

$$-\frac{1}{2}\cos 2n+\frac{3}{8}m\cos(2\tau-2n)+e\cos(\xi-2n)-e\cos(\xi+2n)\Big\}$$

$$+\gamma^2\Big\{\left\{\frac{85}{12}m^3+\frac{75}{8}m e^2\right\}\cos 2\tau+m^2 e\cos\xi+\frac{143}{16}m^2 e\cos(2\tau-\xi)$$

$$+\frac{51}{8}m^2 e\cos(2\tau+\xi)-3m^2 e_{,}\cos\xi_{,}+\frac{77}{8}m^2 e_{,}\cos(2\tau-\xi_{,})$$

$$-\frac{11}{8}m^2 e_{,}\cos(2\tau+\xi_{,})+\frac{5}{2}e^2\cos 2\xi-\frac{21}{4}m e e_{,}\cos(\xi+\xi_{,})$$

$$+\frac{35}{4}m e e_{,}\cos(2\tau-\xi-\xi_{,})+\frac{21}{4}m e e_{,}\cos(\xi-\xi_{,})$$

$$-\frac{15}{4}m e e_{,}\cos(2\tau-\xi+\xi_{,})\Big\}\left\{\frac{1}{2}-\frac{1}{2}\cos 2n\right\}$$

$$\frac{d\lambda'}{n\,dt}=\Big\{-\frac{3}{8}m+\left\{-\frac{3}{32}+\frac{11}{8}-\frac{3}{8}-1-1\right\}m^2$$

$$+\left\{\frac{3}{4}+\frac{3}{4}-\frac{3}{4}-\frac{15}{16}-\frac{15}{4}+\frac{75}{16}\right\}m e^2+\frac{15}{16}m e_{,}^2\Big\}\gamma^2\cos 2\tau$$

$$+\left\{-\frac{15}{8}+\frac{3}{4}-\frac{3}{8}+\frac{15}{8}-\frac{15}{8}\right\}m\gamma^2 e\cos(2\tau-\xi)$$

$$+\left\{-\frac{3}{4}-\frac{3}{8}\right\}m\gamma^2 e\cos(2\tau+\xi)$$

$$+\Big\{-\frac{7}{8}m+\left\{-\frac{7}{2}-\frac{55}{32}-\frac{21}{16}-\frac{7}{2}+\frac{77}{16}\right\}m^2\Big\}\gamma^2 e_{,}\cos(2\tau-\xi_{,})$$

$$+\Big\{\frac{3}{8}m+\left\{\frac{1}{2}+\frac{51}{32}+\frac{3}{16}+\frac{1}{2}-\frac{11}{16}\right\}m^2\Big\}\gamma^2 e_{,}\cos(2\tau+\xi_{,})$$

$$+\left\{\frac{21}{8}+\frac{63}{8}-\frac{21}{8}\right\}m e e_{,}\gamma^2\cos(\xi+\xi_{,})$$

$$+\left\{-\frac{35}{8}+\frac{7}{4}-\frac{35}{8}-\frac{7}{8}+\frac{35}{8}\right\}m e e_{,}\gamma^2\cos(2\tau-\xi-\xi_{,})$$

$$+\left\{\frac{3}{4}+\frac{3}{8}\right\}m \text{[illegible]} (2\tau+\xi+\xi_{,})$$

$$+\Big\{-\frac{21}{8} \text{[illegible]} (\xi-\xi_{,})$$

$$+\left\{\frac{15}{8}-\frac{3}{4}+\frac{15}{8}+\frac{3}{8}-\frac{15}{8}\right\} m\,e\,e_{,}\,\gamma^2 \cos\left(2\,\tau-\xi+\xi_{,}\right)$$

$$+\left\{-\frac{7}{4}-\frac{7}{8}\right\} m\,e\,e_{,}\,\gamma^2 \cos\left(2\,\tau+\xi-\xi_{,}\right)-\frac{51}{32} m\,e_{,}^2\,\gamma^2 \cos\left(2\,\tau-2\,\xi_{,}\right)$$

$$+\frac{9}{32} m\,e_{,}^2\,\gamma^2 \cos\left(2\,\tau+2\,\xi_{,}\right)+\left\{-\frac{1}{2}+\left\{-\frac{1}{6}+1\right\} m^2\right\} \gamma^2 \cos 2\,\eta$$

$$+\left\{-\frac{11}{16}-\frac{11}{16}\right\} m^2\,\gamma^2 \cos\left(2\,\tau+2\,\eta\right)$$

$$+\left\{1-\frac{1}{2}-\frac{5}{4}+\frac{135}{32} m\right\} e\,\gamma^2 \cos\left(\xi-2\,\eta\right)$$

$$+\left\{-1-\frac{1}{2}\right\} e\,\gamma^2 \cos\left(\xi+2\,\eta\right)$$

$$+\left\{\frac{15}{8}-\frac{21}{16}+\frac{3}{8}-\frac{15}{16}-\frac{3}{8}\right\} m\,e\,\gamma^2 \cos\left(2\,\tau-\xi-2\,\eta\right)$$

$$+\left\{-\frac{15}{8}-\frac{15}{16}\right\} m\,e\,\gamma^2 \cos\left(2\,\tau-\xi+2\,\eta\right)$$

$$+\left\{-\frac{3}{8}-\frac{33}{32}+\frac{3}{8}\right\} m\,e\,\gamma^2 \cos\left(2\,\tau+\xi-2\,\eta\right)$$

$$+\left\{-\frac{3}{8} m+\left\{-\frac{9}{64}+\frac{3}{2}+\frac{3}{4}\right\} m^2\right\} e_{,}\,\gamma^2 \cos\left(\xi_{,}-2\,\eta\right)$$

$$+\left\{\frac{3}{8} m+\left\{\frac{69}{64}+\frac{3}{2}+\frac{3}{4}\right\} m^2\right\} e_{,}\,\gamma^2 \cos\left(\xi_{,}+2\,\eta\right)$$

$$\left\{-\frac{77}{32}-\frac{77}{32}\right\} m^2\,\gamma^2\,e_{,} \cos\left(2\,\tau-\xi_{,}+2\,\eta\right)$$

$$+\left\{\frac{11}{32}+\frac{11}{32}\right\} m^2\,e_{,}\,\gamma^2 \cos\left(2\,\tau+\xi_{,}+2\,\eta\right)$$

$$+\left\{-\frac{15}{8}-\frac{3}{8}+\frac{75}{32}+\frac{21}{16}\right\} m\,e\,e_{,}\,\gamma^2 \cos\left(\xi+\xi_{,}-2\,\eta\right)$$

$$+\left\{\frac{27}{8}+\frac{3}{8}+\frac{21}{16}\right\} m\,e\,e_{,}\,\gamma^2 \cos\left(\xi+\xi_{,}+2\,\eta\right)$$

$$+\left\{\frac{35}{8}-\frac{49}{16}+\frac{7}{8}-\frac{7}{8}-\frac{35}{16}\right\} m\,e\,e_{,}\,\gamma^2 \cos\left(2\,\tau-\xi-\xi_{,}-2\,\eta\right)$$

$$+\left\{-\frac{35}{8}-\frac{35}{16}\right\} m\,e\,e_{,}\,\gamma^2 \cos\left(2\,\tau-\xi-\xi_{,}+2\,\eta\right)$$

$$+\left\{-\frac{3}{8}+\frac{33}{32}\right\} m\,e\,e_{,}\,\gamma^2 \cos\left(2\,\tau+\xi+\xi_{,}-2\,\eta\right)$$

$$+\left\{\frac{15}{8}-\frac{75}{32}+\frac{3}{8}-\frac{21}{16}\right\} m\,e\,e_{,}\,\gamma^2 \cos\left(\xi-\xi_{,}-2\,\eta\right)$$

$$+ \left\{-\frac{27}{8} \quad \frac{3}{8}-\frac{21}{16}\right\} m\, e\, e_{,}\, \gamma^2 \cos (\xi-\xi_{,}+2\, n)$$

$$+ \left\{-\frac{15}{8}+\frac{21}{16}-\frac{3}{8}+\frac{3}{8}+\frac{15}{16}\right\} m\, e\, e_{,}\, \gamma^2 \cos (2\, \tau-\xi+\xi_{,}-2\, n)$$

$$+ \left\{\frac{15}{8}+\frac{15}{16}\right\} m\, e\, e_{,}\, \gamma^2 \cos (2\, \tau-\xi+\xi_{,}+2\, n)$$

$$+ \left\{\frac{7}{8}-\frac{77}{32}\right\} m\, e\, e_{,}\, \gamma^2 \cos (2\, \tau+\xi-\xi_{,}-2\, n)$$

$$-\frac{9}{32} m\, e_{,}^2\, \gamma^2 \cos (2\, \xi_{,} - \; 2\, n)+\frac{9}{32} m\, e_{,}^2\, \gamma^2 \cos (2\, \xi_{,}+2\, n)$$

$$+ \left\{-\frac{33}{64}-\frac{33}{64}\right\} m^3\, \gamma^2 \cos 4\, \tau$$

$$-\frac{9}{128} m^2\, \gamma^2 \cos (4\, \tau-2\, n)$$

$$-\frac{21}{64} m^2\, \gamma^2\, e \cos (4\, \tau-\xi_{,}-2\, n)+\frac{9}{64} m^2\, \gamma^2\, e_{,} \cos (4\, \tau+\xi_{,}-2\, n)$$

From the expression for $\frac{d\, \lambda^{\backprime}}{n\, d\, t}$ it is easy to infer, by integration, it the longitude contains the following terms:

$$= \left\{-\frac{3}{16} m-\frac{47}{64} m^2+\frac{3}{8} m\, e^2+\frac{15}{32} m\, e_{,}^2\right\} \gamma^2 \sin 2\, \tau$$

$$-\frac{3}{2} m\, \gamma^2\, e \sin (2\, \tau-\xi)-\frac{3}{8} m\, \gamma^2\, e \sin (2\, \tau+\xi)$$

$$+ \left\{-\frac{7}{16} m-\frac{209}{64} m^2\right\} \gamma^2\, e_{,} \sin (2\, \tau-\xi_{,})$$

$$+ \left\{\frac{3}{16} m+\frac{73}{64} m^2\right\} \gamma^2\, e_{,} \sin (2\, \tau+\xi_{,})$$

$$+\frac{63}{8} m\, e\, e_{,}\, \gamma^2 \sin (\xi+\xi_{,})-\frac{7}{2} m\, e\, e_{,}\, \gamma^2 \sin (2\, \tau-\xi-\xi_{,})$$

$$+\frac{3}{8} m\, e\, e_{,}\, \gamma^2 \sin (2\, \tau+\xi+\xi_{,})-\frac{63}{8} m\, e\, e_{,}\, \gamma^2 \sin (\xi-\xi_{,})$$

$$+\frac{3}{2} m\, e\, e_{,}\, \gamma^2 \sin (2\, \tau-\xi+\xi_{,})-\frac{7}{8} m\, e\, e_{,}\, \gamma^2 \sin (2\, \tau+\xi-\xi_{,})$$

$$-\frac{51}{64} m\, e_{,}^2\, \gamma^2 \sin (2\, \tau-2\, \xi_{,})+\frac{9}{64} m\, e_{,}^2 \sin (2\, \tau+2\, \xi_{,})$$

$$- \left\{ -\frac{1}{4} + \frac{29}{48} m^2 \right\} \gamma^2 \sin 2\eta - \frac{11}{64} m^2 \gamma^2 \sin (2\tau + 2\eta)$$

$$+ \left\{ \frac{3}{4} - \frac{135}{32} m \right\} e \gamma^2 \sin (\xi - 2\eta) - \frac{1}{2} e \gamma^2 \sin (\xi + 2\eta)$$

$$+ \frac{3}{8} m e \gamma^2 \sin (2\tau - \xi - 2\eta) - \frac{15}{16} m e \gamma^2 \sin (2\tau - \xi + 2\eta)$$

$$- \frac{33}{32} m e \gamma^2 \sin (2\tau + \xi - 2\eta) + \left\{ \frac{3}{16} m - \frac{123}{128} m^2 \right\} e_{,} \gamma^2 \sin (\xi_{,} - 2\eta)$$

$$+ \left\{ \frac{3}{16} m + \frac{201}{128} m^2 \right\} e_{,} \gamma^2 \sin (\xi_{,} + 2\eta)$$

$$- \frac{77}{64} m^2 \gamma^2 e_{,} \sin (2\tau - \xi_{,} + 2\eta)$$

$$+ \frac{11}{64} m^2 \gamma^2 e_{,} \sin (2\tau + \xi_{,} + 2\eta) - \frac{45}{32} m e e_{,} \gamma^2 \sin (\xi + \xi_{,} - 2\eta)$$

$$+ \frac{27}{16} m e e_{,} \gamma^2 \sin (\xi + \xi_{,} + 2\eta) + \frac{7}{8} m e e_{,} \gamma^2 \sin (2\tau - \xi - \xi_{,} - 2\eta)$$

$$- \frac{35}{16} m e e_{,} \gamma^2 \sin (2\tau - \xi - \xi_{,} + 2\eta) + \frac{5}{8} m e e_{,} \gamma^2 \sin (2\tau + \xi + \xi_{,} - 2\eta)$$

$$+ \frac{45}{32} m e e_{,} \gamma^2 \sin (\xi - \xi_{,} - 2\eta) - \frac{27}{16} m e e_{,} \gamma^2 \sin (\xi - \xi_{,} + 2\eta)$$

$$- \frac{3}{8} m e e_{,} \gamma^2 \sin (2\tau - \xi + \xi_{,} - 2\eta)$$

$$+ \frac{15}{16} m e e_{,} \gamma^2 \sin (2\tau - \xi + \xi_{,} + 2\eta)$$

$$- \frac{49}{32} m e e_{,} \gamma^2 \sin (2\tau + \xi - \xi_{,} - 2\eta)$$

$$+ \frac{9}{64} m e_{,}^2 \gamma^2 \sin (2\xi_{,} - 2\eta) + \frac{9}{64} m e_{,}^2 \gamma^2 \sin (2\xi_{,} + 2\eta)$$

I shall now endeavour to obtain certain terms from the equation

$$\frac{d\lambda'}{dt} = \frac{h(1+s^2)}{2r^2} + \frac{\mu}{hr} - \frac{\mu}{2ha} - \frac{d r^2}{2h\,dt^2} - \frac{r^2}{2h(1+s^2)^2} \frac{d s^2}{dt^2}$$

$$- \frac{1}{h} \int dR - \frac{(1+s^2)}{2hr^2} \left\{ \int \frac{dR}{d\lambda'} dt \right\}^2$$

In order to procure the two first terms in λ_{63} it is sufficient to take

$$\frac{\mathrm{n}\, r^2}{h} = 1 - \left\{2\, m^2 + \frac{19}{3} m^3\right\} \cos 2\,\tau$$

$$\frac{h}{\mathrm{n}\, r^2} = 1 + \left\{2\, m^2 + \frac{19}{3} m^3\right\} \cos 2\,\tau$$

From the expression for R, p. 33, it may be inferred that the portion of $\mathrm{d}\,R$ which arises from the differentiation with respect to $\mathrm{n}\,t = -\frac{9}{16} m^4\,\gamma^2 \cos(2\,\tau - 2\,\eta)$. In order to obtain the portion of $\mathrm{d}\,R$ which arises from the differentiation with respect to $\mathrm{n}_{,}t$, in as much as this quantity is contained in $\delta\,r$, $\delta\,\lambda$, and $\delta\,s$, it will be found sufficient to take $R = \frac{3}{4}\frac{m_{,}\,r^2}{r_{,}^3}\,s^2$, $s^2 = \frac{3}{8}\,m \cos(2\,\tau - 2\,\eta)$. Hence $\mathrm{d}\,R = 0$, considering Arg. 63 and the term multiplied by m^4 inclusive.

$$\begin{aligned}\frac{\mathrm{d}\,\lambda'}{\mathrm{n}\,\mathrm{d}\,t} &= \frac{1}{2}\, s_{146}\; s_{147} - \frac{1}{2}\, s_{146}\; s_{147}\, g\,(2 - 2\,m - g)\\ &\quad + \left\{m^2 + \frac{19}{6}\, m^3\right\}\left\{-\frac{1}{4} + \frac{1}{4}\right\} - \frac{3}{2}\, m^2 - \frac{9}{8}\, m^3\\ &= -\frac{9}{8}\, m^2 + \frac{61}{32}\, m^3 \qquad\qquad \lambda_{63} = \frac{9}{16}\, m - \frac{11}{8}\, m^2\end{aligned}$$

In the same way it may easily be shown from the same equation, considering only the first term, that

$$\lambda_{73} = -\left\{\frac{3\,m}{2}\, s_{157} + r_{73}\right\}\frac{1}{3\,m} = \frac{21}{16}\, m$$

$$\lambda_{75} = -\left\{\frac{m}{2}\, s_{159} + r_{75}\right\}\frac{1}{m} = -\frac{9}{16}\, m$$

I shall now endeavour to obtain λ_{77}. This argument results from the following combinations:

$\{62,8\}$ $\{63,9\}$ $\{64,10\}$ $\{65,2\}$ $\{68,4\}$ $\{69,3\}$ $\{77,0\}$

Since the eccentricity $e_{,}$ may be neglected, it is sufficient to take

$$\begin{aligned}\int \mathrm{d}\,R &= R - \int \frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda_{,}}\,\mathrm{d}\,\lambda_{,}\\ &= R + \int \frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\, m\,\mathrm{n}\,\mathrm{d}\,t = R + m\,\mathrm{n}\,h - m\,\mathrm{n}\,r'^2\frac{\mathrm{d}\,\lambda'}{\mathrm{d}\,t}\end{aligned}$$

In order to obtain R_{77} it is sufficient to take

$$R = -\frac{m_{,} r^2}{4\, a_{,}^3}\{1 - 3\, s^2\}$$

$$\frac{r^2}{\mathrm{a}^2} = 1 - 2\, e \cos \xi - \frac{1}{2} e^2 \cos 2\, \xi + \frac{5}{4} e\, \gamma^2 \cos (\xi - 2\, \eta)$$

$$s^2 = -\frac{\gamma^2}{2} + \gamma^2\, e \cos (\xi - 2\, \eta) + \frac{1}{4} \gamma^2\, e^2 \cos (2\, \xi - 2\, \eta)$$

$$R_{77} = \frac{15}{32} + \frac{3}{16} - \frac{3}{4} + \frac{3}{32} = 0$$

In order to find the term in $r'^2 \frac{\mathrm{d}\, \lambda}{\mathrm{d}\, t}$ multiplied by m, it is sufficient to take

$$s^2 = -\frac{\gamma^2}{2} \cos 2\eta + \gamma^2\, e \cos(\xi - 2\eta) + \left\{\frac{1}{4} - \frac{135}{64} m\right\} \gamma^2\, e^2 \cos(2\xi - 2\eta)$$

$$\frac{r^2}{\mathrm{a}^2} = 1 - 2\, e \cos \xi - \frac{e^2}{2} \cos 2\, \xi + \left\{\frac{5}{4} - \frac{135}{32} m\right\} e\, \gamma^2 \cos (\xi - 2\, \eta)$$
$$+ \left\{-\frac{15}{8} + \frac{405}{64} m\right\} e^2\, \gamma^2 \cos (2\, \xi - 2\, \eta)$$

$$\frac{r'^2}{\mathrm{a}^2} = \frac{r^2 (1 - s^2)}{\mathrm{a}^2}$$
$$= 1 - 2\, e \cos \xi - \frac{e^2}{2} \cos 2\, \xi + \frac{\gamma^2}{2} \cos 2\, \eta$$
$$+ \left\{-\frac{1}{4} - \frac{135}{32} m\right\} e\, \gamma^2 \cos (\xi - 2\, \eta)$$
$$+ \left\{-\frac{5}{4} + \frac{135}{16} m\right\} \gamma\, e^2 \cos (2\, \xi - 2\, \eta)$$

$$\frac{\mathrm{d}\, \lambda'}{\mathrm{n}\, \mathrm{d} t} = 1 + 2\, e \cos \xi + \frac{5}{2} e^2 \cos 2\, \xi$$
$$+ \left\{-\frac{3}{4} + \frac{135}{32} m\right\} e\, \gamma^2 \cos (\xi - 2\, \eta) - \frac{\gamma^2}{2} \cos 2\, \eta$$

$$\frac{r'^2}{\mathrm{a}^2 \mathrm{n}} \frac{\mathrm{d}\, \lambda'}{\mathrm{d}\, t} = \left\{\frac{1}{8} - \frac{1}{4} + \frac{5}{8} + \frac{3}{4} - \frac{5}{4}\right.$$
$$\left. + \left\{\frac{135}{16} \cdot \frac{135}{32} - \frac{135}{32}\right\} m \right\} e^2\, \gamma^2 \cos (2\, \xi - 2\, \eta)$$
$$= 0$$

Hence I conclude that $\frac{r\,\mathrm{d}R}{\mathrm{d}r}$, $\int \mathrm{d}R$ and $\frac{\mathrm{a}}{r}$ contain no term (Arg. 77) multiplied by m^2.

I propose only to seek at present the term in $\frac{\mathrm{d}\lambda'}{\mathrm{n}\,\mathrm{d}t}$ multiplied by m^2; in this approximation $\int \mathrm{d}R$ and r_{77} may be taken equal to zero; and if the quantities between brackets denote as before the values when γ^2 is neglected, it is sufficient to take, by the method of p. 219,

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t} = \left[\frac{h}{2r^2}\right] s^2 - \left[\frac{r^2}{2h}\right]\frac{\mathrm{d}s^2}{\mathrm{d}t^2}$$

$$+ \left[\frac{h}{r}\right] \delta\,\frac{1}{r} - \left[\frac{\mathrm{d}r}{h\,\mathrm{d}t}\right]\frac{\mathrm{d}\,.\,\delta r}{\mathrm{d}t} - \frac{1}{2h\mathrm{a}^2}\left\{\int \frac{\mathrm{d}R}{\mathrm{d}\lambda'}\,\mathrm{d}t\right\}^2$$

$$\left[\frac{h}{\mathrm{n}r^2}\right] = 1 + \underset{[2]}{\{2 + m^2\}\,e\cos\xi}$$

$$+ \underset{[8]}{\left\{\frac{5}{2} + \frac{7}{2}m^2\right\} e^2\cos 2\xi} + \underset{[9]}{\frac{15}{8}m\,e^2\cos(2\tau - 2\xi)}$$

$$s^2 = \underset{[62]}{\left\{-\frac{1}{2} - \frac{m^2}{6}\right\}\gamma^2\cos 2\eta} + \underset{[63]}{\frac{3}{8}m\gamma^2\cos(2\tau - 2\eta)}$$

$$+ \underset{[65]}{\left\{1 - \frac{2}{3}m^2\right\}\gamma^2 e\cos(\xi - 2\eta)}$$

$$+ \underset{[77]}{\left\{s_{161} - \frac{1}{2} + \frac{3497}{1536}m^2\right\}\gamma^2 e^2\cos(2\xi - 2\eta)}$$

$$\left[\frac{\mathrm{n}r^2}{h}\right] = 1 - \underset{[2]}{2e\cos\xi} - \underset{[3]}{\frac{15}{4}m\,e\cos(2\tau - \xi)}$$

$$+ \underset{[8]}{\left\{-\frac{1}{2} - \frac{3}{2}m^2\right\} e^2\cos 2\xi} + \underset{[9]}{\frac{45}{8}m\,e^2\cos(2\tau - 2\xi)}$$

$$\frac{\mathrm{d}s^2}{\mathrm{n}^2\mathrm{d}t^2} = \underset{[62]}{\left\{\frac{1}{2} + \frac{11}{12}m^2\right\}\gamma^2\cos 2\eta} + \underset{[63]}{\frac{3}{8}m\gamma^2\cos(2\tau - 2\eta)}$$

$$-\frac{3}{32}m^2 e\gamma^2\cos(\xi-2\eta)+\frac{3}{4}\,m e\gamma^2\cos(2\tau+\xi-2\eta)$$

[65] [69]

$$+\left\{s_{161}+\frac{883}{512}m^2\right\}\gamma^2 e^2\cos(2\xi-2\eta)$$

[77]

$$\sqrt{\frac{\mathrm{a}}{\mu}}\left[\frac{h}{r}\right]=\left(1+\frac{m^2}{3}\right)e\cos\xi+e^2\cos 2\xi+\frac{15}{8}m e\cos(2\tau-\xi)$$

$$\mathrm{a}\,\delta\frac{1}{r}=\frac{m^2}{2}\gamma^2\cos 2\eta+r_{65}\,e\gamma^2\cos(\xi-2\eta)-\frac{33}{64}m e\gamma^2\cos(2\tau+\xi-2\eta)$$

$$\sqrt{\frac{\mu}{\mathrm{a}}}\left[\frac{\mathrm{d}\,r}{h\,\mathrm{n}\,\mathrm{d}\,t}\right]=\left(1-\frac{1}{12}m^2\right)e\sin\xi+e^2\sin 2\xi$$

$$+\frac{15}{8}m e\sin(2\tau-\xi)$$

$$\frac{\mathrm{d}\,.\,\delta r}{\mathrm{a}\,\mathrm{n}\,\mathrm{d}\,t}=m^2\gamma^2\sin 2\eta+\left\{-r_{65}+\frac{163}{96}m^2\right\}e\gamma^2\sin(\xi-2\eta)$$

$$-\frac{33}{64}m e\gamma^2\sin(2\tau+\xi-2\eta)$$

$$-\frac{\mathrm{a}}{\mathrm{n}\,\mu}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\mathrm{d}\,t=-\frac{15}{8}m e^2\cos(2\tau-2\xi)-\frac{3}{8}m\gamma^2\cos(2\tau-2\eta)$$

$$\frac{\mathrm{d}\,\lambda'}{\mathrm{n}\,\mathrm{d}\,t}=\left\{\frac{1}{4}\left\{\frac{5}{2}+\frac{7}{2}m^2\right\}\left\{-\frac{1}{2}-\frac{m^2}{6}\right\}-\frac{1}{4}\left\{-\frac{1}{2}-\frac{3}{2}m^2\right\}\left\{\frac{1}{2}+\frac{11}{12}m^2\right\}\right.$$

$$+\frac{45}{256}m^2-\frac{135}{256}m^2+\frac{1}{4}\{2+m^2\}\left\{1-\frac{2}{3}m^2\right\}-\frac{3}{64}m^2+\frac{45}{64}m^2$$

$$+\frac{1}{2}\left\{s_{161}-\frac{1}{2}+\frac{3497}{1536}m^2\right\}-\frac{1}{2}\left\{s_{161}+\frac{883}{512}m^2\right\}+\frac{m^2}{4}-\frac{m^2}{2}$$

$$+\frac{1}{2}\left\{r_{65}-\frac{5}{24}m^2\right\}+\frac{1}{2}\left\{-r_{65}+\frac{163}{96}m^2\right\}\left\{1-\frac{1}{12}m^2\right\}$$

$$\left.-\frac{495}{1024}m^2+\frac{495}{1024}m^2-\frac{45}{128}m^2\right\}e^2\gamma^2\cos(2\xi-2\eta)$$

$$=\left\{-\frac{5}{16}+\frac{1}{16}+\frac{1}{2}-\frac{1}{4}\right.$$

$$+\left\{-\frac{5}{48}-\frac{7}{16}+\frac{3}{16}+\frac{11}{96}+\frac{45}{256}-\frac{135}{256}-\frac{1}{3}+\frac{1}{4}-\frac{3}{64}+\frac{45}{64}\right.$$

$$\left.\left.+\frac{3497}{3072}-\frac{883}{1024}+\frac{1}{4}-\frac{1}{2}-\frac{5}{48}+\frac{79}{96}-\frac{45}{128}\right\}m^2\right\}e^2\gamma^2\cos(2\xi-2\eta)$$

$$= \frac{3}{8} m^2 e^2 \gamma^2 \cos (2\xi - 2\eta)$$

$$\lambda_{77} = -\frac{1}{8}$$

The quantities s_{161} and r_{65} have been left indeterminate, in order to show that they are eliminated by the nature of the reductions in this method. This inequality is the subject of a Memoir by Laplace in the *Connoissance de Temps* for 1824, which is not free from inaccuracy. M. Poisson has shown that the coefficient may be obtained with great facility by the method of the variation of the elliptic constants.

M. Plana has $+ \frac{153}{128} m^2 e_{\prime} \gamma^2 \sin (\xi_{\prime} + 2\eta)$

[72]

$$+ \frac{33}{32} m e e_{\prime} \gamma^2 \sin (2\tau + \xi + \xi_{\prime} - 2\eta) - \frac{77}{32} m e e_{\prime} \gamma^2 \sin (2\tau + \xi - \xi_{\prime} - 2\eta)$$

[87] [93]

instead of the corresponding terms which I have given in p. 268; the other discrepancies are removed by substituting $e\left(1 - \frac{m^2}{2}\right)$ for e and $\gamma\left(1 - \frac{m^2}{6}\right)$ for γ in my expression, agreeably to the remark in p. 248.

Calculation of terms in the reciprocal of the radius vector, and in the longitude multiplied by $\frac{\mathrm{a}}{a_{\prime}}$.

It is easy to show, as a first approximation, that

$$\mathrm{a}\delta\frac{1}{r} = -\frac{15}{16} m \frac{\mathrm{a}}{a_{\prime}} \cos\tau + \frac{25}{64} m^2 \frac{\mathrm{a}}{a_{\prime}} \cos 3\tau \qquad \delta\lambda = -\frac{15}{8} m \frac{\mathrm{a}}{a_{\prime}} \sin\tau$$

In order to find the next terms in $\mathrm{a}\delta\frac{1}{r}$, Arg. 101 and 116, it is sufficient to take

$$\frac{\mathrm{a}\delta R}{\mu} = -m^2 \left\{\frac{1}{2} + \frac{3}{2} m^2 \cos 2\tau\right\} \frac{15}{16} m \frac{\mathrm{a}}{a_{\prime}} \cos\tau$$

$$-\frac{3}{2} m^2 \sin 2\tau \left\{\frac{15}{8} m \frac{\mathrm{a}}{a_{\prime}} \sin\tau\right\}$$

$$= -\frac{165}{64} m^3 \frac{\mathrm{a}}{a_{,}} \cos \tau + \frac{45}{64} m^3 \frac{\mathrm{a}}{a_{,}} \cos 3\tau$$

$$\mathrm{a}^2 \left(\delta \frac{1}{r}\right)^2 = -\left\{\frac{m^2}{3} + 2 m^2 \cos 2\tau\right\} \frac{15}{16} m \frac{\mathrm{a}}{a_{,}} \cos \tau$$

$$= -\frac{5}{4} m^3 \frac{\mathrm{a}}{a_{,}} \cos \tau - \frac{15}{16} m^3 \frac{\mathrm{a}}{a_{,}} \cos 3\tau$$

$$\mathfrak{r}_{101} = -\frac{15}{8} m^3 \frac{\mathrm{a}}{a_{,}} \qquad \mathfrak{r}_{116} = -\frac{45}{32} m^3 \frac{\mathrm{a}}{a_{,}}$$

Let $r_{101} = -\frac{15}{16} m \frac{\mathrm{a}}{a_{,}} + A m^2 \frac{\mathrm{a}}{a_{,}}$, then

$$\left\{-\frac{15}{16} m + A m^2\right\}\{-2m + m^2\} = -\frac{15}{8} m^3$$

$$+ \{5 + 2m\} \frac{3}{8} m^2 + \frac{165}{16} m^3$$

$$-\frac{15}{16} - 2A = -\frac{15}{8} + \frac{165}{16} + \frac{3}{4}$$

$$A = -\frac{81}{16}$$

Let $r_{116} = \frac{25}{64} m^2 \frac{\mathrm{a}}{a_{,}} + A m^3 \frac{\mathrm{a}}{a_{,}}$, then

$$\left\{\frac{25}{64} m^2 + A m^3\right\}\{8 - 18m\} = -\frac{9 . 45}{32} m^3$$

$$+ \{5 + 2m\} \frac{5}{8} m^3 - \frac{45}{16} m^3$$

$$-\frac{225}{32} + 8A = -\frac{405}{32} + \frac{5}{4} - \frac{45}{16}$$

$$A = -\frac{115}{128}$$

It may easily be inferred, from the development of R in p. 34, that $-\frac{\mathrm{a}}{\mathrm{n}\mu}\int \frac{\mathrm{d} R}{\mathrm{d}\lambda} \mathrm{d} t$ contains the terms

$$\frac{3}{8} m^2 \frac{\mathrm{a}}{a_{,}} \cos \tau + \frac{5}{8} m^2 \frac{\mathrm{a}}{a_{,}} \cos 3\tau$$

Hence

$$\frac{\mathrm{d}\lambda}{\mathrm{n\,d}t}=\left\{-\frac{15}{8}m+\left(-\frac{81}{8}+\frac{3}{8}\right)m^2\right\}\frac{\mathrm{a}}{a_{,}}\cos\tau$$

$$+\left(\frac{25}{32}+\frac{5}{8}\right)m^2\frac{\mathrm{a}}{a_{,}}\cos 3\tau$$

$$=\left\{-\frac{15}{8}m-\frac{39}{4}m^2\right\}\frac{\mathrm{a}}{a_{,}}\cos\tau+\frac{45}{32}m^2\frac{\mathrm{a}}{a_{,}}\cos 3\tau$$

$$\lambda=\left\{-\frac{15}{8}m-\frac{93}{8}m^2\right\}\frac{\mathrm{a}}{a_{,}}\sin\tau+\frac{15}{32}m^2\frac{\mathrm{a}}{a_{,}}\sin 3\tau$$

In order to find the next term in $\mathrm{a}\,\delta\frac{1}{r}$, Arg. 101, I take

$$\frac{\mathrm{a}\,\delta R}{\mu}=m^2\left\{\frac{1}{2}+\frac{3}{2}\cos 2\tau\right\}\left\{-\frac{81}{16}m^2\frac{\mathrm{a}}{a_{,}}\cos\tau+\frac{25}{64}m^2\frac{\mathrm{a}}{a_{,}}\cos 3\tau\right\}$$

$$+\frac{3}{2}m^2\sin 2\tau\left\{-\frac{93}{8}m^2\frac{\mathrm{a}}{a_{,}}\sin\tau+\frac{15}{32}m^2\frac{\mathrm{a}}{a_{,}}\sin 3\tau\right\}$$

$$+m^2\left\{\frac{9}{8}\frac{\mathrm{a}}{a_{,}}\cos\tau+\frac{15}{8}\frac{\mathrm{a}}{a_{,}}\cos 3\tau\right\}\left\{\frac{m^2}{6}+m^2\cos 2\tau\right\}$$

$$+m^2\left\{\frac{3}{8}\frac{\mathrm{a}}{a_{,}}\sin\tau+\frac{15}{8}\frac{\mathrm{a}}{a_{,}}\sin 3\tau\right\}\frac{11}{8}m^2\sin 2\tau$$

$$=\left\{-\frac{81}{32}+\frac{75}{256}-\frac{243}{64}-\frac{279}{32}+\frac{45}{128}\right.$$

$$\left.+\frac{3}{16}+\frac{9}{16}+\frac{15}{16}+\frac{33}{128}+\frac{165}{128}\right\}m^4\frac{\mathrm{a}}{a_{,}}\cos\tau$$

$$=-\frac{2859}{256}m^4\frac{\mathrm{a}}{a_{,}}\cos\tau$$

$$\frac{r\,\mathrm{d}R}{\mathrm{d}r}=-\frac{2445}{128}m^4\frac{\mathrm{a}}{a_{,}}\cos\tau$$

$$\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2=\left\{\frac{m^2}{3}+\left\{2m^2+\frac{19}{3}m^3\right\}\cos 2\tau\right\}$$

$$\left\{\left\{-\frac{15}{16}m-\frac{81}{16}m^2\right\}\frac{\mathrm{a}}{a_{,}}\cos\tau+\frac{25}{16}m^2\frac{\mathrm{a}}{a_{,}}\cos 3\tau\right\}$$

$$= \left\{ -\frac{27}{16} - \frac{81}{16} - \frac{95}{32} + \frac{25}{16} \right\} m^4 \frac{\mathrm{a}}{a_{\prime}} \cos \tau$$

$$= -\frac{261}{32} m^4 \frac{\mathrm{a}}{a_{\prime}} \cos \tau$$

It is sufficient to take

$$\int \mathrm{d} R = R + \int \frac{\mathrm{d} R}{\mathrm{d} \lambda} m \, \mathrm{n} \, \mathrm{d} t$$

$$\frac{\mu}{\mathrm{a}} \delta \frac{\mathrm{d} R}{\mathrm{d} \lambda} = -3 m^2 \sin 2 \tau \left(-\frac{15}{16} m \frac{\mathrm{a}}{a_{\prime}} \cos \tau \right)$$

$$+ 3 m^3 \cos 2 \tau \left(-\frac{15}{8} m \frac{\mathrm{a}}{a_{\prime}} \sin \tau \right)$$

$$= \left\{ \frac{45}{32} + \frac{45}{16} \right\} m^3 \frac{\mathrm{a}}{a_{\prime}} \sin \tau + \left\{ \frac{45}{32} - \frac{45}{16} \right\} m^3 \frac{\mathrm{a}}{a_{\prime}} \sin 3 \tau$$

$$= \frac{135}{32} m^3 \frac{\mathrm{a}}{a_{\prime}} \sin \tau - \frac{45}{32} m^3 \frac{\mathrm{a}}{a_{\prime}} \sin 3 \tau$$

$$-\frac{\mathrm{a}}{\mathrm{n} \mu} \int \frac{\mathrm{d} R}{\mathrm{d} \lambda} \mathrm{d} t = \left\{ \frac{3}{8} m^2 + \frac{135}{32} m^3 \right\} (1 + m) \frac{\mathrm{a}}{a_{\prime}} \cos \tau$$

$$+ \left\{ \frac{15}{8} m^2 - \frac{45}{32} m^3 \right\} \frac{1}{3} (1 + m) \frac{\mathrm{a}}{a_{\prime}} \cos 8 \tau$$

$$= \left\{ \frac{3}{8} m^2 + \frac{147}{32} m^3 \right\} \frac{\mathrm{a}}{a_{\prime}} \cos \tau$$

$$+ \left\{ \frac{5}{8} m^2 + \frac{5}{32} m^3 \right\} \frac{\mathrm{a}}{a_{\prime}} \cos 3 \tau$$

$$\frac{\mathrm{a}}{\mu} \int \mathrm{d} R = \left\{ -\frac{3}{8} m^2 - \frac{165}{64} m^3 - \frac{2859}{256} m^4 - \frac{3}{8} m^3 - \frac{147}{32} m^4 \right\} \frac{\mathrm{a}}{a_{\prime}} \cos \tau$$

$$= \left\{ -\frac{3}{8} m^2 - \frac{189}{64} m^3 - \frac{4035}{256} m^4 \right\} \frac{\mathrm{a}}{a_{\prime}} \cos \tau$$

Let $r_{101} = -\frac{15}{16} m \frac{\mathrm{a}}{a_{\prime}} - \frac{81}{16} m^2 \frac{\mathrm{a}}{a_{\prime}} + A m^3 \frac{\mathrm{a}}{a_{\prime}}$, then

$$\left\{ -\frac{15}{16} m - \frac{81}{16} m^2 + A m^3 \right\} \left\{ -2 m + m^2 \right\} = (1 - 2 m) \left\{ -\frac{15}{8} m^3 - \frac{783}{64} m^4 \right\}$$

$$+ \frac{9}{8} m^2 + \frac{165}{32} m^3 + \frac{2445}{128} m^4 + \frac{3}{4} m^2 + \frac{189}{32} m^3 + \frac{4035}{128} m^4$$

$$-\frac{81}{16}-2A=-\frac{783}{64}+\frac{15}{4}+\frac{2445}{128}+\frac{4035}{128}$$

$$A=-\frac{3021}{128}\quad \text{M. Plana has } -\frac{5857}{256}.$$

I shall now endeavour to find the terms multiplied by m^3 in λ_{101} and λ_{116}

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,t}=\left\{\frac{1}{\mathrm{r}^2}+\frac{2}{\mathrm{r}}\,\delta\,\frac{1}{r}+\left(\delta\,\frac{1}{r}\right)^2\right\}\left\{h-\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t\right\}$$

$$=\left\{1+\left\{-\frac{15}{8}m-\frac{81}{8}m^2-\frac{3021}{64}m^3\right\}\frac{\mathrm{a}}{a_{,}}\cos\tau\right.$$

$$\left.+\left\{\frac{25}{32}m^2-\frac{115}{64}m^3\right\}\frac{\mathrm{a}}{a_{,}}\cos 3\tau-\frac{5}{4}m^3\frac{\mathrm{a}}{a_{,}}\cos\tau-\frac{15}{16}m^3\frac{\mathrm{a}}{a_{,}}\cos 3\tau\right\}$$

$$\left\{1-\frac{m^2}{3}+\frac{3}{4}m^2\cos 2\tau+\left\{\frac{3}{8}m^2+\frac{147}{32}m^3\right\}\frac{\mathrm{a}}{a_{,}}\cos\tau\right.$$

$$\left.+\left\{\frac{5}{8}m^2+\frac{5}{32}m^3\right\}\frac{\mathrm{a}}{a_{,}}\cos 3\tau\right\}$$

$$\frac{\mathrm{d}\,\lambda}{\mathrm{n}\,\mathrm{d}\,t}=\left\{\frac{5}{8}-\frac{45}{64}+\frac{147}{32}-\frac{5}{4}-\frac{3021}{64}\right\}m^3\frac{\mathrm{a}}{a_{,}}\cos\tau$$

$$+\left\{-\frac{45}{64}-\frac{15}{16}-\frac{115}{64}+\frac{5}{32}\right\}m^3\frac{\mathrm{a}}{a_{,}}\cos 3\tau$$

$$\lambda=\left\{-\frac{15}{8}m-\frac{39}{4}m^2-\frac{703}{16}m^3\right\}(1+m+m^2)\frac{\mathrm{a}}{a_{,}}\cos\tau$$

$$+\left\{\frac{45}{32}m^2-\frac{105}{32}m^3\right\}\frac{1}{3}(1+m)\frac{\mathrm{a}}{a_{,}}\cos 3\tau$$

$$=\left\{-\frac{15}{8}m-\frac{93}{8}m^2-\frac{889}{16}m^3\right\}\frac{\mathrm{a}}{a_{,}}\sin\tau$$

$$+\left\{\frac{15}{32}m^2-\frac{5}{8}m^3\right\}\frac{\mathrm{a}}{a_{,}}\sin 3\tau$$

M. Plana has $-\frac{6927}{128}m^3\,\frac{\mathrm{a}}{a_{,}}\sin\tau$. This discrepancy is accounted for by the discrepancy in r_{101}.

The term $\frac{5\,\mathrm{a}}{4\,a_{,}}\,e_{,}\cos(\tau+\xi_{,})$ in $\frac{\mathrm{a}}{r}$, (See p. 41.)

$_{09}=\frac{15\,\mathrm{a}}{\cdot}$ and hence it may be easily shown that

$$r_{109} = \frac{5\,\mathrm{a}}{2\,\mathrm{a}_{,}} \qquad\qquad \lambda_{109} = \frac{25}{8}\,\frac{\mathrm{a}}{a_{,}}$$

I shall now consider the terms which arise in

$\frac{\mathrm{a}\,\delta R}{\mu}$ from

$$\mathrm{a}\,\delta r = \frac{5\,\mathrm{a}}{4\,a_{,}}\,e_{,}\cos(\tau + \xi_{,}) + \frac{5\,\mathrm{a}}{2\,a_{,}}\,e\,e_{,}\cos(\tau + \xi + \xi_{,})$$

$$\delta\lambda = \frac{5\,\mathrm{a}}{2\,a_{,}}\,e_{,}\sin(\tau + \xi_{,}) + \frac{25\,\mathrm{a}}{8\,a_{,}}\,e\,e_{,}\sin(\tau + \xi + \xi_{,})$$

$$+ \lambda_{110}\frac{\mathrm{a}}{a_{,}}\,e\,e_{,}\sin(\tau - \xi + \xi_{,})$$

These are the only terms of this order independent of m.

$$\mathrm{r}\,\delta\frac{1}{r} = \frac{5\,\mathrm{a}}{4\,a_{,}}\,e_{,}\cos(\tau + \xi_{,}) + \frac{15\,a}{8\,a_{,}}\,e\,e_{,}\cos(\tau + \xi + \xi_{,})$$

$$- \frac{5\,\mathrm{a}}{8\,a_{,}}\,e\,e_{,}\cos(\tau - \xi + \xi_{,})$$

$$\begin{aligned}
\frac{\mathrm{a}\,\delta R}{\mu} = {} & \Big\{\frac{m^2}{2} + \frac{3}{2}m^2\cos 2\tau - m^2 e\cos\xi_{,}) - \frac{9}{2}m^2 e\cos(2\tau - \xi) \\
& + \frac{3}{2}m^2 e\cos(2\tau + \xi) + \frac{3}{2}m^2 e_{,}\cos\xi_{,} + \frac{21}{4}m^2 e_{,}\cos(2\tau - \xi_{,}) \\
& - \frac{3}{4}m^2 e_{,}\cos(2\tau + \xi_{,})\Big\}\Big\{\frac{5\,\mathrm{a}}{4\,a_{,}}\,e_{,}\cos(\tau + \xi_{,}) \\
& + \frac{15\,\mathrm{a}}{8\,a_{,}}\,e\,e_{,}\cos(\tau + \xi + \xi_{,}) - \frac{5\,\mathrm{a}}{8\,a_{,}}\,e\,e_{,}\cos(\tau - \xi + \xi_{,})\Big\} \\
& + \Big\{\frac{3}{2}m^2\sin\tau - \frac{9}{2}m^2 e\sin(\tau - \xi) + \frac{3}{2}m^2 e\sin(\tau + \xi) \\
& + \frac{21}{4}m^2 e_{,}\sin(\tau - \xi_{,}) - \frac{3}{4}m^2 e_{,}\sin(\tau + \xi_{,})\Big\} \\
& \Big\{\frac{5\,\mathrm{a}}{2\,a_{,}}\,e_{,}\sin(\tau + \xi_{,}) + \frac{25\,\mathrm{a}}{8\,a_{,}}\,e\,e_{,}\sin(\tau + \xi + \xi_{,}) \\
& + \lambda_{110}\frac{\mathrm{a}}{a_{,}}\,e\,e_{,}\sin(\tau - \xi + \xi_{,})\Big\}
\end{aligned}$$

$$m^2\frac{a}{a_{,}}\left\{\left\{\frac{15}{16}-\frac{15}{32}-\frac{15}{16}\right\}e_{,}^2\cos 2\tau+\left\{\frac{15}{8}+\frac{15}{16}\right\}e_{,}\cos(\tau-\xi_{,})\right.$$

$$+\frac{5}{8}e_{,}\cos(\tau+\xi_{,})+\left\{\frac{45}{32}-\frac{45}{16}+\frac{75}{32}-\frac{45}{8}\right\}e\,e_{,}\cos(\tau-\xi-\xi_{,})$$

$$+\left\{\frac{15}{16}-\frac{5}{8}\right\}e\,e_{,}\cos(\tau+\xi+\xi_{,})+\left\{-\frac{5}{16}-\frac{5}{8}\right\}e\,e_{,}\cos(\tau-\xi+\xi_{,})$$

$$+\left\{-\frac{15}{32}+\frac{15}{16}+\frac{15}{8}+\frac{3}{4}\lambda_{110}\right\}e\,e_{,}\cos(\tau+\xi-\xi_{,})$$

$$+\left\{\frac{105}{32}+\frac{105}{16}\right\}e_{,}^2\cos(\tau-2\xi_{,})+\frac{15}{16}e_{,}^2\cos(\tau+2\xi_{,})$$

$$+\left\{\frac{105}{32}-\frac{105}{16}\right\}e_{,}^2\cos 3\tau+\left\{\frac{15}{16}-\frac{15}{8}\right\}e_{,}\cos(3\tau+\xi_{,})$$

$$+\left\{\frac{45}{32}+\frac{15}{16}-\frac{75}{32}-\frac{15}{8}\right\}e\,e_{,}\cos(3\tau+\xi+\xi_{,})$$

$$+\left\{-\frac{15}{32}-\frac{45}{16}+\frac{45}{8}-\frac{3}{4}\lambda_{110}\right\}e\,e_{,}\cos(3\tau-\xi+\xi_{,})$$

$$+\left\{-\frac{15}{32}+\frac{15}{16}\right\}e_{,}^2\cos(3\tau+2\xi_{,})$$

$$\frac{\delta R}{\mu}=\underset{[101]}{-\frac{15}{16}m^2\frac{a}{a_{,}}e_{,}^2\cos\tau}+\underset{[104]}{\frac{45}{16}m^2\frac{a}{a_{,}}e_{,}\cos(2\tau-\xi_{,})}$$

$$+\underset{[105]}{\frac{5}{8}m^2\frac{a}{a_{,}}e_{,}\cos(\tau+\xi_{,})}-\underset{[108]}{\frac{75}{16}m^2\frac{a}{a_{,}}e\,e_{,}\cos(\tau-\xi-\xi_{,})}$$

$$+\underset{[109]}{\frac{5}{16}m^2\frac{a}{a_{,}}e\,e_{,}\cos(\tau+\xi+\xi_{,})}-\underset{[110]}{\frac{15}{16}m^2\frac{a}{a_{,}}e\,e_{,}\cos(\tau-\xi+\xi_{,})}$$

$$+\underset{[111]}{\left\{\frac{75}{32}+\frac{3}{4}\lambda_{110}\right\}m^2\frac{a}{a_{,}}e\,e_{,}\cos(\tau+\xi-\xi_{,})}$$

$$+\underset{[112]}{\frac{315}{32}m^2\frac{a}{a_{,}}e_{,}\cos(\tau-2\xi_{,})}$$

$$+\underset{[113]}{\frac{15}{16}m^2\frac{a}{a_{,}}e_{,}^2\cos(\tau+2\xi_{,})}-\underset{[116]}{\frac{105}{32}m^2\frac{a}{a_{,}}e_{,}^2\cos 3\tau}$$

$$-\frac{15}{16}m^2\frac{\mathrm{a}}{a_{,}}e_{,}\cos(3\tau+\xi_{,})-\frac{15}{8}m^2\frac{\mathrm{a}}{a_{,}}e\,e_{,}\cos(3\tau+\xi+$$

[120] [124]

$$+\left\{\frac{75}{32}-\frac{3}{4}\lambda_{110}\right\}m^2\frac{\mathrm{a}}{a_{,}}e\,e_{,}\cos(3\tau-\xi+\xi_{,})$$

[125]

$$+\frac{15}{32}m^2\frac{\mathrm{a}}{a_{,}}e_{,}^2\cos(3\tau+2\xi_{,})$$

[128]

By doubling these terms the corresponding development $\dfrac{r\,\mathrm{d}R}{\mathrm{d}r}$ is at once obtained.

$$\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2=\left\{\frac{m^2}{3}+2m^2\cos 2\tau+\left\{\frac{15}{4}m+\frac{187}{16}m^2\right\}e\cos(2\tau-\xi)\right.$$

$$+\frac{33}{8}m^2e\cos(2\tau+\xi)-3m^2e_{,}\cos\xi_{,}+7m^2e_{,}\cos(2\tau-\xi_{,})$$

$$\left.-m^2e_{,}\cos(2\tau+\xi_{,})\right\}\left\{\frac{5\,\mathrm{a}}{4\,a_{,}}e_{,}\cos(\tau+\xi_{,})+\frac{5\,\mathrm{a}}{2\,a_{,}}e\,e_{,}\cos(\tau+\xi+\xi_{,})\right\}$$

$$=-\frac{5}{4}m^2\frac{\mathrm{a}}{a_{,}}e_{,}^2\cos\tau+\frac{5}{4}m^2\frac{\mathrm{a}}{a_{,}}e_{,}\cos(\tau-\xi_{,})$$

[101] [104]

$$+\frac{5}{12}m^2\frac{\mathrm{a}}{a_{,}}e_{,}\cos(\tau+\xi_{,})$$

[105]

$$+\left\{\frac{75}{32}m+\frac{1255}{128}m^2\right\}\frac{\mathrm{a}}{a_{,}}e\,e_{,}\cos(\tau-\xi-\xi_{,})$$

[108]

$$+\frac{5}{6}m^2\frac{\mathrm{a}}{a_{,}}e\,e_{,}\cos(\tau+\xi+\xi_{,})$$

[109]

$$+\frac{165}{64}m^2\frac{\mathrm{a}}{a_{,}}e\,e_{,}\cos(\tau+\xi-\xi_{,})$$

[111]

$$+\frac{35}{8}m^2\frac{\mathrm{a}}{a_{,}}e_{,}^2\cos(\tau-2\xi_{,})-\frac{15}{8}m^2\frac{\mathrm{a}}{a_{,}}e_{,}^2\cos(\tau+2\xi_{,})$$

[112] [113]

$$+ \frac{35}{8} m^{3} \frac{\mathrm{a}}{a_{,}} e_{,}^{3} \cos 3\tau + \frac{5}{4} m^{3} \frac{\mathrm{a}}{a_{,}} e_{,} \cos (3\tau + \xi_{,})$$

[116] [120]

$$+ \frac{325}{64} m^{2} \frac{\mathrm{a}}{a_{,}} e\, e_{,} \cos (3\tau + \xi + \xi_{,})$$

[124]

$$+ \left\{ \frac{75}{32} m + \frac{935}{128} m^{2} \right\} \frac{\mathrm{a}}{a_{,}} e\, e_{,} \cos (3\tau - \xi + \xi_{,})$$

[125]

$$r_{101} = - \frac{15}{16} m \frac{\mathrm{a}}{a_{,}} \qquad r_{102} = \frac{45}{16} m^{3} \frac{\mathrm{a}}{a_{,}}$$

$$\mathfrak{r}_{103} = - \frac{45}{32} m \frac{\mathrm{a}}{a_{,}} \qquad r_{103} = - \frac{15}{8} m \frac{\mathrm{a}}{a_{,}} \qquad \lambda_{103} = - \frac{75}{32} m \frac{\mathrm{a}}{a_{,}}$$

$$_{104} \{ (1 - 2m)^{2} - 1 \} = (1 - 2m)^{2} \mathfrak{r}_{104} + \frac{5 . 9}{8} m^{2} \frac{\mathrm{a}}{a_{,}} - \frac{45}{4} m^{3} \frac{\mathrm{a}}{a_{,}}$$

$$- 4 m r_{104} = \frac{15}{8} m^{2} \frac{\mathrm{a}}{a_{,}} + \frac{45}{8} m^{2} \frac{\mathrm{a}}{a_{,}} - \frac{45}{4} m^{2} \frac{\mathrm{a}}{a_{,}}$$

$$r_{104} = \frac{15}{16} m \frac{\mathrm{a}}{a_{,}} \qquad \lambda_{104} = \frac{15}{8} m \frac{\mathrm{a}}{a_{,}}$$

$$\mathfrak{r}_{106} = \frac{3}{2} r_{102} = \frac{135}{32} m^{2} \frac{\mathrm{a}}{a_{,}} \qquad \mathfrak{r}_{107} = \frac{3}{2} r_{103} = - \frac{45}{16} m \frac{\mathrm{a}}{a_{,}}$$

$$m\, r_{106} = \frac{135}{32} m^{3} \frac{\mathrm{a}}{a_{,}} + \frac{5 . 33}{64} m^{2} \frac{\mathrm{a}}{a_{,}} \qquad r_{106} = \frac{435}{128} m \frac{\mathrm{a}}{a_{,}} \qquad \lambda_{106} = - \frac{435}{64} m \frac{\mathrm{a}}{a}$$

$$8\, r_{107} = - \frac{9 . 45}{16} m \frac{\mathrm{a}}{a_{,}} \qquad r_{107} = - \frac{405}{128} m \frac{\mathrm{a}}{a_{,}} \qquad \lambda_{107} = - \frac{195}{64} m \frac{\mathrm{a}}{a}$$

$$\mathfrak{r}_{111} = \frac{45}{32} m \frac{\mathrm{a}}{a_{,}} \qquad r_{111} = \frac{15}{8} m \frac{\mathrm{a}}{a_{,}} \qquad \lambda_{111} = \frac{75}{32} m \frac{\mathrm{a}}{a_{,}}$$

$$\mathfrak{r}_{112} = \frac{105}{16} m^{2} \frac{\mathrm{a}}{a_{,}} \qquad - 6 m\, r_{112} = \frac{105}{16} m^{3} \frac{\mathrm{a}}{a_{,}} + \frac{5 . 159}{64} m^{2} \frac{\mathrm{a}}{a_{,}} - \frac{315}{8} m^{3} \frac{\mathrm{a}}{a}$$

$$r_{112} = \frac{435}{128} m \frac{\mathrm{a}}{a_{,}} \qquad \lambda_{112} = \frac{435}{64} m\, e_{,}^{3} \frac{\mathrm{a}}{a_{,}}$$

$$\mathfrak{r}_{113} = - \frac{45}{16} m^{3} \frac{\mathrm{a}}{a_{,}} \qquad 2 m\, r_{113} = - \frac{45}{16} m^{3} \frac{\mathrm{a}}{a_{,}} + \frac{5 . 33}{64} m^{2} \frac{\mathrm{a}}{a_{,}} - \frac{15}{4} m^{2} \frac{\mathrm{a}}{a_{,}}$$

$$r_{113} = -\frac{255}{128} m \frac{\mathrm{a}}{a_{,}} \qquad \lambda_{113} = -\frac{255}{64} m \frac{\mathrm{a}}{a_{,}}$$

$$\mathfrak{r}_{118} = \frac{3}{2} r_{116} = \frac{75}{128} m^2 \frac{\mathrm{a}}{a_{,}}$$

$$15\, r_{118} = \frac{16 \cdot 75}{128} m^2 \frac{\mathrm{a}}{a_{,}} - \frac{5 \cdot 15}{16} m^2 \frac{\mathrm{a}}{a_{,}} \qquad r_{118} = \frac{15}{16} m^2 \frac{\mathrm{a}}{a_{,}}$$

$$8\, r_{119} = \frac{5 \cdot 25}{8} m^2 \frac{\mathrm{a}}{a_{,}} \qquad r_{119} = \frac{125}{64} m^2 \frac{\mathrm{a}}{a_{,}}$$

$$\mathfrak{r}_{120} = \frac{15}{16} m^2 \frac{\mathrm{a}}{a_{,}} \qquad 8\, r_{120} = \frac{9 \cdot 15}{8} m^2 \frac{\mathrm{a}}{a_{,}} - \frac{5 \cdot 5}{8} m^2 \frac{\mathrm{a}}{a_{,}} + \frac{15}{4}$$

$$r_{120} = \frac{35}{16} m^2 \frac{\mathrm{a}}{a_{,}}$$

$$\mathfrak{r}_{125} = \frac{225}{64} m \frac{\mathrm{a}}{a_{,}} \qquad r_{125} = \frac{75}{16} m \frac{\mathrm{a}}{a_{,}}$$

$\mathrm{a}^2 \left(\delta \frac{1}{r}\right)^2$ contains the terms

$$\frac{15}{4} m\, e \cos (2\tau - \xi) \left\{ -\frac{15}{16} m \frac{\mathrm{a}}{a_{,}} \cos \tau \right\}$$

$$= -\frac{225}{128} m^2 e \frac{\mathrm{a}}{a_{,}} \cos (3\tau - \xi) - \frac{225}{128} m^2 e \frac{\mathrm{a}}{a_{,}} \cos (\tau - \xi)$$

$$\mathfrak{r}_{117} = \left\{ \frac{75}{128} - \frac{675}{256} \right\} m^2 \frac{\mathrm{a}}{a_{,}} = -\frac{525}{256} m^2 \frac{\mathrm{a}}{a_{,}}$$

$$3\, r_{117} = -\frac{525}{64} m^2 \frac{\mathrm{a}}{a_{,}} - \frac{135}{16} m^2 \frac{\mathrm{a}}{a_{,}} - \frac{45}{8} m^2 \frac{\mathrm{a}}{a_{,}}$$

$$r_{117} = -\frac{475}{64} m^2 \frac{\mathrm{a}}{a_{,}}$$

Let $r_{103} = -\frac{15}{8} m \frac{\mathrm{a}}{a_{,}} + A m^2 \frac{\mathrm{a}}{a_{,}}$

$$\mathfrak{r}_{103} = -\frac{45}{32} m \frac{\mathrm{a}}{a_{,}} - \frac{243}{32} m^2 \frac{\mathrm{a}}{a_{,}}$$

$$\{3 - 4m\} \left\{ -\frac{15}{8} m + A m^2 \right\} = \{4 - 4m\} \left\{ -\frac{45}{32} m - \frac{243}{32} m^2 \right\} - \frac{5 \cdot 3}{16} m^2$$

$$\frac{15}{2} - 3A = -\frac{243}{8} + \frac{45}{8} - \frac{15}{16} \qquad A = -\frac{177}{16}$$

I shall now calculate the second term in r_{105}. For this purpose I take

$$\frac{\mathrm{a}\,\delta R}{\mu} = m^2\left\{\frac{1}{2}+\frac{3}{2}\cos\tau\right\}\left\{\frac{15}{16}\,m\,\frac{\mathrm{a}}{a_{,}}\,e_{,}\cos(\tau-\xi_{,})+r_{105}\,e_{,}\cos(\tau+\xi_{,})\right\}$$

$$+\,m^2\left\{\frac{3}{2}e_{,}\cos\xi_{,}+\frac{21}{2}\,e_{,}\cos(2\tau-\xi_{,})-\frac{3}{4}\,e_{,}\cos(2\tau+\xi_{,})\right\}\left\{-\frac{15}{16}\,m\,\frac{\mathrm{a}}{a_{,}}\cos\tau\right\}$$

$$+\,\frac{3}{2}\,m^2\sin 2\tau\left\{\frac{15}{8}\,m\,\frac{\mathrm{a}}{a_{,}}\,e_{,}\sin(\tau-\xi)+\lambda_{105}\,e_{,}\sin(\tau+\xi_{,})\right\}$$

$$+\,m^2\left\{\frac{21}{4}e_{,}\sin(2\tau-\xi_{,})-\frac{3}{4}\,e_{,}\sin(2\tau+\xi_{,})\right\}\left\{-\frac{15}{8}\,m\,\frac{\mathrm{a}}{a_{,}}\sin\tau\right\}$$

$$+\,m^2\left\{\frac{3\,\mathrm{a}}{8\,a_{,}}\sin\tau+\frac{15\,\mathrm{a}}{8\,a_{,}}\sin 3\tau\right\}\left\{-3\,m\,e_{,}\sin\xi_{,}\right\}$$

$$=\left\{\frac{1}{2}r_{105}+\left\{\frac{45}{64}-\frac{45}{64}+\frac{45}{128}+\frac{45}{32}+\frac{45}{64}+\frac{9}{16}\right\}m\,\frac{\mathrm{a}}{a_{,}}\right\}m^2\,e_{,}\cos(\tau+\xi_{,})$$

$$\frac{\mathrm{a}\,R}{\mu}=\left\{\frac{1}{2}r_{105}-\frac{3\,\mathrm{a}}{8\,a_{,}}+\frac{387}{128}\,m\,\frac{\mathrm{a}}{a_{,}}\right\}m^2\,e_{,}\cos(\tau+\xi_{,})$$

$$\frac{\mathrm{a}\,r\,\mathrm{d}R}{\mu\,\mathrm{d}r}=\left\{r_{105}-\frac{9\,\mathrm{a}}{8\,a_{,}}+\frac{423}{64}\,m\,\frac{\mathrm{a}}{a_{,}}\right\}m^2\,e_{,}\cos(\tau+\xi_{,})$$

$$\frac{\mathrm{a}}{\mu}\int\mathrm{d}R=\frac{\mathrm{a}}{\mu}\,R-\frac{\mathrm{a}}{\mu}\int\left\{\frac{r_{,}\,\mathrm{d}R}{\mathrm{d}r_{,}}\frac{\mathrm{d}r_{,}}{r_{,}}+\frac{\mathrm{d}R}{\mathrm{d}\lambda_{,}}\,\mathrm{d}\lambda_{,}\right\}$$

It is sufficient in this approximation to take

$$\frac{\mathrm{a}\,r_{,}\,\mathrm{d}R}{\mu\,\mathrm{d}r_{,}}=\frac{3}{2}\,\frac{m^2\,\mathrm{a}}{a_{,}}\cos\tau \qquad \frac{\mathrm{d}r_{,}}{r_{,}}=e_{,}\sin\xi_{,}\,m\,\mathrm{n}\,\mathrm{d}t$$

$$\frac{\mathrm{a}\,\mathrm{d}R}{\mu\,\mathrm{d}\lambda_{,}}=-\frac{3}{8}\,m^2\,\frac{\mathrm{a}}{a_{,}}\sin\tau-\frac{3}{8}\,m^2\,\frac{\mathrm{a}}{a_{,}}\,e'\,(\tau+\xi_{,})$$

$$\mathrm{d}\lambda_{,}=\{1+2\,e_{,}\cos\xi_{,}\}\,m\,\mathrm{n}\,\mathrm{d}t$$

$$\frac{\mathrm{a}}{\mu}\int\mathrm{d}R=\left\{-\frac{3}{8}\,\frac{\mathrm{a}}{a_{,}}+\frac{1}{2}r_{105}+\left\{\frac{387}{128}+\frac{3}{4}-\frac{3}{8}-\frac{3}{8}\right\}m\,\frac{\mathrm{a}}{a_{,}}\right\}m^2 e_{,}\cos(\tau+\xi_{,})$$

$$=\left\{-\frac{3}{8}\,\frac{\mathrm{a}}{a_{,}}+\frac{1}{2}r_{105}+\frac{387}{128}\,m\,\frac{\mathrm{a}}{a_{,}}\right\}m^2\,\frac{\mathrm{a}}{a_{,}}\,e_{,}\cos(\tau$$

$$\mathrm{a}^2\left(\delta\,\frac{1}{r}\right)^2=\left\{\frac{m^2}{3}+2\,m^2\cos 2\tau\right\}\left\{\frac{15}{16}\,m\,\frac{\mathrm{a}}{a_{,}}\,e_{,}\cos(\tau-\xi_{,})+r_{105}\,\frac{\mathrm{a}}{a_{,}}\,e_{,}\right.$$

$$+\left\{-3\,m^2\,e_{,}\cos\xi_{,}-m^2\,e_{,}\cos(2\tau-\xi_{,})\right\}\left\{-\frac{15}{16}\,m\right.$$

$$=\left\{\frac{1}{3}\,r_{105}+\left\{\frac{15}{16}+\frac{45}{32}+\frac{15}{32}\right\}m\,\frac{\mathrm{a}}{a_{,}}\right\}m^2\,e_{,}\cos(\tau$$

$$= \left\{ \frac{1}{3} r_{105} + \frac{45}{16} m \frac{\mathrm{a}}{a_{,}} \right\} m^2 e_{,} \cos(\tau + \xi_{,})$$

Let $r_{105} = \frac{5\,\mathrm{a}}{4\,a_{,}} + A\,m\frac{\mathrm{a}}{a_{,}}$, then

$$0 = \frac{5}{8} + \frac{1}{2} A\,m + \frac{135}{32} m + \frac{3}{4} - \frac{5}{4} - A\,m - \frac{387}{64} m - \frac{5}{4} - A\,m + \frac{9}{8} - \frac{423}{64}$$

$$0 = \left\{ \frac{1}{2} - 1 - 1 \right\} A + \frac{135}{32} - \frac{387}{64} - \frac{423}{64}$$

$$A = -\frac{45}{8}$$

$$\lambda_{105} = \left\{ \frac{5}{2} - \frac{45}{4} m \right\} \frac{\mathrm{a}}{a_{,}} \qquad r_{109} = \frac{5}{2} - \frac{45}{4} m$$

$$\lambda_{109} = \left\{ 5 - \frac{45}{2} m + \frac{5}{4} - \frac{45}{8} m \right\} \frac{1}{2} = \frac{25}{8} - \frac{225}{16} m$$

It is easy to obtain the first term in the longitude, (Arg. 110,) by the method of the variation of the elliptic constants.

The disturbing function contains the terms

$$-\frac{3\,m_{,}\,a^2}{8\ \ a_{,}^3} e^2 + \frac{15\,m_{,}\,a^2}{16\ \ a_{,}^3} e\,e_{,} \cos(\tau - \xi + \xi_{,}) \qquad \text{(See. p. 32.)}$$

$$\mathrm{d}\varepsilon = -\frac{e\ \mathrm{d}R}{2\,a\,n\,\mathrm{d}e}\mathrm{d}t + \frac{2\ \mathrm{d}R}{a\,n\,\mathrm{d}a}\mathrm{d}t$$

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -\frac{9}{8} m^2 n\,e^2 + \frac{165}{32} m^2 n \frac{\mathrm{a}}{a_{,}} e\,e_{,} \cos(\tau - \xi + \xi_{,})$$

It is sufficient to take

$\mathrm{d}e = -\frac{\mathrm{d}R}{a^2 n\,e\,\mathrm{d}\xi}\mathrm{d}t$ (see p. 76), which gives

$$\delta e = \frac{5}{4} m^2 \frac{\mathrm{a}}{a_{,}} e_{,} \cos(\tau - \xi + \xi_{,})$$

$$e\,\delta e = \frac{5}{4} m^2 \frac{\mathrm{a}}{a_{,}} e\,e_{,} \cos(\tau - \xi + \xi_{,})$$

$$\mathrm{d}\varepsilon = \frac{75}{32} m^2 n \frac{\mathrm{a}}{a_{,}} e\,e_{,} \cos(\tau - \xi + \xi_{,})\,\mathrm{d}t$$

$$\delta\varepsilon = \frac{25}{8} \frac{\mathrm{a}}{a_{,}} e\,e_{,} \sin(\tau - \xi + \xi_{,})$$

which is the term required in the expression for the longitude.

$\frac{\mathrm{a}}{\mu} R$ contains the terms

$$-\frac{9}{16} m^2 \frac{\mathrm{a}}{a_{,}} \gamma^2 \cos(\tau - 2\eta) - \frac{15}{32} m^2 \frac{\mathrm{a}}{a_{,}} \gamma^2 \cos(3\tau - 2\eta)$$

$$\qquad [114] \qquad\qquad\qquad [129]$$

hence $2 m r_{114} = \frac{45}{16} m^2 \frac{\mathrm{a}}{a_{,}}$ $\qquad r_{114} = \frac{45}{32} m \frac{\mathrm{a}}{a_{,}}$

$- 6 m r_{129} = \frac{75}{32} m^2 \frac{\mathrm{a}}{a_{,}}$ $\qquad r_{129} = -\frac{25}{64} m \frac{\mathrm{a}}{a_{,}}$

Since $\frac{\mathrm{a}}{r}$ contains the term $-\frac{15}{16} m \frac{\mathrm{a}}{a_{,}} \cos\tau$, $\mathbf{Z}$ contains the terms

$$\frac{45}{32} m \frac{\mathrm{a}}{a_{,}} \sin(\iota - \eta) - \frac{45}{32} m \frac{\mathrm{a}}{a_{,}} \sin(\tau + \eta)$$

$$\qquad [185] \qquad\qquad [186]$$

$z_{185} = -\frac{45}{32} m \frac{\mathrm{a}}{a_{,}}$ $\qquad z_{186} = -\frac{45}{32} m \frac{\mathrm{a}}{a_{,}}$

$$s_{185} = \left\{-\frac{45}{32} + \frac{15}{32}\right\} m \frac{\mathrm{a}}{a_{,}} = -\frac{15}{16} m \frac{\mathrm{a}}{a_{,}}$$

$$s_{186} = \left\{-\frac{15}{32} - \frac{15}{32}\right\} m \frac{\mathrm{a}}{a_{,}} = -\frac{15}{16} m \frac{\mathrm{a}}{a_{,}}$$

s^2 contains the terms

$$-\frac{\gamma^2}{2} \cos 2\eta - \frac{15}{16} m \frac{\mathrm{a}}{a_{,}} \gamma^2 \sin(\tau - 2\eta) + \frac{15}{16} m \frac{\mathrm{a}}{a_{,}} \gamma^2 \sin(\tau + 2\eta)$$

$\frac{\mathrm{a}^2}{r^2}$ contains the terms

$$\frac{15}{8} m \frac{\mathrm{a}}{a_{,}} \cos\tau + \frac{45}{16} m \frac{\mathrm{a}}{a_{,}} \gamma^2 \cos(\tau - 2\eta) - \frac{25}{32} m \frac{\mathrm{a}}{a} \gamma^2 \cos(3\tau - 2\eta)$$

Hence

$$\frac{\mathrm{d}\lambda}{\mathrm{n}\,\mathrm{d}t} = \left\{\frac{45}{16} + \frac{15}{32} - \frac{15}{16}\right\} m \frac{\mathrm{a}}{a} \gamma^2 \cos(\tau - 2\eta)$$

$$+ \left\{\frac{15}{16} + \frac{15}{32}\right\} m \frac{\mathrm{a}}{a_{,}} \gamma^2 \cos(\tau + 2\eta)$$

$$\frac{25}{40} m \frac{\mathrm{a}}{a_{,}} \gamma^2 \cos(3\tau - 2\eta)$$

$$\lambda' = -\frac{75}{32} m \frac{\mathrm{a}}{a_{,}} \gamma^2 \sin(\tau - 2\eta) + \frac{15}{32} m \frac{\mathrm{a}}{a_{,}} \gamma^2 \sin(\tau + 2\eta)$$

$$- \frac{25}{32} m \frac{\mathrm{a}}{a_{,}} \gamma^2 \sin(3\tau - 2\eta)$$

I shall now recapitulate the terms in the longitude multiplied by $\frac{\mathrm{a}}{a_{,}}$ which have been obtained in the latter pages.

$$\lambda' = \underset{[101]}{\left\{ -\frac{15}{8} m - \frac{93}{8} m^2 - \frac{889}{16} m^3 \right\} \frac{\mathrm{a}}{a_{,}} \sin \tau} - \underset{[102]}{\frac{165}{32} m e \frac{\mathrm{a}}{a_{,}} \sin(\tau - \xi)}$$

$$+ \underset{[103]}{\left\{ -\frac{75}{32} m - \frac{117}{8} m^2 \right\} e \frac{\mathrm{a}}{a_{,}} \sin(\tau + \xi)} + \underset{[104]}{\frac{15}{8} m e_{,} \frac{\mathrm{a}}{a_{,}} \sin(\tau - \xi_{,})}$$

$$+ \underset{[105]}{\left\{ \frac{5}{2} - \frac{45}{4} m \right\} \frac{\mathrm{a}}{a_{,}} e_{,} \sin(\tau + \xi_{,})} - \underset{[106]}{\frac{435}{64} m e^2 \frac{\mathrm{a}}{a_{,}} \sin(\tau - 2\xi)}$$

$$- \underset{[107]}{\frac{195}{64} m e^2 \frac{\mathrm{a}}{a_{,}} \sin(\tau + 2\xi)}$$

$$+ \underset{[109]}{\left\{ \frac{25}{8} - \frac{225}{16} m \right\} e e_{,} \frac{\mathrm{a}}{a_{,}} \sin(\tau + \xi + \xi_{,})}$$

$$+ \underset{[110]}{\frac{25}{8} e e_{,} \frac{\mathrm{a}}{a_{,}} \sin(\tau - \xi + \xi_{,})} + \underset{[111]}{\frac{75}{32} m e e_{,} \frac{\mathrm{a}}{a_{,}} \sin(\tau + \xi - \xi_{,})}$$

$$+ \underset{[112]}{\frac{435}{64} m e_{,}^2 \frac{\mathrm{a}}{a_{,}} \sin(\tau - 2\xi_{,})} - \underset{[113]}{\frac{255}{64} m e_{,}^2 \frac{\mathrm{a}}{a_{,}} \sin(2\tau + \xi_{,})}$$

$$- \underset{[114]}{\frac{75}{32} m \frac{\mathrm{a}}{a_{,}} \gamma^2 \sin(\tau - 2\eta) + \frac{15}{32} m \gamma^2 \frac{\mathrm{a}}{a_{,}} \sin(\tau + 2\eta)}$$

$$+ \underset{[116]}{\left\{ \frac{15}{32} m^2 - \frac{5}{8} m^3 \right\} \frac{\mathrm{a}}{a_{,}} \sin 3\tau}$$

$$- \underset{[117]}{\frac{2535}{256} m^2 e \frac{\mathrm{a}}{a_{,}} \sin(3\tau - \xi)} + \underset{[118]}{\frac{115}{128} m^2 e \frac{\mathrm{a}}{a_{,}} \sin(3\tau + \xi)}$$

$$-\frac{25}{32}m\frac{\mathrm{a}}{a_{,}}\gamma^{2}\sin(3\tau-2\eta)$$

[129]

Calculation of Terms in the Parallactic Inequality which depend upon the Mass of the Moon.

According to the *Méc. Cél.*, vol. iii. p. 107, and to M. Poisson, $x_{,}+\mathfrak{m}x$, and $y_{,}+\mathfrak{m}y$ must be substituted in the disturbing function, R, $x_{,}$, $y_{,}$ being the elliptic values of the coordinates of the sun, and $\mathfrak{m}$ the ratio of the mass of the moon to that of the earth; so that neglecting the eccentricities,

$$\delta r_{,}=\mathfrak{m}\,\mathrm{a}\cos\tau \qquad \delta\lambda_{,}=\mathfrak{m}\frac{\mathrm{a}}{a_{,}}\sin\tau$$

Hence I take

$$\frac{\mathrm{a}\,\delta R}{\mu}=\left\{\frac{3}{4}m^{2}+\frac{9}{4}m^{2}\cos 2\tau\right\}\mathfrak{m}\frac{\mathrm{a}}{a_{,}}\cos\tau-\frac{3}{2}m^{2}\mathfrak{m}\frac{\mathrm{a}}{a_{,}}\sin 2\tau\sin\tau$$

$$=\frac{9}{8}m^{2}\mathfrak{m}\frac{\mathrm{a}}{a_{,}}\cos\tau+\frac{15}{8}m^{2}\mathfrak{m}\frac{\mathrm{a}}{a_{,}}\cos 3\tau$$

$$\int \mathrm{d}R^{*}=R-\int\left\{\frac{\mathrm{d}R}{\mathrm{d}r_{,}}\mathrm{d}r_{,}+\frac{\mathrm{d}R}{\mathrm{d}\lambda_{,}}\mathrm{d}\lambda_{,}\right\}$$

I take $\mathrm{d}r_{,}=-\mathfrak{m}\,\mathrm{a}(1-m)\sin\tau \qquad \mathrm{d}\lambda_{,}=\mathfrak{m}(1-m)\frac{\mathrm{a}}{a_{,}}\cos\tau$

$$\frac{\mathrm{a}}{\mu}\int\left\{\frac{\mathrm{d}R}{\mathrm{d}r_{,}}\mathrm{d}r_{,}+\frac{\mathrm{d}R}{\mathrm{d}\lambda_{,}}\mathrm{d}\lambda_{,}\right\}=\frac{3}{8}m^{2}\mathfrak{m}\frac{\mathrm{a}}{a_{,}}\cos\tau+\frac{5}{8}m^{2}\mathfrak{m}\frac{\mathrm{a}}{a_{,}}\cos 3\tau$$

$$\{(1-m)^{2}-1\}\,r_{101}=-\frac{3}{2}m^{2}\mathfrak{m}\frac{\mathrm{a}}{a_{,}}-\frac{9}{4}m^{3}\mathfrak{m}\frac{\mathrm{a}}{a_{,}}$$

$$r_{101}=\frac{15}{8}m\,\mathfrak{m}\frac{\mathrm{a}}{a_{,}} \qquad \lambda_{101}=\frac{15}{4}m\,\mathfrak{m}\frac{\mathrm{a}}{a_{,}}$$

In order to procure the variation of R which arises from these values of r_{101} and λ_{101}, I take

$$\frac{\mathrm{a}\,\delta R}{\mu}=m^{2}\left\{\frac{1}{2}+\frac{3}{2}\cos 2\tau\right\}\frac{15}{8}\mathfrak{m}\frac{\mathrm{a}}{a_{,}}\cos\tau$$

---

* d R is of course here restrained to mean the differential of R with regard only to the coordinates of the moon.

$$+\frac{3}{2}m^2\sin 2\tau\left\{\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\sin\tau\right\}$$

$$=\left\{\frac{15}{16}+\frac{45}{32}+\frac{45}{16}\right\}m^3\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\sin\tau$$

$$=\frac{165}{32}m^3\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\cos\tau$$

$$\frac{\mathrm{a}\,\delta\,\mathrm{d}\,R}{\mu\,\delta\,\lambda}=-\frac{9}{2}m^2\sin 2\tau\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\cos\tau-3\,m^2\cos 2\tau\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\sin\tau$$

$$=\left\{-\frac{9}{4}+\frac{3}{2}\right\}m^2\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\sin\tau$$

$$+\left\{-\frac{9}{4}+\frac{3}{2}\right\}m^2\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\sin 3\,\tau$$

$$=-\frac{3}{4}m^2\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\sin\tau-\frac{15}{4}m^2\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\sin 3\,\tau$$

$$\frac{\mathrm{a}}{\mathrm{n}\,\mu}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\mathrm{d}\,t=\frac{3}{4}m^2\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\cos\tau+\frac{5}{4}m^2\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\cos 3\,\tau$$

This value of $\frac{\mathrm{a}}{\mathrm{n}\,\mu}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\mathrm{d}\,t$ gives in $\frac{\mathrm{a}}{\mu}\int\mathrm{d}\,R$ the term

$$\frac{3}{4}m^3\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\cos\tau$$

$$\mathrm{a}^2\left(\delta\,\frac{1}{r}\right)^2=\left\{\frac{m^2}{3}+m^2\cos 2\,\tau\right\}\left\{\frac{15}{8}m\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\cos\tau\right\}$$

$$\left\{\frac{5}{8}+\frac{15}{8}\right\}m^3\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\cos\tau=\frac{5}{2}m^3\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}\cos\tau$$

Hence, if $r_{101}=\frac{15}{8}m\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}+A\,m^2\,\mathfrak{m}\,\frac{\mathrm{a}}{a_{,}}$

$$(-2\,m+m^2)\left\{\frac{15}{8}m+A\,m^2\right\}=\frac{15}{4}m^3-\frac{165}{8}m^3-\frac{3}{2}m^3$$

$$-2\,A+\frac{15}{8}=\frac{15}{4}-\frac{165}{8}-\frac{3}{2}$$

$$A=\frac{81}{8}$$

$$\lambda_{101} = \left\{ \frac{15}{4} m + \left\{ \frac{81}{4} - \frac{3}{4} \right\} m^2 \right\} \mathfrak{m} \frac{\mathrm{a}}{a_{\prime}} (1 + m)$$

$$= \frac{15}{4} m \, \mathfrak{m} \frac{\mathrm{a}}{a_{\prime}} + \frac{93}{4} m^2 \mathfrak{m} \frac{\mathrm{a}}{a_{\prime}}$$

Uniting these to the terms found before, (see p. 286),

$$\lambda_{101} = \left\{ - \frac{15}{8} m - \frac{93}{8} m^2 + \left\{ \frac{15}{4} m + \frac{93}{4} m^2 \right\} \mathfrak{m} \right\} \frac{\mathrm{a}}{a_{\prime}}$$

This result is in accordance with that of M. Plana, but not with that of M. Poisson*.

Calculation of Terms due to the Non-sphericity of the Earth.

If the ellipticity of the earth be called $\alpha \rho$, D its mean radius, $\alpha \phi$ the ratio of the centrifugal force to gravity at the equator, δ the declination of the moon, the quantity to be added to the disturbing function in order to take into account the non-sphericity of the earth, is

$$\left(\alpha \rho - \frac{1}{2} \alpha \phi \right) \frac{\mu D^2}{r^3} \left\{ \sin^2 \delta - \frac{1}{3} \right\}$$

See *Méc. Cél.*, vol. iii. p. 251.

$$\sin \delta = \frac{\sin f \lambda \sin \omega + s \cos \omega}{\sqrt{1 + s^2}}$$

ω being the obliquity of the ecliptic, and $f \lambda$ the longitude of the moon, reckoned from the first point of Aries.

Considering R as a function of r, λ', and s,

---

* "Le coefficient total de l'équation parallactique dépend de la masse de la lune, ou de la quantité que nous avons appellée λ ($\mathfrak{m}$). Il est négatif; et, en le désignant par $-P$, on a $P = (K - \lambda L) m n$; n représentant le rapport de la parallaxe du soleil à celle de la lune; K et L étant des quantités indépendantes de λ et de n. Pour avoir égard à la masse de la lune, il suffirait selon M. Plana de multiplier cette quantité K par $\frac{1 - \lambda}{1 + \lambda}$, ou à très peu près par $1 - 2\lambda$; en sorte que la quantité L serait double de celle de K. Mais cette règle ne convient qu'aux premiers termes des séries que K et L représentent; et en calculant directement les deux premiers termes de la valeur de L, j'ai trouvé $L = \frac{15}{4} + \frac{27}{4} m$."—*Poisson, Mémoire sur le Mouvement de la Lune.*

$$\frac{\mathrm{d}R}{\mathrm{d}z}=\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)\frac{\mathrm{d}s}{\mathrm{d}z}+\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)\frac{\mathrm{d}r}{\mathrm{d}z}$$

$$z=\frac{r\,s}{\sqrt{1+s^2}}\qquad \frac{\mathrm{d}s}{\mathrm{d}z}=\frac{1}{r}\text{ nearly,}\qquad \frac{\mathrm{d}r}{\mathrm{d}z}=\frac{z}{r}$$

In order to have the term in the latitude of which the argument is $f\mathrm{n}\,t$, it is sufficient to take

$$\frac{\mathrm{d}R}{\mathrm{d}z}=2\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{\mu D^2}{\mathrm{a}^4}\sin\omega\cos\omega\sin f\mathrm{n}\,t$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+\frac{\mu z}{r^3}+\frac{m_{,}z}{r_{,}^3}+2\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{\mu D^2}{\mathrm{a}^4}\sin\omega\cos\omega\sin f\mathrm{n}\,t.$$

See p. 227.

Let $z=A\,\mathrm{a}\sin fn\,t$

$$(f^2-1)A=\frac{3}{2}m^2A+2\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{\mu D^2}{\mathrm{a}^2}\sin\omega\cos\omega$$

f may be taken equal to unity, and

$$A=-\frac{4}{3}\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{\mu D^2}{m^2\mathrm{a}^2}\sin\omega\cos\omega$$

$$s=-\frac{4}{3}\left(\alpha\varrho=\frac{1}{2}\alpha\varphi\right)\frac{\mu D^2}{m^2\mathrm{a}^2}\sin\omega\cos\omega\sin f\mathrm{n}\,t$$

This result agrees with that of Laplace (*Méc. Cél.*, vol. iii. p. 251), and with that of M. Plana. In M. Plana's notation,

$$\alpha\varrho=K_{(2)}\qquad \frac{1}{2}\alpha\varphi=\psi$$

In order to obtain the term in the longitude of which the argument is $f\mathrm{n}\,t-\eta$, it is sufficient to take

$$s=\gamma\sin\eta-\frac{4}{3}\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{\mu D^2}{\mathrm{a}^2}\sin\omega\cos\omega\sin f\mathrm{n}\,t$$

$$s^2=-\frac{4}{3}\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{\mu D^2}{\mathrm{a}^2}\gamma\sin\omega\cos\omega\cos(f\mathrm{n}\,t-\eta)$$

$\frac{\mathrm{a}R}{\mu}$ contains the term

$$\frac{3}{4}m^2s^2=-\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{\mu D^2}{\mathrm{a}^2}\gamma\sin\omega\cos\omega\cos(f\mathrm{n}\,t-\eta)$$

$\frac{\mathrm{a} R}{\mu}$ contains also the term

$$2\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{D^2\mathrm{a}}{r^3}\sin\omega\cos\omega\, s\sin f\lambda$$

$$=\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{D^2\mathrm{a}}{r^3}\gamma\sin\omega\cos\omega\cos(f\mathrm{n}\,t-\eta)$$

These terms destroy each other, so that $\frac{\mathrm{a} R}{\mu}=0$

$\frac{\mathrm{a}}{\mu}\int \mathrm{d}R=\frac{\mathrm{a}R}{\mu}$ in this case, therefore $\int\mathrm{d}R=0$.

$$\frac{\mathrm{a}\,r\,\mathrm{d}R}{\mu\,\mathrm{d}r}=\{-2-3\}\left\{\alpha\varrho-\frac{1}{2}\alpha\varphi\right\}\frac{D^2}{\mathrm{a}^2}\gamma\sin\omega\cos\omega\cos(f\mathrm{n}\,t-\eta)$$

$$\frac{\mathrm{a}}{r}=-5\left\{\alpha\varrho-\frac{1}{2}\alpha\varphi\right\}\frac{D^2}{\mathrm{a}^2}\gamma\sin\omega\cos\omega\cos(f\mathrm{n}\,t-\eta)$$

In order to procure the term in the longitude of which the argument is $f\mathrm{n}\,t-\eta$ by the method of p. 219, it is sufficient to take

$$s=\left(1+\frac{m^2}{6}\right)\gamma\sin\eta+A\sin f\mathrm{n}\,t$$

$$\frac{\mathrm{d}s}{\mathrm{n}\,\mathrm{d}t}=\left(1+\frac{11}{12}m^2\right)\gamma\cos\eta+A\cos f\mathrm{n}\,t$$

$$s^2-\frac{\mathrm{d}s^2}{\mathrm{n}^2\mathrm{d}t^2}=\left(1+\frac{m^2}{6}\right)A-\left(1+\frac{11}{12}m^2\right)A$$

$$=\left\{\frac{1}{6}-\frac{11}{12}\right\}m^2A$$

$$=\left(\alpha\varrho-\frac{1}{2}\alpha\varphi\right)\frac{D^2}{\mathrm{a}^2}\gamma\sin\omega\cos\omega\cos(f\mathrm{n}\,t-\eta)$$

Moreover, it is sufficient to take

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t}=\frac{h}{2\mathrm{a}^2}s^2-\frac{\mathrm{a}^2\,\mathrm{d}s^2}{2h\,\mathrm{d}t^2}+\frac{2\sqrt{\mu}}{\sqrt{\mathrm{a}}\,r}$$

$$\frac{\mathrm{d}\lambda'}{\mathrm{n}\,\mathrm{d}t}=\left\{\frac{1}{2}-10\right\}\left\{\alpha\varrho-\frac{1}{2}\alpha\varphi\right\}\frac{D^2}{\mathrm{a}^2}\gamma\sin\omega\cos\omega\cos(f\mathrm{n}\,t-\eta)$$

$$=-\frac{19}{2}\left\{\alpha\varrho-\frac{1}{2}\alpha\varphi\right\}\frac{D^2}{\mathrm{a}^2}\gamma\sin\omega\cos\omega\cos(f\mathrm{n}\,t-\eta)$$

Since $f - g = -\frac{3}{4} m^2$

$$\lambda = \frac{19}{3}\left\{\alpha\varrho - \frac{1}{2}\alpha\varphi\right\}\frac{D^2}{m^2 a^2}\gamma \sin 2\varpi \sin(f n t - \eta)$$

This result agrees with that of Laplace (*Méc. Cél.*, vol. iii. p. 255), and with that of M. Plana.

THE END.

PRINTED BY RICHARD TAYLOR,
RED LION COURT, FLEET STREET.

ON

THE THEORY OF THE MOON,

AND ON

HE PERTURBATIONS OF THE PLANETS.

By J. W. LUBBOCK, Esq., F.R., R.A., and L.SS.

VICE-CHANCELLOR OF THE UNIVERSITY OF LONDON,
MEMBER OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES,
AND OF THE ACADEMY OF PALERMO.

PART III.

LONDON.
1837.

PRINTED BY RICHARD AND JOHN E. TAYLOR,
RED LION COURT, FLEET STREET.

PREFACE.

THE objects which I have in view in these investigations, and the methods I propose to substitute for those hitherto employed, are explained in the Introduction to each of the former parts of this work. However, it did not escape me that one difficulty remained, if possible, to be surmounted, by furnishing some means of calculating with facility the remarkable quantity $\int(\mathrm{d}\,R)$ in the case of inequalities *of long period*; at least unless we are content in this instance to abandon processes analogous to those employed in the rest of my work, and to derive our knowledge of the properties of the function in question entirely from a totally different manner of considering the subject, namely, that employed by M. Poisson. I have at length found that in all cases in the Theory of the Moon when the argument of the inequality is independent of the mean motion of the sun (or of the quantity m), which case includes all inequalities of long period, the following equation obtains rigorously:

$$\int(\mathrm{d}\,R) = R - \frac{\mathrm{d}\,x}{\mathrm{d}\,t}\,\frac{\mathrm{d}'\,x}{\mathrm{d}\,t} - \frac{\mathrm{d}\,y}{\mathrm{d}\,t}\,\frac{\mathrm{d}'\,y}{\mathrm{d}\,t} - \frac{\mathrm{d}\,z}{\mathrm{d}\,t}\,\frac{\mathrm{d}'\,z}{\mathrm{d}\,t}$$

$$= R - \frac{\mathrm{d}\,r'}{\mathrm{d}\,t}\,\frac{\mathrm{d}'\,r'}{\mathrm{d}\,t} - \frac{r'^2\,\mathrm{d}'\,\lambda'}{\mathrm{d}\,t}\,\frac{\mathrm{d}'\,\lambda'}{\mathrm{d}\,t} - \frac{\mathrm{d}\,z}{\mathrm{d}\,t}\,\frac{\mathrm{d}'\,z}{\mathrm{d}\,t}\,;$$

d' denoting the sign of differentiation with regard only to $m\,\mathrm{n}\,t$. [illegible] of any inequality, even when we assume or admit that

$\int(\mathrm{d}\,R) = 0$, we cannot dispense with the calculation of the coefficient of the corresponding argument in R if we wish to arrive at the coefficient of the inequality in longitude. When R is obtained, the supplementary quantity

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t}\,\frac{\mathrm{d}'\,x}{\mathrm{d}\,t} + \frac{\mathrm{d}\,y}{\mathrm{d}\,t}\,\frac{\mathrm{d}'y}{\mathrm{d}\,t} + \frac{\mathrm{d}\,z}{\mathrm{d}\,t}\,\frac{\mathrm{d}'\,z}{\mathrm{d}\,t}$$

necessary for finding the value of $\int(\mathrm{d}\,R)$, and which is of the order of the square of the disturbing mass multiplied by m, is easily found, and thus the difficulty to which I have alluded is completely removed.

In p. 321 I have given a rigorous equation, of which an approximate form is employed by Laplace in the *Méc. Cél.*; and in p. 344 I have applied this equation to the problem of determining the term in λ_9 multiplied by m^3.

The further I proceed the more I find cause to admire the fidelity and skill with which M. Plana's calculations have been executed, although there can be little doubt that some terms in his expressions require correction. Thus, for instance, I find the term

$$\frac{4171}{16}\,m^4 \text{ (Arg. 5.) (See p. 222.)}$$

in the *annual equation* instead of

$$\frac{1261}{4}\,m^4,$$

according to M. Plana. M. de Pontécoulant has not discovered error in my calculation, and if this result be established, M. Plana's coefficient of the *annual equation* (— 668·644) will require to be altered nearly 6″ sex.; after which alteration it will nearly agree with the coefficient of M. Damoiseau (— 673·70). One of M. Plana's coefficients of the latitude seems to require an alteration of more than $15''$, (see p. 335,) arising from an inadvertence in the numerical conversion. I mention these instances in order to show the practical utility of the investigation which I commenced, but which must be greatly extended before the expressions for the longitude and latitude of the moon can be said to have reached the desirable state of perfection. I trust, however, the examples I have furnished will render manifest the superiority of the methods which I have endeavoured to explain, over the indirect and circuitous process previously employed. It is difficult to account for the circumstance that a desire of generalization and a love of symmetry should not have induced mathematicians at an earlier period to seek uniformity in the manner of arriving at the perturbations of the moon and planets. When I began the inquiry (Phil. Trans. 1831. Part II.), even the more elementary developments, as those of different powers of the radius vector, of the disturbing function in terms of the mean longitude, &c., were wanting. I was therefore sensible that to determine the coefficients of all the inequalities of the moon's motion would require much greater labour and skill than I could command, but I hoped to draw attention to the subject and to lay the foundation of improved methods.

As far as I have proceeded I have constantly endeavoured to retain all quantities of a given order, and therefore wherever the numerical coefficient of a given term in the longitude or latitude is found to be inaccurate or incomplete, this circumstance is to be attributed to some involuntary omission. Well knowing the difficulty of avoiding numerical mistakes in such intricate calculations, I have given the details of all the operations, in order that where my results may be found in error the source of the evil may be discovered with the greatest possible facility.

Although the intermediate steps of M. Plana's work were of no assistance to me, yet the form in which M. Plana's final results are presented has been of essential service, by enabling me to verify separately the several terms of which a coefficient is composed. Indeed, I think that the most excellent and remarkable feature of M. Plana's work consists in the form in which the expressions for the longitude and latitude of the moon are exhibited, by means of the development of the denominators which arise from the successive integrations. If calculations of this kind were less subject to error the case would be different; but until methods shall have been devised for obtaining the Lunar inequalities, far easier of execution than any which have yet been imagined, in order to advance securely, we must be content to proceed slowly and cautiously, and that form of result is to be preferred which is most easy of verification.

Burg's Lunar Tables were principally founded upon 3200 observations of the moon made at Greenwich between 1765 and 1793. (*Tables Astronomiques publiées par le Bureau des Longitudes.* Paris, 1806.) Burckhardt's Tables are rather more recent, and are probably more accurate. Unfortunately the form of both is such that they are not directly comparable with the final results of MM. Damoiseau and Plana.

At the last meeting of the British Association for the Advancement of Science, held at Bristol, I suggested the importance of attempting to determine empirically the coefficients of the principal terms in the expressions for the longitude and latitude of the moon. This method is by no means free from difficulty and is attended with great labour; an obstacle exists also at present in the paucity of unexceptionable data, owing to the practice, so much to be regretted, which formerly prevailed of publishing only unreduced observations. The difficulties however which have already been overcome by Burg and Burckhardt would not be increased by seeking expressions in an unobjectionable form. Considerable labour would be avoided in this inquiry by using the expressions of M. Plana (with such corrections as may then have been discovered) as a first approximation, and by only endeavouring to improve (or verify) a certain number of coefficients. Each observation of R.A. and Dec^n. would, of course, furnish equations of the form

$$\delta\,(\text{R.A.}) = A\,\delta\,\lambda + B\,\delta\,s \qquad \delta\,(\text{Dec}^n.) = C\,\delta\,\lambda + D\,\delta\,s;$$

δ (R.A.) and δ (Dec^n.) being the errors of the calculated R.A. and Dec^n.; $\delta\,\lambda$ and $\delta\,s$ the errors of the calculated longitude and latitude and a linear function of the errors of the coefficients; A, B, C, D known quantities depending upon the R.A. and Dec^n.

The construction of Lunar Tables in this manner was formerly recommended by Laplace. (See *Conn. des Temps*, 1823, p. 231.) "La méthode d'approximation, tirée des observations, peut donc être utilement employée. Mais on la rendra plus exacte et plus facile, si l'on y regarde comme autant de données certaines, les coefficiens sur lesquels la théorie ne laisse point d'incertitude et les rapports qu'elle indique avec précision entre ces coefficiens. Par ce moyen, on diminuera considérablement le nombre des coefficiens à déterminer par le comparaison des observations; ce qui simplifiera le calcul, et donnera plus de précision à ses résultats." But it is only since M. Plana's calculations have exposed the divergence of the numerical coefficients of the higher powers of m, that the importance or rather necessity of having recourse to this method can be fully appreciated.

I have endeavoured (chiefly at the instigation of the Astronomer Royal) to transform Burckhardt's expressions for the longitude and latitude of the moon into such a shape that they may become comparable with the final results of M. Plana. Mr. Airy has also executed the troublesome calculation of the longitude independently, and having favoured me with the communication of his result, whilst mine was in the press, I was enabled to render my own more accurate. I confess, however, that I do not place reliance upon the accuracy of Burckhardt's coefficients. Since the formation of these Tables the Lunar Theory has been considerably improved through the labours of MM. Damoiseau and Plana, and less uncertainty now exists with respect to the numerical amount of the coefficients of the smaller inequalities.

I am indebted to M. de Pontécoulant for various numerical corrections. M. de Pontécoulant has kindly communicated to me results which he has obtained, and has frequently enabled me to ascertain the cause of discrepancies between my results and those of M. Plana. Thus, for example, M. de Pontécoulant has obtained by the method of p. 205 the same term in λ_3 multiplied by m^4 as that given by M. Plana. This agreement in a term arrived at with so much difficulty would alone suffice to prove the extraordinary skill with which M. Plana's calculations have been conducted.

29 *Eaton Place,*
August, 1837.

CONTENTS.

on of terms in the latitude of the moon . . *page* 293 to 314.

of terms in (λ_2), the coefficient of $e \sin \xi$ in
ession for the longitude . . . . *page* 314 to 320.

of terms in the longitude of Arguments 24
. . . . *page* 320 to 326.

ardt's expressions for the longitude and lati-
he moon . . . . *page* 326 to 336.

culation of λ_2, λ_3, and λ_4 . . . . *page* 336 to 340.

culation of λ_j . . . . *page* 340 to 349.

ntity $\int(\mathrm{d} R)$ . . . . *page* 349 to 354.

CORRECTIONS.

Page 241, line 8, *read* $-\frac{15}{16}\, m\, e_{\prime}^{2}$; this correction gives in p. 309

$$-\left.\frac{15}{4}\, m\, e_{\prime}^{2}\right\}\, e\, \gamma \sin\,(2\,\tau - \xi - \eta), \text{ and } -\left.\frac{15}{16}\, m\, e_{\prime}^{2}\right\}\, e\, \gamma \sin\,(2\,\tau + \xi - \eta),$$

$$[151] \qquad\qquad\qquad\qquad [153]$$

agreeing with M. Plana. M. de Pontécoulant has also found the terms p. 309

$$\frac{2893}{1536}\, m^3\, e\, \gamma \sin\,(2\,\tau + \xi - \eta) + \frac{303}{256}\, m^2\, e^2\, \gamma \sin\,(2\,\tau + 2\,\xi - \eta),$$

$$[156] \qquad\qquad\qquad\qquad [165]$$

but in the case of other terms marked with an asterisk in that page my calculations
require revision.

ON

THE THEORY OF THE MOON.

Calculation of Terms in the Latitude of the Moon.

HE method adopted in this work is explained p. 227, to which I fer.

$\frac{3\,\mathrm{a}^3}{\mathrm{r}^3}\,\delta\,\frac{1}{r}$ contains the terms

$$\cos\xi\,\Big\{\frac{15}{4}\,m\,e^2\cos 2\tau-\frac{21}{8}\,m\,e\,e_{,}\cos(\xi+\xi_{,})$$

$$+\frac{35}{8}\,m\,e\,e_{,}\cos(2\tau-\xi-\xi_{,})+\frac{21}{8}\,m\,e\,e_{,}\cos(\xi-\xi_{,})-\frac{15}{8}\,m\,e\,e_{,}\cos(2\tau-\xi+\xi_{,})$$

$$+\Big\{-\frac{15}{16}\,m-\frac{81}{16}\,m^2\Big\}\,\frac{\mathrm{a}}{a_{,}}\cos\tau+\frac{25}{64}\,m^2\,\frac{\mathrm{a}}{a_{,}}\cos 3\tau\Big\}$$

$$+\Big\{\frac{3}{2}\,e^2+\frac{15}{2}\,e^2\cos 2\xi\Big\}\;\Big\{\frac{15}{8}\,m\,e\cos(2\tau-\xi)\Big\}$$

$$-\frac{225}{16}\,m\,e\,e_{,}^2\cos(2\tau-\xi)+\frac{1215}{64}\,m\,e^3\cos(2\tau+\xi)+\frac{105}{4}\,m\,e^2\,e_{,}\cos(2\tau-\xi_{,})$$

$$-\frac{45}{4}\,m\,e^2\,e_{,}\cos(2\tau+\xi_{,})-\frac{315}{64}\,m\,e^3\cos(2\tau-3\xi)-\frac{63}{4}\,m\,e^2\,e_{,}\cos(2\xi+\xi_{,})$$

$$+\frac{63}{4}\,m\,e^2\,e_{,}\cos(2\xi-\xi_{,})-\frac{189}{32}\,m\,e\,e_{,}^2\cos(\xi+2\xi_{,})+\frac{765}{32}\,m\,e\,e_{,}^2\cos(2\tau-\xi-2\xi_{,})$$

$$+\frac{189}{32}\,m\,e\,e_{,}^2\cos(\xi-2\xi_{,})-\frac{135}{32}\,m\,e\,e_{,}^2\cos(2\tau-\xi+2\xi_{,})$$

$$+\Big\{-\frac{45}{16}\,m-\frac{243}{16}\,m^2\Big\}\,\frac{\mathrm{a}}{a_{,}}\cos\tau+\frac{135}{16}\,m^2\,\frac{\mathrm{a}}{a_{,}}\,e\cos(\tau-\xi)$$

$$+\Big\{-\frac{45}{8}\,m-\frac{531}{16}\,m^2\Big\}\,\frac{\mathrm{a}}{a_{,}}\,e\cos(\tau+\xi)+\frac{45}{16}\,m\,\frac{\mathrm{a}}{a_{,}}\,e_{,}\cos(\tau-\xi_{,})$$

$$+\left\{\frac{15}{4}-\frac{135}{8}m\right\}\frac{\mathrm{a}}{a_{,}}e_{,}\cos(\tau+\xi_{,})+\frac{75}{64}m^2\frac{\mathrm{a}}{a_{,}}\cos 3\tau$$

$$-\frac{1425}{64}m^2\frac{\mathrm{a}}{a_{,}}e\cos(3\tau-\xi)+\frac{45}{16}m^2\frac{\mathrm{a}}{a_{,}}e\cos(3\tau+\xi)$$

$\mathrm{a}^2\left(\delta\frac{1}{r}\right)^2$ contains the terms

$$-\frac{225}{128}m^2 e\frac{\mathrm{a}}{a_{,}}\cos(\tau-\xi)-\frac{225}{128}m^2\frac{\mathrm{a}}{a_{,}}e\cos(3\tau-\xi) \qquad \text{See p. 282.}$$

$\frac{\mathrm{a}^3}{r^3}$ contains the following terms:

$$+\frac{27}{8}e^3\cos\xi+\left\{\frac{225}{16}e^2-\frac{225}{16}e_{,}^2\right\}m\,e\cos(2\tau-\xi)+\frac{2385}{64}m\,e^3\cos(2\tau+\xi$$

$$+\frac{315}{8}m\,e^2e_{,}\cos(2\tau-\xi_{,})-\frac{135}{8}m\,e^2e_{,}\cos(2\tau+\xi_{,})+\frac{53}{8}e^3\cos 3\xi$$

$$+\frac{135}{64}m\,e^3\cos(2\tau-3\xi)-\frac{189}{8}m\,e^2e_{,}\cos(2\xi+\xi_{,})+\frac{105}{8}m\,e^2e_{,}\cos(2\tau-2\xi-$$

$$+\frac{189}{8}m\,e^2e_{,}\cos(2\xi-\xi_{,})-\frac{45}{8}m\,e^2e_{,}\cos(2\tau-2\xi+\xi_{,})$$

$$-\frac{189}{32}m\,e\,e_{,}^2\cos(\xi+2\xi_{,})+\frac{765}{32}m\,e\,e_{,}^2\cos(2\tau-\xi-2\xi_{,})$$

$$+\frac{189}{32}m\,e\,e_{,}^2\cos(\xi-2\xi_{,})-\frac{135}{32}m\,e\,e_{,}^2\cos(2\tau-\xi+2\xi_{,})$$

$$+\left\{-\frac{45}{16}m-\frac{243}{16}m^2\right\}\frac{\mathrm{a}}{a_{,}}\cos\tau+\left\{-\frac{45}{16}m-\frac{1089}{128}m^2\right\}\frac{\mathrm{a}}{a_{,}}e\cos(\tau-\xi)$$

$$+\left\{-\frac{135}{16}m-\frac{387}{8}m^2\right\}\frac{\mathrm{a}}{a_{,}}e\cos(\tau+\xi)+\frac{45}{16}m\frac{\mathrm{a}}{a_{,}}e_{,}\cos(\tau-\xi_{,})$$

$$+\left\{\frac{15}{4}-\frac{135}{8}m\right\}\frac{\mathrm{a}}{a_{,}}e_{,}\cos(\tau+\xi)+\frac{75}{64}m^2\frac{\mathrm{a}}{a_{,}}\cos 3\tau$$

$$-\frac{2925}{128}m^2\frac{\mathrm{a}}{a_{,}}e\cos(3\tau-\xi)+\frac{255}{64}m^2\frac{\mathrm{a}}{a_{,}}e\cos(3\tau+\xi)$$

An approximate expression for z is given p. 241, which oug to contain the term

$$+\frac{45}{128}m^2\,\mathrm{a}\,\gamma\,e\sin(4\tau-\xi-\eta),$$

omitted by mistake, and the terms

$$-\frac{57}{64}m^2\,\mathrm{a}\,\gamma\,e_{,}\sin(\xi_{,}-\eta)-\frac{27}{8}m\,\mathrm{a}\,\gamma\,e\,e_{,}\sin(\xi+\xi_{,}-\eta)$$

instead of

$$-\frac{57}{32}m^2\,\mathrm{a}\,\gamma\,e_{,}\sin(\xi_{,}-\eta)-\frac{15}{4}m\,\mathrm{a}\,\gamma\,e\,e_{,}\sin(\xi+\xi_{,}-\eta)$$

Moreover $\frac{a^3}{r^3} + \frac{m_{,} a^3}{\mu r_{,}^3}$ contains the term $\frac{9}{4} m^2 e_{,}^2$ instead of $\frac{3}{2} m^2 e_{,}^2$

$\frac{2025}{128} m^2 e^2 \cos(4\tau - 2\xi)$ instead of $\frac{675}{128} m^2 e^2 \cos(4\tau - 2\xi)$.

Z contains the following terms:

$$\text{in } \eta \left\{ -\frac{9}{32} m^4 + \frac{963}{128} m^2 e^2 + \frac{9}{4} m^2 e_{,}^2 + \left\{ \frac{137}{6} m^4 + \frac{1941}{32} m^2 e^2 - \frac{15}{2} m^2 e_{,}^2 \right\} \cos 2\tau \right.$$

$$+ \left\{ \frac{27}{8} e^2 + \frac{45}{8} m^3 \right\} e \cos \xi + \left\{ \frac{38937}{512} m^3 + \frac{225}{16} m e^2 - \frac{225}{16} m e_{,}^2 \right\} e \cos(2\tau - \xi)$$

$$+ \left\{ \frac{455}{16} m^3 + \frac{2385}{64} m e^2 \right\} e \cos(2\tau + \xi) + \left\{ \frac{471}{8} m^3 + \frac{315}{8} m e^2 \right\} e_{,} \cos(2\tau - \xi_{,})$$

$$+ \left\{ -\frac{91}{8} m^3 - \frac{135}{8} m e^2 \right\} e_{,} \cos(2\tau + \xi_{,}) + \frac{33}{4} m^2 e^2 \cos 2\xi$$

$$+ \frac{321}{32} m^2 e^2 \cos(2\tau - 2\xi) + \frac{327}{16} m^2 e^2 \cos(2\tau + 2\xi) - \frac{2799}{64} m^2 e e_{,} \cos(\xi + \xi_{,})$$

$$+ \frac{4479}{64} m^2 e e_{,} \cos(2\tau - \xi - \xi_{,}) - \frac{147}{32} m^2 e e_{,} \cos(2\tau + \xi + \xi_{,})$$

$$+ \frac{3051}{64} m^2 e e_{,} \cos(\xi - \xi_{,}) - \frac{387}{64} m^2 e e_{,} \cos(2\tau - \xi + \xi_{,})$$

$$+ \frac{1029}{32} m^2 e e_{,} \cos(2\tau + \xi - \xi_{,}) - \frac{9}{4} m^2 e_{,}^2 \cos 2\xi_{,} + \frac{51}{2} m^2 e_{,}^2 \cos(2\tau - 2\xi_{,})$$

$$+ \frac{53}{8} e^3 \cos 3\xi + \frac{135}{64} m e^3 \cos(2\tau - 3\xi) - \frac{189}{8} m e^2 e_{,} \cos(2\xi + \xi_{,})$$

$$+ \frac{105}{8} m e^2 e_{,} \cos(2\tau - 2\xi - \xi_{,}) + \frac{189}{8} m e^2 e_{,} \cos(2\xi - \xi_{,})$$

$$- \frac{45}{8} m e^2 e_{,} \cos(2\tau - 2\xi + \xi_{,}) - \frac{189}{32} m e e_{,}^2 \cos(\xi + 2\xi_{,})$$

$$+ \frac{765}{32} m e e_{,}^2 \cos(2\tau - \xi - 2\xi_{,}) + \frac{189}{32} m e e_{,}^2 \cos(\xi - 2\xi_{,})$$

$$- \frac{135}{32} m e e_{,}^2 \cos(2\tau - \xi + 2\xi_{,}) + \frac{33}{8} m^4 \cos 4\tau + \frac{2925}{128} m^3 e \cos(4\tau - \xi)$$

$$+ \frac{2025}{128} m^2 e^2 \cos(4\tau - 2\xi) + \left\{ -\frac{45}{16} m - \frac{219}{16} m^2 \right\} \frac{a}{a_{,}} \cos \tau$$

$$- \frac{45}{16} m \frac{a}{a_{,}} e \cos(2\tau - \xi) - \frac{135}{16} m \frac{a}{a_{,}} e \cos(\tau + \xi) + \frac{45}{16} m \frac{a}{a_{,}} e_{,} \cos(\tau - \xi_{,})$$

$$\left. + \left\{ \frac{15}{4} - \frac{135}{8} m \right\} \frac{a}{a_{,}} e_{,} \cos(\tau + \xi_{,}) + \frac{75}{64} m^2 \frac{}{a_{,}} \cos 3\tau \right\}$$

$$+ \gamma \left\{ \frac{3}{8} m \sin (2\tau - \eta) + \frac{3}{2} e \sin (\xi - \eta) + \frac{1}{2} e \sin (\xi + \eta) \right\}$$

$$\Big\{ \frac{3}{2} e^2 + \Big\{ \frac{19}{2} m^3 + \frac{135}{8} m e^2 \Big\} \cos 2\tau + \Big\{ \frac{27}{8} e^2 + 3 m^2 \Big\} e \cos \xi$$

$$+ \frac{657}{32} m^2 e \cos (2\tau - \xi) + \frac{147}{16} m^2 e \cos (2\tau + \xi) - \frac{3}{2} m^2 e_{,} \cos \xi_{,}$$

$$+ \frac{21}{2} m^2 e_{,} \cos (2\tau - \xi_{,}) - \frac{3}{2} m^2 e_{,} \cos (2\tau + \xi_{,}) + \frac{9}{2} e^2 \cos 2\xi$$

$$+ \frac{45}{8} m e^2 \cos (2\tau - 2\xi) - \frac{63}{8} m e e_{,} \cos (\xi + \xi_{,})$$

$$+ \frac{105}{8} m e e_{,} \cos (2\tau - \xi - \xi_{,}) + \frac{63}{8} m e e_{,} \cos (\xi - \xi_{,}) - \frac{45}{8} m e e \cos (2\tau - \xi +$$

$$+ \frac{53}{8} e^3 \cos 3\xi - \frac{45}{16} m \frac{a}{a_{,}} \cos \tau \Big\}$$

$$+ \gamma \Big\{ - \frac{e^2}{2} \sin \eta + \frac{41}{32} m^2 \sin (2\tau - \eta) + \frac{3}{16} m^2 \sin (2\tau + \eta)$$

$$+ \frac{9}{4} m e \sin (2\tau - \xi - \eta) + \frac{15}{16} m e \sin (2\tau - \xi + \eta) + \frac{3}{16} m e \sin (2\tau + \xi -$$

$$- \frac{3}{8} m e_{,} \sin (\xi_{,} - \eta) - \frac{3}{8} m e_{,} \sin (\xi_{,} + \eta) + \frac{7}{8} m e_{,} \sin (2\tau - \xi_{,} - \eta)$$

$$- \frac{3}{8} m e_{,} \sin (2\tau + \xi_{,} - \eta) + \frac{e^2}{2} \sin (2\xi - \eta) + \frac{3}{8} e^2 \sin (2\xi + \eta)$$

$$- \frac{45}{32} m \frac{a}{a_{,}} \sin (\tau - \eta) - \frac{15}{32} m \frac{a}{a_{,}} \sin (\tau + \eta) \Big\}$$

$$\Big\{ \frac{3}{2} e^2 + \frac{3}{2} m^2 + 3 m^2 \cos 2\tau + \frac{45}{8} m e \cos (2\tau - \xi) + \frac{9}{2} e^2 \cos 2\xi \Big\}$$

$$+ \gamma \Big\{ - \frac{e^2}{2} \sin \eta + \Big\{ \frac{5389}{1536} m^3 + \frac{3}{32} m e^2 + \frac{15}{16} m e_{,}^2 \Big\} \sin (2\tau - \eta)$$

$$+ \Big\{ \frac{7}{8} m^3 + \frac{45}{32} m e^2 \Big\} \sin (2\tau + \eta) - \frac{231}{128} m^2 e \sin (\xi - \eta)$$

$$+ \frac{3}{4} m^2 e \sin (\xi + \eta) + \frac{291}{32} m^2 e \sin (2\tau - \xi - \eta) + \frac{337}{64} m^2 e \sin (2\tau - \xi + \eta)$$

$$+ \frac{23}{64} m^2 e \sin (2\tau + \xi - \eta) + \frac{3}{8} m^2 e \sin (2\tau + \xi + \eta) - \frac{57^{*}}{64} m^2 e_{,} \sin (\xi_{,} - \eta)$$

---

* This term is calculated correctly in p. 236, but is wrongly transcribed p. 241.

$$- \frac{21}{64} m^2 e_{\prime} \sin (\xi_{\prime} + \eta) + \frac{367}{64} m^2 e_{\prime} \sin (2 \tau - \xi_{\prime} - \eta) + \frac{21}{32} m^2 e_{\prime} \sin (2 \tau - \xi_{\prime} + \eta)$$

$$- \frac{131}{64} m^2 e_{\prime} \sin (2 \tau + \xi_{\prime} - \eta) - \frac{3}{32} m^2 e_{\prime} \sin (2 \tau + \xi_{\prime} + \eta)$$

$$+ \left\{ \frac{1}{2} - \frac{135}{64} m \right\} e^2 \sin (2 \xi - \eta) + \frac{3}{8} e^2 \sin (2 \xi + \eta)$$

$$+ \frac{93}{64} m e^2 \sin (2 \tau - 2 \xi - \eta) + \frac{45}{64} m e^2 \sin (2 \tau - 2 \xi + \eta)$$

$$+ \frac{9}{64} m e^2 \sin (2 \tau + 2 \xi - \eta) - \frac{27}{8} m e e_{\prime} \sin (\xi + \xi_{\prime} - \eta)$$

$$- \frac{3}{2} m e e_{\prime} \sin (\xi + \xi_{\prime} + \eta) + \frac{21}{4} m e e_{\prime} \sin (2 \tau - \xi - \xi_{\prime} - \eta)$$

$$+ \frac{35}{16} m e e_{\prime} \sin (2 \tau - \xi - \xi_{\prime} + \eta) - \frac{3}{16} m e e_{\prime} \sin (2 \tau + \xi + \xi_{\prime} - \eta)$$

$$+ \frac{27}{8} m e e_{\prime} \sin (\xi - \xi_{\prime} - \eta) + \frac{3}{2} m e e_{\prime} \sin (\xi - \xi_{\prime} + \eta)$$

$$- \frac{9}{4} m e e_{\prime} \sin (2 \tau - \xi + \xi_{\prime} - \eta) - \frac{15}{16} m e e_{\prime} \sin (2 \tau - \xi + \xi_{\prime} + \eta)$$

$$+ \frac{7}{16} m e e_{\prime} \sin (2 \tau + \xi - \xi_{\prime} - \eta) - \frac{9}{32} m e_{\prime}^2 \sin (2 \xi_{\prime} - \eta) - \frac{9}{32} m e_{\prime}^2 \sin (2 \xi_{\prime} + \eta)$$

$$+ \frac{51}{32} m e_{\prime}^2 \sin (2 \tau - 2 \xi_{\prime} - \eta) - \frac{9}{32} m e_{\prime}^2 \sin (2 \tau + 2 \xi_{\prime} - \eta)$$

$$\left. + \frac{9}{128} m^3 \sin (4 \tau - \eta) + \frac{45}{128} m^2 e \sin (4 \tau - \xi - \eta) \right\}$$

$$\left\{ 3 e \cos \xi \right\}$$

Therefore $\mathbb{Z}$ contains the following terms:

Terms contained in $\mathbb{Z}$

$$\left\{ \frac{963}{128} - \frac{405}{128} - \frac{9}{4} + \frac{3}{4} - \frac{3}{4} \quad \frac{405}{64} + \frac{675}{256} + \frac{693}{256} + \frac{9}{8} \right\} m^2 e^2$$

$$\left. + \frac{9}{4} m^2 e_{\prime}^2 \right\} \gamma \sin \eta$$

$$\left\{ \frac{137}{12} + \frac{9}{32} \right\} m^4 + \left\{ \frac{1941}{64} - \frac{441}{64} + \frac{657}{128} - \frac{3}{4} + \frac{9}{32} + \frac{1011}{128} + \frac{9}{16} \right\} m^2 e^2$$

$$\left. - \frac{15}{4} m^2 e_{\prime}^2 \right\} \gamma \sin (2 \tau + \eta)$$

$$\left[\left\{ - \frac{45}{16} + \frac{1971}{512} + \frac{1845}{512} - \frac{45}{32} + \frac{9}{32} \right\} m^3 + \left\{ - \frac{27}{16} + \frac{9}{4} - \frac{9}{8} + \frac{3}{4} + \frac{3}{4} \right\} e^2 \right.$$

$$\left. - \frac{405}{128} m e^2 \right\} e \gamma \sin (\xi - \eta)$$

Terms ontained in $\mathbb{Z}$

$$+\left\{\left\{\frac{45}{16}-\frac{441}{256}+\frac{135}{256}-\frac{27}{8}\right\}m^3+\left\{\frac{27}{16}+\frac{3}{4}-\frac{27}{8}-\frac{3}{4}+\frac{9}{16}\right\}e^2\right\}e\,\gamma\sin($$

$$+\left\{\left\{-\frac{38937}{1024}-\frac{19}{8}+\frac{9}{16}+\frac{27}{8}+\frac{5389}{1024}\right\}m^3\right.$$

$$+\left\{-\frac{225}{32}+\frac{81}{128}+\frac{27}{8}-\frac{135}{32}+\frac{27}{64}+\frac{135}{32}+\frac{9}{64}+\frac{279}{128}+\frac{45}{32}\right\}m\,e^2$$

$$\left.+\left\{\frac{225}{32}+\frac{45}{64}\right\}m\,e_{,}^2\right\}e\sin(2\tau-\xi-\eta)$$

$$+\left\{\left\{\frac{38937}{1024}-\frac{57}{8}+\frac{45}{32}+\frac{21}{16}\right\}m^3\right.$$

$$+\left\{\frac{225}{32}+\frac{45}{32}-\frac{405}{32}+\frac{45}{32}-\frac{45}{32}+\frac{135}{128}+\frac{135}{64}\right\}m\,e^2$$

$$\left.-\frac{225}{32}m\,e_{,}^2\right\}e\sin(2\tau-\xi+\eta)$$

$$+\left\{\left\{-\frac{455}{32}+\frac{57}{8}+\frac{9}{16}+\frac{9}{32}+\frac{5389}{1024}\right\}m^3\right.$$

$$+\left\{-\frac{2385}{128}+\frac{81}{128}+\frac{405}{32}+\frac{81}{16}+\frac{45}{32}+\frac{9}{64}+\frac{27}{128}\right\}me^2$$

$$\left.+\frac{45}{64}m\,e_{,}^2\right\}e\sin(2\tau+\xi-\eta)$$

$$+\left\{\left\{\frac{455}{32}+\frac{19}{8}+\frac{21}{16}\right\}m^3+\left\{\frac{2385}{128}+\frac{9}{32}+\frac{135}{32}+\frac{135}{64}+\frac{135}{128}\right.\right.$$

$$\left.\left.+\frac{135}{64}\right\}m\,e^2\right\}e\,\gamma\sin(2\tau+\xi+\eta)$$

$$+\left\{-\frac{9}{16}+\frac{63}{32}+\frac{189}{32}-\frac{81}{16}-\frac{9}{4}\right\}^{*}m\,e^2\,e_{,}\,\gamma\sin(\xi_{,}-\eta)$$

$$+\left\{-\frac{9}{16}+\frac{189}{32}+\frac{63}{32}-\frac{9}{4}-\frac{81}{16}\right\}m\,e^2\,e_{,}\sin(\xi_{,}+\eta)$$

$$+\left\{-\frac{315}{16}+\frac{21}{16}+\frac{315}{32}+\frac{63}{8}+\frac{21}{32}\right\}m\,e^2\,e_{,}\sin(2\tau-\xi_{,}-\eta)$$

$$+\left\{\left\{\frac{471}{16}+\frac{9}{16}\right\}m^3+\left\{\frac{315}{16}+\frac{105}{32}+\frac{105}{32}\right\}m\,e^2\right\}e_{,}\,\gamma\sin(2\tau-\xi_{,}+\eta)$$

---

[*] See p. 236 for the terms in $\mathbb{Z}$ multiplied by m^3 of the arguments 155, 156, and 159. These are required from the nature of the divisors in order to procure terms in s of those arguments multiplied by m^2.

Terms contained in

Z

$$-\frac{9}{16}-\frac{135}{32}-\frac{9}{32}-\frac{27}{8}\Big\}\, m\, e^2 e_, \sin(2\,\tau+\xi_,-\eta)$$

$$\frac{91}{16}-\frac{9}{16}\Big\}\, m^3+\Big\{-\frac{135}{16}-\frac{45}{32}-\frac{45}{32}\Big\}\, m\, e^2\Big\}\, e_,\gamma \sin(2\,\tau+\xi_,+\eta)$$

$$\frac{3}{4}+\frac{9}{16}+\frac{9}{8}\Big\}^{*}\, m^2 e^2 \gamma \sin(2\,\xi+\eta)$$

$$\frac{7}{2}+\frac{441}{64}+\frac{69}{128}+\frac{3}{4}+\frac{369}{128}\Big\}\, m^2 e^2 \sin(2\,\tau+2\,\xi-\eta)$$

$$+\frac{147}{64}+\frac{9}{16}+\frac{9}{16}+\frac{27}{64}\Big\}\, m^2 e^2 \sin(2\,\tau+2\,\xi+\eta)$$

$$+\frac{315}{128}-\frac{9}{8}-\frac{135}{128}-\frac{171}{128}\Big\}\, m^2 e\, e_, \sin(\xi+\xi_,-\eta)$$

$$\frac{799}{28}-\frac{3}{8}-\frac{63}{128}\Big\}\, m^2 e\, e_, \sin(\xi+\xi_,+\eta)$$

$$\frac{179}{28}-\frac{189}{128}-\frac{21}{8}+\frac{135}{128}+\frac{1101}{128}\Big\}\, m^2 e\, e_, \sin(2\,\tau-\xi-\xi_,-\eta)$$

$$-\frac{63}{8}+\frac{135}{128}+\frac{63}{64}\Big\}\, m^2 e\, e_, \sin(2\,\tau-\xi-\xi_,+\eta)$$

$$-\frac{189}{128}-\frac{9}{8}-\frac{398}{128}\Big\}\, m^2 e\, e_, \sin(2\,\tau+\xi+\xi_,+\eta)$$

$$\frac{47}{4}-\frac{3}{8}-\frac{9}{64}\Big\}\, m^2 e\, e_, \sin(2\,\tau+\xi+\xi_,+\eta)$$

$$\frac{051}{28}-\frac{135}{128}-\frac{9}{8}+\frac{315}{128}+\frac{63}{128}\Big\}\, m^2 e\, e_, \sin(\xi-\xi_,-\eta)$$

$$-\frac{3}{8}+\frac{171}{128}\Big\}\, m^2 e\, e_, \sin(\xi-\xi_,+\eta)$$

$$+\frac{189}{128}+\frac{3}{8}-\frac{135}{128}-\frac{393}{128}\Big\}\, m^2 e\, e_, \sin(2\,\tau-\xi+\xi_,-\eta)$$

$$\frac{87}{28}+\frac{9}{8}-\frac{135}{128}-\frac{9}{64}\Big\}\, m^2 e\, e_, \sin(2\,\tau-\xi+\xi_,+\eta)$$

$$\frac{029}{64}+\frac{189}{128}+\frac{63}{8}+\frac{1101}{128}\Big\}\, m^2 e\, e_, \sin(2\,\tau+\xi-\xi_,-\eta)$$

$$+\frac{21}{8}+\frac{63}{64}\Big\}\, m^2 e\, e_, \sin(2\,\tau+\xi-\xi_,+\eta)$$

$$-\frac{135}{256}\Big\}\, m^2 \frac{a}{a_,}\gamma \sin(\tau-\eta)-\frac{219}{32}\, m^2 \frac{a}{a_,}\gamma \sin(\tau+\eta)$$

$$-\frac{135}{64}+\frac{45}{64}\Big\}\, m\, \frac{a}{a_,} e\,\gamma \sin(\tau-\xi-\eta)$$

---

* For terms of the arguments 161, 163, and 240, see p. 46.

Terms contained in
Z

$$+\left\{-\frac{45}{32}-\frac{45}{64}+\frac{135}{64}\right\} m\frac{a}{a_{,}} e\gamma\sin(\tau-\xi+\eta)$$

$$+\left\{\frac{135}{32}-\frac{135}{64}-\frac{135}{64}\right\} m\frac{a}{a_{,}} e\gamma\sin(\tau+\xi-\eta)$$

$$+\left\{-\frac{135}{32}-\frac{45}{64}-\frac{45}{64}\right\} m\frac{a}{a_{,}} e\gamma\sin(\tau+\xi+\eta)$$

$$-\frac{45}{32} m\frac{a}{a_{,}} e_{,}\gamma\sin(\tau-\xi_{,}-\eta)+\frac{45}{32} m\frac{a}{a_{,}} e_{,}\gamma\sin(\tau-\xi_{,}+\eta)$$

$$+\left\{\frac{15}{8}-\frac{135}{16}m\right\}\frac{a}{a_{,}} e\gamma\sin(\tau+\xi_{,}-\eta)+\left\{\frac{15}{8}-\frac{135}{16}m\right\}\frac{a}{a_{,}} e_{,}\gamma\sin(\tau+$$

$$+\left\{-\frac{53}{16}+\frac{27}{8}+\frac{3}{4}-\frac{405}{128}m\right\} e^3\gamma\sin(3\xi-\eta)$$

$$+\left\{\frac{53}{16}+\frac{9}{8}+\frac{9}{16}\right\} e^3\gamma\sin(3\xi+\eta)$$

$$+\left\{-\frac{135}{128}+\frac{159}{128}+\frac{81}{16}-\frac{45}{32}-\frac{135}{128}+\frac{279}{128}\right\} m e^3\gamma\sin(2\tau-3\xi-\eta)$$

$$+\left\{\frac{135}{128}+\frac{135}{64}-\frac{135}{32}-\frac{45}{32}+\frac{135}{128}-\frac{45}{32}\right\} m e^3\gamma\sin(2\tau-3\xi+\eta)$$

$$+\left\{\frac{27}{64}+\frac{159}{128}+\frac{27}{128}\right\} m e^3\gamma\sin(2\tau+3\xi-\eta)$$

$$+\left\{\frac{189}{16}-\frac{27}{32}-\frac{189}{32}-\frac{81}{16}\right\} m e^2 e_{,}\gamma\sin(2\xi+\xi_{,}-\eta)$$

$$+\left\{-\frac{189}{16}-\frac{27}{32}-\frac{63}{32}-\frac{9}{4}\right\} m e^2 e_{,}\gamma\sin(2\xi+\xi_{,}+\eta)$$

$$+\left\{-\frac{105}{16}+\frac{63}{32}-\frac{105}{32}+\frac{63}{8}\right\} m e^2 e_{,}\gamma\sin(2\tau-2\xi-\xi_{,}-\eta)$$

$$+\left\{\frac{105}{16}-\frac{315}{32}+\frac{105}{32}\right\} m e^2 e_{,}\gamma\sin(2\tau-2\xi-\xi_{,}+\eta)$$

$$+\left\{-\frac{27}{32}-\frac{9}{32}\right\} m e^2 e_{,}\gamma\sin(2\tau+2\xi+\xi_{,}-\eta)$$

$$+\left\{-\frac{189}{16}+\frac{27}{32}+\frac{189}{32}+\frac{81}{16}\right\} m e^2 e_{,}\gamma\sin(2\xi-\xi_{,}-\eta)$$

$$+\left\{\frac{189}{16}+\frac{27}{32}+\frac{63}{32}+\frac{9}{4}\right\} m e^2 e_{,}\gamma\sin(2\xi-\xi_{,}+\eta)$$

$$+\left\{\frac{45}{16}-\frac{27}{32}+\frac{45}{32}-\frac{27}{8}\right\} m e^2 e_{,}\gamma\sin(2\tau-2\xi+\xi_{,}-\eta)$$

$$+\left\{-\frac{45}{16}+\frac{135}{32}-\frac{45}{32}\right\} m e^2 e_{,}\gamma\sin(2\tau-2\xi+\xi_{,}+\eta)$$

$$+\left\{\frac{63}{32}+\frac{21}{32}\right\} m e^2 e_{,}\gamma\sin(2\tau+2\xi-\xi_{,}-\eta)$$

Terms contained $\mathbb{Z}$

$$+\left\{\frac{189}{64}-\frac{27}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(\xi+2\,\xi_{,}-\eta)$$

$$+\left\{-\frac{189}{64}-\frac{27}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(\xi+2\,\xi_{,}+\eta)$$

$$+\left\{-\frac{765}{64}+\frac{153}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(2\,\tau-\xi-2\,\xi_{,}-\eta)$$

$$+\frac{765}{64}\,m\,e\,e_{,}^2\,\gamma \sin(2\,\tau-\xi-2\,\xi_{,}+\eta)-\frac{27}{64}\,m\,e\,e_{,}^2\,\gamma \sin(2\,\tau+\xi+2\,\xi_{,}+\eta)$$

$$+\left\{-\frac{189}{64}+\frac{27}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(\xi-2\,\xi_{,}-\eta)$$

$$+\left\{\frac{189}{64}+\frac{27}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(\xi-2\,\xi_{,}+\eta)$$

$$+\left\{\frac{135}{64}-\frac{27}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(2\,\tau-\xi+2\,\xi_{,}-\eta)$$

$$-\frac{135}{64}\,m\,e\,e_{,}^2\,\gamma \sin(2\,\tau-\xi+2\,\xi_{,}+\eta)+\frac{153}{64}\,m\,e\,e_{,}^2\,\gamma \sin(2\,\tau+\xi-2\,\xi_{,}-\eta)$$

$$+\left\{-\frac{75}{128}-\frac{135}{256}\right\} m^2\,\frac{\mathrm{a}}{a_{,}}\,\gamma \sin(3\,\tau-\eta)+\frac{75}{128}\,m^2\,\frac{\mathrm{a}}{a_{,}}\,\gamma \sin(3\,\tau+\eta)$$

$$+\left\{\left\{-\frac{33}{16}+\frac{57}{32}+\frac{123}{64}\right\} m^4+\left\{\frac{405}{128}+\frac{135}{256}+\frac{135}{256}\right\} m^2 e^2\right\} \gamma \sin(4\,\tau-\eta)$$

$$+\left\{\frac{33}{16}+\frac{9}{32}\right\} m^4\,\gamma \sin(4\,\tau+\eta)+\left\{\frac{1971}{512}+\frac{27}{8}+\frac{1845}{512}+\frac{27}{256}\right.$$

$$\left.-\frac{2925}{256}\right\} m^3\,e\,\gamma \sin(4\,\tau-\xi-\eta)$$

$$+\left\{\frac{45}{32}+\frac{2925}{256}+\frac{135}{256}\right\} m^3\,e\,\gamma \sin(4\,\tau-\xi+\eta)$$

$$+\left\{\frac{441}{256}+\frac{9}{32}+\frac{27}{256}\right\} m^3\,e\,\gamma \sin(4\,\tau+\xi-\eta)$$

$$+\left\{\frac{63}{32}+\frac{21}{16}\right\} m^3\,e_{,}\,\gamma \sin(4\,\tau-\xi_{,}-\eta)+\left\{-\frac{9}{32}-\frac{9}{16}\right\} m^3\,e_{,}\,\gamma \sin(4\,\tau+\xi_{,}\quad\eta)$$

$$+\left\{-\frac{2025}{256}+\frac{135}{128}+\frac{405}{64}+\frac{135}{256}\right\} m^2\,e^2\,\gamma \sin(4\,\tau-2\,\xi-\eta)$$

$$+\left\{\frac{2025}{256}+\frac{675}{256}\right\} m^2\,e^2\,\gamma \sin(4\,\tau-2\,\xi+\eta)$$

$$+\left\{-\frac{135}{128}-\frac{135}{128}\right\} m^2\,e\,e_{,}\,\gamma \sin(4\,\tau-\xi+\xi_{,}-\eta)$$

$$+\left\{\frac{315}{128}+\frac{315}{128}\right\} m^2\,e\,e_{,}\,\gamma \sin(4\,\tau-\xi-\xi_{,}-\eta)$$

In addition to $\frac{z}{\mathrm{a}}$ the latitude of the moon contains the follo terms:

$$\gamma \sin \eta \Big\{ -\frac{179}{288} m^4 + \frac{m^2 e_,^2}{2} + \Big\{ \frac{131}{18} m^4 + \frac{189}{16} m^2 e_,^2 \quad \frac{5}{2} m^2 e_,^2 \Big\} \cos 2\tau$$

$$+ \Big\{ \frac{33113}{1536} m^3 - \frac{75}{16} m e_,^2 \Big\} e \cos (2\tau - \xi) + \Big\{ \frac{101}{16} m^3 + \frac{405}{64} m e^2 \Big\} e \cos (2\tau + \xi)$$

$$+ \Big\{ \frac{157}{8} m^3 + \frac{35}{4} m e^2 \Big\} e_, \cos (2\tau - \xi_,) + \Big\{ -\frac{91}{24} m^3 - \frac{15}{4} m e^2 \Big\} e \cos (2\tau + \xi_,)$$

$$+ \frac{5}{3} m^2 e^2 \cos 2\xi - \frac{15}{4} m^2 e^2 \cos (2\tau - 2\xi) + \frac{7}{2} m^2 e^2 \cos (2\tau + 2\xi)$$

$$- \frac{837}{64} m^2 e e_, \cos (\xi + \xi_,) + \frac{1269}{64} m^2 e e_, \cos (2\tau - \xi - \xi_,) - \frac{33}{32} m^2 e e_, \cos (2\tau + \xi + \xi_,)$$

$$+ \frac{1113}{64} m^2 e e_, \cos (\xi - \xi_,) - \frac{97}{64} m^2 e e_, \cos (2\tau - \xi + \xi_,) + \frac{231}{32} m^2 e e_, \cos (2\tau + \xi - \xi_,)$$

$$- \frac{9}{4} m^2 e_,^2 \cos 2\xi_, + \frac{17}{2} m^2 e_,^2 \cos (2\tau - 2\xi_,) + \frac{9}{8} e^3 \cos 3\xi - \frac{105}{64} m e^3 \cos (2\tau - 3\xi)$$

$$- \frac{21}{4} m e^2 e_, \cos (2\xi + \xi_,) + \frac{21}{4} m e^2 e_, \cos (2\xi - \xi_,) - \frac{63}{32} m e e_,^2 \cos (\xi + 2\xi_,)$$

$$+ \frac{255}{32} m e e_,^2 \cos (2\tau - \xi - 2\xi_,) + \frac{63}{32} m e e_,^2 \cos (\xi - 2\xi_,)$$

$$- \frac{45}{32} m e e_,^2 \cos (2\tau - \xi + 2\xi_,) + \frac{7}{8} m^4 \cos 4\tau + \frac{495}{128} m^3 e \cos (4\tau - \xi)$$

$$+ \frac{225}{64} m^2 e^2 \cos (4\tau - 2\xi)$$

$$- \frac{81}{16} m^2 \frac{a}{a_,} \cos \tau + \frac{25}{64} m^2 \frac{a}{a_,} \cos 3\tau \Big\}$$

$$+ \gamma \Big\{ \frac{3}{8} m \sin (2\tau - \eta) + \frac{3}{2} e \sin (\xi - \eta) + \frac{1}{2} e \sin (\xi + \eta) \Big\}$$

$$\Big\{ \Big\{ \frac{19}{6} m^3 + \frac{15}{4} m e^2 \Big\} \cos 2\tau + \Big\{ -\frac{e^2}{8} + \frac{2}{3} m^2 \Big\} e \cos \xi + \frac{187}{32} m^2 e \cos (2\tau - \xi)$$

$$+ \frac{33}{16} m^2 e \cos (2\tau + \xi) - \frac{3}{2} m^2 e_, \cos \xi_, + \frac{7}{2} m^2 e_, \cos (2\tau - \xi_,)$$

$$- \frac{m^2}{2} e_, \cos (2\tau + \xi_,) + e^2 \cos 2\xi - \frac{21}{8} m e e_, (\xi + \xi_,) + \frac{35}{8} m e e_, \cos (2\tau - \xi - \xi_,)$$

$$+ \frac{21}{8} m e e_, \cos (\xi - \xi_,) - \frac{15}{8} m e e_, \cos (2\tau - \xi + \xi_,) + \frac{9}{8} e^3 \cos 3\xi \Big\}$$

$$+ \gamma \Big\{ -\frac{e^2}{2} \sin \eta + \frac{41}{32} m^2 \sin (2\tau - \eta) + \frac{3}{16} m^2 \sin (2\tau + \eta)$$

$$+\frac{9}{4}\,m\,e\sin(2\,\tau-\xi-\eta)+\frac{15}{16}m\,e\sin(2\,\tau-\xi+\eta)+\frac{3}{16}m\,e\sin(2\,\tau+\xi-\eta)$$

$$-\frac{3}{8}\,m\,e_{,}\sin(\xi_{,}-\eta)-\frac{3}{8}m\,e_{,}\sin(\xi_{,}+\eta)+\frac{7}{8}m\,e_{,}\sin(2\,\tau-\xi_{,}-\eta)$$

$$-\frac{3}{8}m\,e_{,}\sin(2\,\tau+\xi_{,}-\eta)+\frac{1}{2}e^2\sin(2\,\xi-\eta)+\frac{3}{8}\,e^2\sin(2\,\xi+\eta)$$

$$-\frac{45}{32}m\,\frac{\mathrm{a}}{a_{,}}\sin(\tau-\eta)-\frac{15}{32}m\,\frac{\mathrm{a}}{a_{,}}\sin(\tau+\eta)\Big\}$$

$$\Big\{\frac{m^2}{6}+m^2\cos 2\,\tau+\frac{15}{8}m\,e\cos(2\,\tau-\xi)+e^2\cos 2\,\xi\Big\}$$

$$+\gamma\Bigg\{-\frac{e^2}{2}\sin\eta+\Big\{\frac{5389}{1536}m^3+\frac{3}{32}m\,e^2+\frac{15}{16}\,m\,e_{,}^{2}\Big\}\sin(2\,\tau-\eta)$$

+ terms which are recapitulated in p. 296.$\Bigg\}$

$$\{e\cos\xi\}$$

Hence the latitude contains the following terms:

Terms
the latitu
the mo

$$+\Bigg\{\Big\{-\frac{179}{288}-\frac{19}{32}-\frac{41}{64}+\frac{3}{32}\Big\}m^4+\Big\{-\frac{45}{64}-\frac{1}{2}+\frac{1}{6}-\frac{1}{12}-\frac{135}{64}+\frac{225}{256}$$

$$+\frac{231}{256}+\frac{3}{8}\Big\}m^2e^2+\frac{m^2e_{,}^{2}}{4}\Bigg\}\gamma\sin\eta$$

$$+\Bigg\{z_{147}+\Big\{-\frac{131}{36}+\frac{41}{192}\Big\}m^4+\Big\{-\frac{189}{32}+\frac{561}{128}-\frac{33}{64}+\frac{1}{4}+\frac{23}{128}+\frac{291}{64}$$

$$+\frac{69}{128}\Big\}m^2e^2+\frac{5}{4}m^2e_{,}^{2}\Bigg\}\gamma\sin(2\,\tau-\eta)$$

$$+\Bigg\{\Big\{\frac{989}{384}+\frac{131}{36}+\frac{1}{32}\Big\}m^4+\Big\{\frac{855}{128}+\frac{189}{32}-\frac{99}{64}+\frac{187}{128}-\frac{1}{4}+\frac{337}{128}+\frac{3}{16}\Big\}m^2e^2$$

$$+\Big\{-\frac{15}{32}-\frac{5}{4}\Big\}\,m^2e_{,}^{2}\Bigg\}\gamma\sin(2\,\tau+\eta)$$

$$+\Bigg\{\Big\{-\frac{225}{64}+\frac{561}{512}+\frac{615}{512}-\frac{15}{32}+\frac{3}{32}\Big\}m^3+\Big\{-\frac{15}{16}+\frac{1}{16}-\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\Big\}e^2$$

$$+\Big\{\frac{405}{128}-\frac{135}{128}\Big\}m\,e^2\Bigg\}e\,\gamma\sin(\xi-\eta)$$

Term m
ıe latitude of
the moon.

$$+ \left\{ \left\{ \frac{1173}{128} - \frac{99}{256} + \frac{45}{256} - \frac{9}{8} \right\} m^3 + \left\{ - \frac{3}{8} - \frac{1}{16} - \frac{3}{4} - \frac{1}{4} \right. \right.$$

$$\left. \left. + \frac{3}{16} \right\} e^2 \right\} e \gamma \sin (\xi + \eta)$$

$$+ \left\{ \left\{ \frac{9433}{256} - \frac{33113}{3072} + \frac{1}{8} - \frac{19}{24} + \frac{3}{8} + \frac{5389}{3072} \right\} m^3 \right.$$

$$+ \left\{ - \frac{9}{32} - \frac{15}{16} + \frac{3}{32} - \frac{3}{128} + \frac{3}{64} + \frac{93}{128} + \frac{15}{32} \right\} m e^2$$

$$\left. + \left\{ - \frac{495}{64} + \frac{75}{32} + \frac{15}{32} \right\} m e_{,}^2 \right\} e \gamma \sin (2 \tau - \xi - \eta)$$

$$+ \left\{ \left\{ \frac{67961}{3072} + \frac{33113}{3702} - \frac{19}{8} + \frac{5}{32} + \frac{7}{16} \right\} m^3 + \left\{ - \frac{45}{128} - \frac{45}{16} + \frac{45}{128} - \frac{15}{32} + \frac{45}{64} \right\} m \right.$$

$$\left. + \left\{ - \frac{75}{32} - \frac{75}{32} \right\} m e_{,}^2 \right\} e \gamma \sin (2 \tau - \xi + \eta)$$

$$+ \left\{ \left\{ \frac{2317}{3072} - \frac{101}{32} + \frac{1}{8} + \frac{19}{8} + \frac{1}{32} + \frac{5389}{3072} \right\} m^3 \right.$$

$$+ \left\{ \frac{63}{128} - \frac{405}{128} + \frac{45}{16} + \frac{9}{8} - \frac{3}{128} + \frac{3}{64} + \frac{9}{128} + \frac{15}{32} \right\} m e^2$$

$$\left. + \left\{ \frac{15}{64} + \frac{15}{32} \right\} m e_{,}^2 \right\} e \gamma \sin (2 \tau + \xi - \eta)$$

$$+ \left\{ \left\{ \frac{51}{32} + \frac{101}{32} + \frac{19}{24} + \frac{7}{16} \right\} m^3 + \left\{ \frac{303}{160} + \frac{405}{128} + \frac{15}{16} \right. \right.$$

$$\left. \left. + \frac{15}{32} + \frac{45}{128} + \frac{45}{64} \right\} m e^2 \right\} e \gamma \sin (2 \tau + \xi + \eta)$$

$$+ \left\{ z_{155} + \left\{ \frac{21}{32} - \frac{1}{16} - \frac{3}{16} \right\} m^3 + \left\{ \frac{21}{32} + \frac{63}{32} - \frac{27}{16} - \frac{3}{4} \right\} m e^2 \right\} e_{,} \gamma \sin (\xi_{,} - \eta)$$

$$+ \left\{ z_{156} + \left\{ \frac{3}{32} - \frac{1}{16} - \frac{7}{16} \right\} m^3 + \left\{ \frac{63}{32} + \frac{21}{32} - \frac{3}{4} - \frac{27}{16} \right. \right.$$

$$\left. \left. - \frac{105}{128} \right\} m e^2 \right\} e_{,} \gamma \sin (\xi_{,} + \eta)$$

$$+ \left\{ z_{157} + \left\{ - \frac{157}{16} - \frac{9}{32} + \frac{7}{48} + \frac{3}{16} \right\} m^3 + \left\{ - \frac{35}{8} + \frac{105}{32} + \frac{21}{8} \right. \right.$$

$$\left. \left. + \frac{7}{32} \right\} m e^2 \right\} e_{,} \gamma \sin (2 \tau - \xi_{,} - \eta)$$

Terms i
the latitud
the moo

$$+\left\{\left\{\frac{669}{128}+\frac{157}{16}+\frac{3}{16}\right\}m^3+\left\{\frac{105}{32}+\frac{35}{8}+\frac{35}{32}\right.\right.$$

$$\left.\left.+\frac{35}{32}\right\}m\,e^2\right\}e_{,}\gamma\sin(2\tau-\xi_{,}+n)$$

$$+\left\{z_{159}+\left\{\frac{91}{48}-\frac{9}{32}-\frac{1}{16}-\frac{3}{16}\right\}m^3+\left\{\frac{15}{8}-\frac{45}{32}-\frac{3}{32}\right.\right.$$

$$\left.\left.-\frac{9}{8}\right\}m\,e^2\right\}e_{,}\gamma\sin(2\tau+\xi_{,}-n)$$

$$+\left\{\left\{-\frac{109}{128}-\frac{91}{48}-\frac{3}{16}\right\}m^3+\left\{-\frac{45}{32}-\frac{15}{8}-\frac{15}{32}-\frac{15}{32}\right\}m\,e^2\right\}e_{,}\gamma\sin(2\tau+\xi_{,}+n)$$

$$+\left\{z_{161}-\frac{5}{6}+\frac{1}{2}+\frac{45}{256}+\frac{1}{12}-\frac{231}{256}\right\}m^2e^2\sin(2\xi-n)$$

$$+\left\{\frac{33}{32}+\frac{5}{6}+\frac{1}{6}+\frac{1}{16}+\frac{3}{8}\right\}m^2e^2\sin(2\xi+n)$$

$$+\left\{z_{163}+\frac{15}{8}-\frac{187}{128}+\frac{41}{64}-\frac{3}{16}+\frac{291}{64}\right\}m^2e^2\sin(2\tau-2\xi-n)$$

$$+\left\{z_{164}-\frac{15}{8}-\frac{561}{128}+\frac{3}{32}+\frac{337}{128}-\frac{1}{4}\right\}m^2e^2\sin(2\tau-2\xi+n)$$

$$+\left\{\frac{81}{256}-\frac{7}{4}+\frac{99}{64}+\frac{41}{64}+\frac{1}{4}+\frac{23}{128}\right\}m^2e^3\gamma\sin(2\tau+2\xi-n)$$

$$+\left\{\frac{75}{128}+\frac{7}{4}+\frac{33}{64}+\frac{3}{32}+\frac{3}{16}+\frac{3}{16}\right\}m^2e^3\gamma\sin(2\tau+2\xi+n)$$

$$+\left\{-\frac{333}{16}+\frac{837}{128}+\frac{105}{128}-\frac{9}{8}-\frac{45}{128}-\frac{57}{128}\right\}m^2e\,e_{,}\gamma\sin(\xi+\xi_{,}-n)$$

$$+\left\{-\frac{357}{64}-\frac{837}{128}-\frac{3}{8}-\frac{21}{128}\right\}m^2e\,e_{,}\gamma\sin(\xi+\xi_{,}+n)$$

$$+\left\{\frac{471}{16}-\frac{1269}{128}-\frac{63}{128}-\frac{7}{8}+\frac{45}{128}+\frac{367}{128}\right\}m^2e\,e_{,}\gamma\sin(2\tau-\xi-\xi_{,}-n)$$

$$+\left\{\frac{591}{32}+\frac{1269}{128}-\frac{21}{8}+\frac{45}{128}+\frac{21}{64}\right\}m^2e\,e_{,}\gamma\sin(2\tau-\xi-\xi_{,}+n)$$

$$+\left\{-\frac{11}{8}+\frac{33}{64}-\frac{63}{128}-\frac{3}{8}-\frac{131}{128}\right\}m^2e\,e_{,}\gamma\sin(2\tau+\xi+\xi_{,}-n)$$

$$+\left\{-\frac{3}{16}-\frac{33}{64}-\frac{1}{8}-\frac{3}{64}\right\}m^2e\,e_{,}\gamma\sin(2\tau+\xi+\xi_{,}+n)$$

$$+\left\{\frac{369}{16}-\frac{1113}{128}-\frac{45}{[illegible]}-\frac{9}{8}+\frac{105}{128}+\frac{21}{128}\right\}m^2e\,e_{,}\gamma\sin(\xi\quad\xi_{,}-n)$$

$$+\left\{\frac{657}{64}\ [illegible]\ \frac{57}{[illegible]}\right\}\ [illegible]\ \xi-\xi_{,}+n)$$

Terms in the latitude of the moon.

$$+\left\{-\frac{3}{4}+\frac{97}{128}+\frac{63}{128}+\frac{1}{8}-\frac{45}{128}-\frac{131}{128}\right\} m^2 e e_{,} \gamma \sin (2\tau-\xi+\xi_{,}-\eta)$$

$$+\left\{-\frac{73}{32}-\frac{97}{128}+\frac{3}{8}-\frac{45}{128}-\frac{3}{64}\right\} m^2 e e_{,} \gamma \sin (2\tau-\xi+\xi_{,}+\eta)$$

$$+\left\{\frac{27}{8} \quad \frac{231}{64}+\frac{63}{128}+\frac{21}{8}+\frac{367}{128}\right\} m^2 e e_{,} \gamma \sin (2\tau+\xi \quad \xi_{,}-\eta)$$

$$+\left\{\frac{21}{16}+\frac{231}{64}+\frac{7}{8}+\frac{21}{64}\right\} m^2 e e_{,} \gamma \sin (2\tau+\xi \quad \xi_{,}+\eta)$$

$$+\left\{z_{179}+\frac{9}{8}\right\} m^2 e_{,}^2 \gamma \sin (2\xi_{,}-\eta)$$

$$+\left\{z_{180}-\frac{9}{8}\right\} m^2 e_{,}^2 \gamma \sin (2\xi_{,}+\eta)$$

$$+\left\{z_{181}-\frac{17}{4}\right\} m^2 e_{,}^2 \gamma \sin (2\tau \quad 2\xi_{,}-\eta)$$

$$+\left\{\frac{51}{32}+\frac{17}{4}\right\} m^2 e_{,}^2 \gamma \sin (2\tau-2\xi_{,}+\eta)$$

$$+\left\{-\frac{1425}{256}+\frac{81}{32} \quad \frac{45}{256}\right\} m^2 \frac{a}{a_{,}} \gamma \sin (\tau-\eta)$$

$$+\left\{-\frac{85}{32}-\frac{81}{32}\right\} m^2 \frac{a}{a_{,}} \gamma \sin (\tau+\eta)$$

$$+\left\{z_{187}+\frac{15}{64}-\frac{45}{64}\right\} m e \gamma \frac{a}{a_{,}} \sin (\tau-\xi-\eta)$$

$$+\left\{z_{188}-\frac{45}{64}-\frac{15}{64}\right\} m e \gamma \frac{a}{a_{,}} \sin (\tau-\xi+\eta)$$

$$+\left\{z_{189}-\frac{15}{16}-\frac{135}{64}-\frac{45}{64}\right\} m e \gamma \frac{a}{a_{,}} \sin (\tau+\xi-\eta)$$

$$+\left\{-\frac{45}{64}-\frac{15}{16}-\frac{15}{64}-\frac{15}{64}\right\} m e \gamma \frac{a}{a_{,}} \sin (\tau+\xi+\eta)$$

$$+\left\{\frac{45}{32}-\frac{15}{32}\right\} m e_{,} \gamma \frac{a}{a_{,}} \sin (\tau-\xi_{,}-\eta)$$

$$+\left\{\frac{15}{32}+\frac{15}{32}\right\} m e_{,} \gamma \frac{a}{a_{,}} \sin (\tau-\xi_{,}+\eta)$$

$$+\left\{\frac{15}{8}-\frac{5}{8}+\left\{-\frac{135}{16}+\frac{45}{16}\right\} m\right\} e_{,} \gamma \frac{a}{a_{,}} \sin (\tau+\xi_{,}-\eta)$$

$$+\left\{\frac{5}{8}+\frac{5}{8}+\left\{-\frac{45}{16}-\frac{45}{16}\right\} m\right\} e_{,} \gamma \frac{a}{a_{,}} \sin (\tau+\xi_{,}+\eta)$$

$$+\left\{\frac{13}{48}-\frac{9}{16}+\frac{3}{4}+\frac{1}{4}+\left\{-\frac{135}{128}-\frac{135}{128}\right\} m\right\} e^3 \gamma \sin (3\xi-\eta)$$

Terms in
the latitude
the moon,

$$\left.\frac{1}{3}+\frac{9}{16}+\frac{1}{4}+\frac{3}{16}\right\} e^3\,\gamma \sin(3\,\xi+\eta)$$

$$\left.\frac{53}{32}+\frac{105}{128}+\frac{27}{128}+\frac{9}{8}+\frac{93}{128}-\frac{45}{128}\right\} m\,e^3\,\gamma \sin(2\,\tau-3\,\xi-\eta)$$

$$\left.\frac{45}{32}-\frac{105}{128}+\frac{15}{32}+\frac{45}{128}\quad\frac{15}{32}\right\} m\,e^3\,\gamma \sin(2\,\tau-3\,\xi+\eta)$$

$$\left.\frac{1}{8}+\frac{3}{32}+\frac{27}{128}+\frac{9}{128}\right\} m\,e^3\,\gamma \sin(2\,\tau+3\,\xi-\eta)$$

$$\left.z_{201}+\frac{21}{8}-\frac{3}{16}-\frac{63}{32}-\frac{27}{16}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\xi+\xi_{,}-\eta)$$

$$\left.-\frac{135}{64}-\frac{21}{8}-\frac{3}{16}-\frac{21}{32}-\frac{3}{4}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\xi+\xi_{,}+\eta)$$

$$\left.z_{203}+\frac{7}{16}-\frac{35}{32}+\frac{21}{8}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\tau-2\,\xi-\xi_{,}-\eta)$$

$$\left.z_{204}-\frac{105}{32}+\frac{35}{32}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\tau-2\,\xi-\xi_{,}+\eta)$$

$$\left.-\frac{9}{64}-\frac{3}{16}-\frac{3}{32}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\tau+2\,\xi+\xi_{,}-\eta)$$

$$\left.z_{207}-\frac{21}{8}+\frac{3}{16}+\frac{63}{32}+\frac{27}{16}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\xi-\xi_{,}-\eta)$$

$$\left.\frac{135}{64}+\frac{21}{8}+\frac{3}{16}+\frac{21}{32}+\frac{3}{4}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\xi-\xi_{,}+\eta)$$

$$\left.z_{209}-\frac{3}{16}+\frac{15}{32}-\frac{9}{8}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\tau-2\,\xi+\xi_{,}-\eta)$$

$$\left.z_{210}+\frac{45}{32}-\frac{15}{32}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\tau-2\,\xi+\xi_{,}+\eta)$$

$$\left.\frac{21}{64}+\frac{7}{16}+\frac{7}{32}\right\} m\,e^2\,e_{,}\,\gamma \sin(2\,\tau+2\,\xi-\xi_{,}-\eta)$$

$$\left.-\frac{81}{32}+\frac{63}{64}-\frac{9}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(\xi+2\,\xi_{,}-\eta)$$

$$\left.-\frac{9}{8}-\frac{63}{64}-\frac{9}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(\xi+2\,\xi_{,}+\eta)$$

$$\left.\frac{153}{16}-\frac{255}{64}+\frac{51}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(2\,\tau-\xi-2\,\xi_{,}-\eta)$$

$$\left.\frac{255}{64}+\frac{255}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(2\,\tau-\xi-2\,\xi_{,}+\eta)$$

$$\left.-\frac{9}{64}-\frac{9}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(2\,\tau+\xi+2\,\xi_{,}-\eta)$$

$$\left.\frac{81}{32}-\frac{63}{64}+\frac{9}{64}\right\} m\,e\,e_{,}^2\,\gamma \sin(\xi-2\,\xi_{,}-\eta)$$

Terms in the latitude of the moon.

$$+\left\{\frac{9}{8}+\frac{63}{64}+\frac{9}{64}\right\} m\,e\,e_{,}^{2}\sin(\xi-2\xi_{,}+\eta)$$

$$+\left\{-\frac{27}{16}+\frac{45}{64}-\frac{9}{64}\right\} m\,e\,e_{,}^{2}\sin(2\tau-\xi_{,}+2\xi_{,}-\eta)$$

$$+\left\{-\frac{45}{64}-\frac{45}{64}\right\} m\,c\,e_{,}^{2}\sin(2\tau-\xi+2\xi_{,}+\eta)$$

$$+\left\{\frac{51}{64}+\frac{51}{64}\right\} m\,e\,e_{,}^{2}\sin(2\tau+\xi-2\xi_{,}-\eta)$$

$$+\left\{-\frac{95}{256}-\frac{25}{128}-\frac{45}{256}\right\} m^{2}\,\frac{\mathrm{a}}{a_{,}}\sin(3\tau-\eta)$$

$$+\left\{\frac{5}{128}+\frac{25}{128}\right\} m^{2}\,\frac{\mathrm{a}}{a_{,}}\,\gamma\sin(3\tau+\eta)$$

$$+\left\{\left\{\frac{213}{512}-\frac{7}{16}+\frac{19}{32}+\frac{41}{64}\right\} m^{4}+\left\{\frac{135}{256}+\frac{45}{64}+\frac{45}{256}+\frac{45}{256}\right\} m^{2}e^{2}\right\}\gamma\sin(4$$

$$+\left\{\frac{25}{256}+\frac{7}{16}+\frac{3}{32}\right\} m^{4}\gamma\sin(4\tau+\eta)$$

$$+\left\{\frac{219}{128}-\frac{495}{256}+\frac{561}{512}+\frac{615}{512}+\frac{9}{8}+\frac{9}{256}\right\} m^{3}e\,\gamma\sin(4\tau-\xi-\eta)$$

$$+\left\{\frac{57}{64}+\frac{495}{256}+\frac{45}{256}+\frac{15}{32}\right\} m^{3}e\,\gamma\sin(4\tau-\xi+\eta)$$

$$+\left\{\frac{9}{64}+\frac{99}{256}+\frac{3}{32}+\frac{9}{256}\right\} m^{3}e\,\gamma\sin(4\tau+\xi-\eta)$$

$$+\left\{\frac{105}{256}+\frac{21}{32}+\frac{7}{16}\right\} m^{3}e_{,}\gamma\sin(4\tau-\xi_{,}-\eta)$$

$$+\left\{-\frac{27}{256}-\frac{3}{32}-\frac{3}{16}\right\} m^{3}e_{,}\gamma\sin(4\tau+\xi_{,}-\eta)$$

$$+\left\{z-\frac{225}{128}+\frac{135}{64}+\frac{45}{256}\right\} m^{2}e^{2}\gamma\sin(4\tau-2\xi\quad\eta)$$

$$+\left\{\frac{675}{512}+\frac{225}{128}+\frac{225}{256}\right\} m^{2}e^{2}\gamma\sin(4\tau-2\xi+\eta)$$

$$+\left\{-\frac{45}{64}-\frac{45}{128}-\frac{45}{128}\right\} m^{2}e\,e_{,}\gamma\sin(4\tau-\xi+\xi_{,}-\eta)$$

$$+\left\{\frac{105}{64}+\frac{105}{128}+\frac{105}{128}\right\} m^{2}e\,e_{,}\gamma\sin(4\tau-\xi-\xi_{,}-\eta)$$

Hence the latitude of the moon contains the following terms

$$\left\{-\frac{487}{288}m^{4}-\frac{103}{96}m^{2}e^{2}+\frac{m^{2}e_{,}^{2}}{4}\right\}\gamma\sin\eta$$

Terms in the latitude the moon

$$+\left\{\frac{7195}{1152}m^4+\frac{1929}{128}m^2e^2-\frac{55}{32}m^2e_{\prime}^2\right\}\gamma\sin(2\tau+\eta)$$

[148]

$$+\left\{-\frac{5}{8}e^2-\frac{51^*}{32}m^3+\frac{135}{64}me^2\right\}e\gamma\sin(\xi-\eta)$$

[149]

$$+\left\{-\frac{5}{4}e^2+\frac{189^*}{64}m^3\right\}e\gamma\sin(\xi+\eta)$$

[150]

$$+\left\{\frac{881^*}{32}m^3+\frac{3^*}{32}me^2-\frac{315^*}{64}me_{\prime}^2\right\}e\gamma\sin(2\tau-\xi-\eta)$$

[151]

$$+\left\{\frac{47801}{1536}m^3-\frac{165}{64}me^2-\frac{75}{16}me_{\prime}^2\right\}e\gamma\sin(2\tau-\xi+\eta)$$

[152]

$$+\left\{\frac{2893^*}{1536}m^3+\frac{405^*}{128}me^2+\frac{45^*}{64}me_{\prime}^2\right\}e\gamma\sin(2\tau+\xi-\eta)$$

[153]

$$+\left\{\frac{287}{48}m^3+\frac{753^*}{160}me^2\right\}e\gamma\sin(2\tau+\xi+\eta)$$

[154]

$$+\left\{\frac{1949}{128}m^3+\frac{315}{32}me^2\right\}e_{\prime}\gamma\sin(2\tau-\xi_{\prime}+\eta)$$

[158]

$$+\left\{-\frac{1127}{384}m^3-\frac{135}{32}me^2\right\}e_{\prime}\gamma\sin(2\tau+\xi_{\prime}+\eta)$$

[160]

$$+\frac{79}{32}m^2e^2\gamma\sin(2\xi+\eta)$$

[162]

$$+\frac{303^*}{256}m^2e^2\gamma\sin(2\tau+2\xi-\eta)$$

[165]

$$+\frac{425}{128}m^2e^2\gamma\sin(2\tau+2\xi+\eta)-\frac{123}{8}m^2ee_{\prime}\gamma\sin(\xi+\xi_{\prime}-\eta)$$

[166] [167]

---

* The terms marked with an asterisk are inconsistent with M. Plana's expression.

Terms in e latitude of the moon.

$$-\frac{405}{32}m^2 e e_{,}\gamma \sin(\xi+\xi_{,}+\eta)+\frac{171}{8}m^2 e e_{,}\gamma \sin(2\tau-\xi-\xi_{,}-\eta)$$

[168] [169]

$$+\frac{423}{16}m^2 e e_{,}\gamma \sin(2\tau-\xi-\xi_{,}+\eta)$$

[170]

$$-\frac{11}{4}m^2 e e_{,}\gamma \sin(2\tau+\xi+\xi_{,}-\eta)$$

[171]

$$-\frac{7}{8}m^2 e e_{,}\gamma \sin(2\tau+\xi+\xi_{,}+\eta)+\frac{111}{8}m^2 e e_{,}\gamma \sin(\xi-\xi_{,}-\eta)$$

[172] [173]

$$+\frac{609}{32}m^2 e e_{,}\gamma \sin(\xi-\xi_{,}+\eta)-\frac{3}{4}{}^{*}m^2 e e_{,}\gamma \sin(2\tau-\xi+\xi_{,}-\eta)$$

[174] [175]

$$-\frac{49}{16}m^2 e e_{,}^2\gamma \sin(2\tau-\xi+\xi_{,}+\eta)$$

[176]

$$+\frac{19}{4}m^2 e e_{,}\gamma \sin(2\tau+\xi-\xi_{,}-\eta)$$

[177]

$$+\frac{49}{8}m^2 e e_{,}\gamma \sin(2\tau+\xi-\xi_{,}+\eta)$$

[178]

$$+\frac{187}{32}m^2 e_{,}^2\gamma \sin(2\tau-2\xi_{,}+\eta)$$

[182]

$$-\frac{411}{128}m^2\gamma\frac{a}{a_{,}}\sin(\tau-\eta)-\frac{83}{16}m^2\gamma\frac{a}{a_{,}}\sin(\tau+\eta)$$

[185] [186]

$$-\frac{135}{64}m e\gamma\frac{a}{a_{,}}\sin(\tau+\xi+\eta)+\frac{15}{16}{}^{*}m e_{,}\gamma\frac{a}{a_{,}}\sin(\tau-\xi_{,}-\eta)$$

[190] [191]

$$+\frac{15}{16}m e_{,}\gamma\frac{a}{a_{,}}\sin(\tau-\xi_{,}+\eta)$$

[192]

---

* The terms marked with an asterisk are inconsistent with M. Plana's expression.

Terms i
the latitud
the moo

$$+\left\{\frac{5}{4}-\frac{45}{8}m\right\}e_{\prime}\gamma\frac{a}{a_{\prime}}\sin(\tau+\xi_{\prime}-\eta)$$

[193]

$$+\left\{\frac{5}{4}-\frac{45}{8}m\right\}e_{\prime}\gamma\frac{a}{a_{\prime}}\sin(\tau+\xi_{\prime}+\eta)$$

[194]

$$+\left\{\frac{17}{24}-\frac{135}{64}m\right\}e^{3}\gamma\sin(3\xi-\eta)$$

[195]

$$+\frac{4}{3}e^{3}\gamma\sin(3\xi+\eta)+\frac{67^{*}}{16}m\,e^{3}\gamma\sin(2\tau-3\xi-\eta)$$

[196] [197]

$$+\frac{15}{16}m\,e^{3}\gamma\sin(2\tau-3\xi+\eta)+\frac{1}{2}m\,e^{3}\gamma\sin(2\tau+3\xi-\eta)$$

[198] [199]

$$-\frac{405}{64}m\,e^{2}e_{\prime}\gamma\sin(2\xi+\xi_{\prime}+\eta)$$

[202]

$$-\frac{27}{64}m\,e^{2}e_{\prime}\gamma\sin(2\tau+2\xi+\xi_{\prime}-\eta)$$

[205]

$$+\frac{405}{64}m\,e^{2}e_{\prime}\gamma\sin(2\xi-\xi_{\prime}+\eta)+\frac{63}{64}m\,e^{2}e_{\prime}\gamma\sin(2\tau+2\xi-\xi_{\prime}-\eta)$$

[208] [211]

$$-\frac{27}{16}m\,e\,e_{\prime}^{2}\gamma\sin(\xi+2\xi_{\prime}-\eta)-\frac{9}{4}m\,e\,e_{\prime}^{2}\gamma\sin(\xi+2\xi_{\prime}+\eta)$$

[213] [214]

$$+\frac{51}{8}m\,e\,e_{\prime}^{2}\gamma\sin(2\tau-\xi-2\xi_{\prime}-\eta)$$

[215]

$$+\frac{255}{32}m\,e\,e_{\prime}^{2}\sin(2\tau-\xi-2\xi_{\prime}+\eta)$$

[216]

$$-\frac{9}{32}m\,e\,e_{\prime}^{2}\gamma\sin(2\tau+\xi+2\xi_{\prime}-\eta)+\frac{27}{16}m\,e\,e_{\prime}^{2}\gamma\sin(\xi-2\xi_{\prime}-\eta)$$

[217] [219]

Terms in the latitude of the moon.

$$+\frac{9}{4} m e e_{,}^{2} \gamma \sin(\xi - 2\xi_{,} + \eta) - \frac{9}{8} m e e_{,}^{2} \gamma \sin(2\tau - \xi + 2\xi_{,} - \eta)$$

[220] [221]

$$-\frac{45}{32} m e e_{,}^{2} \gamma \sin(2\tau - \xi + 2\xi_{,} + \eta)$$

[222]

$$+\frac{51}{32} m e e_{,}^{2} \gamma \sin(2\tau + \xi - 2\xi_{,} - \eta)$$

[223]

$$-\frac{95}{128} m^2 \gamma \frac{\mathrm{a}}{a_{,}} \sin(3\tau - \eta) + \frac{15}{64} m^2 \gamma \frac{\mathrm{a}}{a_{,}} \sin(3\tau + \eta)$$

[231] [232]

$$+\left\{\frac{621}{512} m^4 + \frac{405}{256} m^2 e^2\right\} \gamma \sin(4\tau - \eta) + \frac{161}{256} m^4 \gamma \sin(4\tau + \eta)$$

[241] [242]

$$+\frac{207}{64}{}^{*} m^3 e \gamma \sin(4\tau - \xi - \eta) + \frac{357}{64}{}^{*} m^3 e \gamma \sin(4\tau - \xi + \eta)$$

[243] [244]

$$+\frac{21}{32} m^3 e \gamma \sin(4\tau + \xi - \eta) + \frac{385}{256} m^3 e_{,} \gamma \sin(4\tau - \xi_{,} - \eta)$$

[245] [246]

$$-\frac{99}{256} m^3 e_{,} \gamma \sin(4\tau + \xi_{,} - \eta) + \frac{2025}{512} m e^2 \gamma \sin(4\tau - 2\xi + \eta)$$

[247]

$$-\frac{45}{32} m^2 e e_{,} \gamma \sin(4\tau - \xi + \xi_{,} - \eta)$$

$$+\frac{105}{32} m^2 e e_{,} \gamma \sin(4\tau - \xi - \xi_{,} - \eta)$$

I have ascertained that the only terms in the preceding expression inconsistent with the expression for the latitude given by M. Plana are those marked with an asterisk.

It may be worth while to remark that the terms in g multiplied by $m^2 e^3$ and $m^2 e_{,}^2$, found from the last expression for $\mathbb{Z}_{,}$ agree with those obtained more easily in p. 81 by the method of the variation of the elliptic constants.

---

* The terms marked with an asterisk are inconsistent with M. Plana's expression.

$$\mathbf{Z} = \left\{\frac{963}{128} - \frac{405}{128} - \frac{9}{4} + \frac{3}{4} - \frac{3}{4} - \frac{405}{64} + \frac{675}{256} + \frac{693}{256} + \frac{9}{8}\right\} m^2 e^2 + \frac{9}{4} m^2 e_{\prime}^2$$

$$= \frac{9}{4} m^2 e^2 + \frac{9}{4} m^2 e_{\prime}^2.$$

Let $g = 1 + \frac{3}{4} m^2 + A m^2 e^2 + B m^2 e_{\prime}^2$

$$\left(1 - \frac{e^2}{2} + \&\text{c.}\right) g^2 - 1) = \frac{9}{4} m^2 e^2 + \frac{9}{4} m^2 e_{\prime}^2$$

$$A = \frac{3}{2} \qquad\qquad B = \frac{9}{8}.$$

Example of the Calculation of s_{148}.

$\mathbf{Z}$ contains the term $\left\{\frac{137}{12} + \frac{9}{32}\right\} m^4 \gamma \sin(2\tau + \eta)$ See p. 297.

$$= \frac{1123}{96} m^4 \gamma \sin(2\tau + \eta)$$

Let $z_{148} = \frac{3}{16} m^2 + \frac{7}{8} m^3 + A m^4$ See p. 241.

$$\left\{8 - 12m + \frac{17}{2} m^2\right\} \left\{\frac{3}{16} m^2 + \frac{7}{8} m^3 + A m^4\right\} = \frac{1123}{96} m^4$$

Equating coefficients

$$8A - \frac{21}{2} + \frac{51}{32} = \frac{1123}{96} \qquad\qquad A = \frac{989}{384}$$

Hence $s_{148} = \left\{\frac{989}{384} + \frac{131}{36} + \frac{1}{32}\right\} m^4$

$$= \frac{7195}{1152} m^4$$

Hence the latitude contains the terms

$$\left\{\frac{11}{16} m^2 + \frac{59}{24} m^3 + \frac{7195}{1152} m^4\right\} \gamma \sin(2\tau + \eta)$$

Substituting $\gamma\left(1 - \frac{m^2}{6}\right)$ for γ in my expression, agreeably to the remark in p. 248, for the coefficient of m^4, instead of $\frac{7195}{1152}$

$\frac{7063}{1152}$ is obtained, which is identical with the result given by M. Plana.

In various instances where the corresponding term of the order $m e^3$ or $m e_{\prime}^2 e$ in the latitude would arise from a further approximation to the value of z, $\mathbb{Z}_n$ is equal to zero in the preceding development; thus, (see p. 298,)

$$\mathbb{Z}_{157} = -\frac{315}{16} + \frac{21}{16} + \frac{315}{32} + \frac{63}{8} + \frac{21}{32} = 0$$

I have not thought it necessary to give at length the expression for z as in p. 241, but the coefficient of the corresponding term in z is always placed first amongst the terms constituting the expression for terms in the latitude which begins p. 303.

Where the terms in the expression for the latitude, p. 309, are marked with an asterisk, I have failed in arriving at M. Plana's coefficients. I do not impute error to M. Plana; my computation may be in fault; but where I have succeeded in arriving at the same figures I think error is extremely improbable.

I shall now revert to the consideration of terms in the reciprocal of the radius vector and of the longitude.

Calculation of terms in (λ_2), *the coefficient of* e sin ξ *in the expression for the longitude.*

Considering only the constant portion of the equation

$$\frac{d^2 . r^2}{2\,d t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int d R + \frac{r\,d R}{d r} = 0$$

and taking $r\frac{d R}{d r} = 2 R$

$$\frac{\mu}{r} = \frac{\mu}{a} + 2 R \qquad \frac{a}{r} = 1 + \frac{2 a}{\mu} R = 1 - \frac{m^2}{2} + \frac{3}{4} m^2 \gamma^2 + \&c.$$

In extending the last equation to other terms a must not be confounded with a.

$$\frac{a}{r} = 1 + \frac{2 a \mathrm{a}}{\mathrm{a} \mu} R$$

$$= 1 + 2\left\{1 - \frac{2}{3} m^2 + \&c.\right\}\left\{-\frac{m^2}{4} + \frac{179}{96} m^4 + \&c.\right\}$$

$$= 1 - \frac{m^2}{2} + \frac{65}{16} m^4 + \&c. \quad \text{See p. 134.}$$

The relation between the constants h and a is to be obtained from the equations

$$\frac{\mathrm{d}\,r^2}{\mathrm{d}\,t^2} + \frac{r^2}{(1+s^2)^2}\frac{\mathrm{d}\,s^2}{\mathrm{d}\,t^2} + \frac{r'^2\,\mathrm{d}\,\lambda'^2}{\mathrm{d}\,t^2} - \frac{2\,\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}\,R = 0$$

$$\frac{r'^4\,\mathrm{d}\,\lambda'^2}{\mathrm{d}\,t^2} = h^2 - 2\,h\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\mathrm{d}\,t + \left\{\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda'}\,\mathrm{d}\,t\right\}^2 \quad .$$

In order to obtain the term in h^2 multiplied by $m^2\,\gamma^2$ it is sufficient to take

$$h^2 = \frac{2\,\mu\,r'^2}{r} - \frac{\mu\,r'^2}{a} - \frac{r^4\,\mathrm{d}\,s^2}{\mathrm{d}\,t^2}$$

$$= 2\,\mu\,r - \frac{\mu\,r^2}{a} - 2\,\mu\,r\,s^2 + \frac{\mu\,r^2\,s^2}{a} - \frac{r^4\,\mathrm{d}\,s^2}{\mathrm{d}\,t^2}$$

$$s = \left(1 + \frac{m^2}{6}\right)\gamma\sin\eta + \frac{3}{8}\,m\,\gamma\sin(2\,\tau - \eta)$$

$$\eta = \left(1 + \frac{3}{4}\,m^2\right)\mathrm{n}\,t = \left(1 - \frac{m^2}{4}\right)n\,t$$

With the addition of the terms already found,

$$h^2 = \mu\,a\left\{1 - e^2 - \gamma^2 - e^2\,\gamma^2 - \frac{145}{32}\,m^4 - \frac{241}{64}\,m^2\,e^2 - \frac{235}{192}\,m^2\,\gamma^2\right\}$$

$$n = \sqrt{\frac{\mu}{a^3}} \qquad\qquad \mathrm{n} = \sqrt{\frac{\mu}{\mathrm{a}^3}}$$

n being the unperiodic or constant term in $\frac{\mathrm{d}\,\lambda'}{\mathrm{d}\,t}$.

$$\mathrm{n} = \sqrt{\frac{\mu}{a^3}}\left\{1 - m^2 + \frac{471}{64}\,m^4 - \frac{9}{8}\,m^2\,e^2 - \frac{3}{2}\,m^2\,e_{\prime}^2 + \frac{9}{8}\,m^2\,\gamma^2\right\}$$

Some of the terms in the value of n can be easily obtained by the method of the variation of constants. See M. Poisson's Mémoire sur le Mouvement de la Lune, p. 69, Equation (F.). Moreover

$$\mathrm{a} = a\left\{1 + \frac{2}{3}\,m^2 - \frac{1253}{288}\,m^4 + \frac{3}{4}\,m^2\,e^2 + m^2\,e_{\prime}^2 - \frac{3}{4}\,m^2\,\gamma^2\right\}$$

$$a = \mathrm{a}\left\{1 - \frac{2}{3}\,m^2 + \frac{1445}{288}\,m^4 - \frac{3}{4}\,m^2\,e^2 - m^2\,e_{\prime}^2 + \frac{3}{4}\,m^2\,\gamma^2\right\}$$

$$h = \sqrt{\mu\,\mathrm{a}}\left\{1 - \frac{e^2}{2} - \frac{\gamma^2}{2} - \frac{3}{4}e^2\gamma^2 - \frac{e^4}{8} - \frac{3}{8}\gamma^4 - \frac{m^2}{3} + \frac{11}{144}m^4 - \frac{803}{384}m^2e^2 - \frac{m^2}{2}e_{\prime}^2 - \frac{9}{128}m^2\gamma^2\right\}.$$

The term in h multiplied by m^4 was found in p. 135.

I shall now endeavour to obtain the terms in the reciprocal of the radius vector and in the longitude, of Argument 2 multiplied by m^2e^3, $m^2e_{\prime}^2e$, and $m^2\gamma^2e$.

Since in the elliptic theory

$$\frac{a}{r} = 1 + e\left(1 - \frac{e^2}{8} + \&\text{c.}\right)\cos\xi + \&\text{c.}$$

and since the constant quantity denoted by e in this work is such that the coefficient of $\cos\xi$ in the expression for $\frac{a}{r}$ has still the same form when the disturbing function is taken into account, then considering only terms belonging to Arguments 0 and 2,

$$\frac{a}{r} = 1 - \frac{m^2}{2} + \frac{65}{16}m^4 - \frac{3}{4}m^2e^2 - \frac{3}{4}m^2e_{\prime}^2 + \frac{3}{4}m^2\gamma^2 + \left\{1 - \frac{e^2}{8}\right\}e\cos\xi.$$

Multiplying this expression for $\frac{a}{r}$ by $\frac{\mathrm{a}}{a}$,

$$\frac{\mathrm{a}}{r} = 1 + \frac{m^2}{6} - \frac{179}{288}m^4 + \frac{1}{4}m^2e_{\prime}^2 + \left\{1 - \frac{e^2}{8} + \frac{2}{3}m^2 + \frac{2}{3}m^2e^2 + m^2e_{\prime}^2 - \frac{3}{4}m^2\gamma^2\right\}e\cos\xi.$$

Neglecting for the present terms multiplied by γ^2,

$$\frac{d\lambda}{dt} = \frac{1}{\mathrm{a}^2}\left\{\frac{\mathrm{a}^2}{\mathrm{r}^2} + \frac{2\,\mathrm{a}^2}{\mathrm{r}}\,\delta\frac{1}{r} + \mathrm{a}^2\left(\delta\frac{1}{r}\right)^2\right\}\left\{h - \int\frac{dR}{d\lambda}\,dt\right\}$$

$$\frac{\mathrm{a}^2}{r^2} = 1 + \frac{m^2}{3} + \frac{2}{3}m^2e^2 + \frac{m^2}{2}e_{\prime}^2 + \left\{2 + \frac{5}{3}m^2 + \frac{3}{4}e^2 + \frac{341}{32}m^2e^2 + \frac{5}{2}m^2e_{\prime}^2\right\}e\cos\xi + \frac{15}{4}m\,e\cos(2\tau - \xi)$$

It is sufficient in order to obtain the terms in the longitude of Arg. 2 multiplied by $m^2 e^3$ and $m^2 e_{\prime}^2 e$ to take

$$\frac{1}{a^2}\left\{h - \int \frac{dR}{d\lambda} dt\right\} = n\left\{1 - \frac{e^2}{2} - \frac{m^2}{3} - \frac{803}{384} m^2 e^2 \right.$$

$$\left. - \frac{m^2}{2} e_{\prime}^2 - \frac{15}{8} m e^2 \cos(2\tau - 2\xi)\right\}$$

$$\frac{d\lambda}{n\,dt} = \left[2 + m^2 - \frac{e^2}{4} + \left\{\frac{341}{32} - \frac{5}{6} - \frac{1}{4} - \frac{803}{192} - \frac{225}{64}\right\} m^2 e^2 \right.$$

$$\left. + \left\{\frac{5}{2} - 1\right\} m^2 e_{\prime}^2 \right] e \cos \xi$$

$$= \left\{2 + m^2 - \frac{e^2}{4} + \frac{15}{8} m^2 e^2 + \frac{3}{2} m^2 e_{\prime}^2\right\} e \cos \xi$$

$$c = 1 - \frac{3}{4} m^2 + \frac{3}{8} m^2 e^2 - \frac{9}{8} m^2 e_{\prime}^2 + \frac{3}{2} m^2 \gamma^2. \quad \text{See p. 80.}$$

$$\lambda = \left\{2 + m^2 - \frac{e^2}{4} + \frac{15}{8} m^2 e^2 + \frac{3}{2} m^2 c_{\prime}^2\right\}$$

$$\left\{1 + \frac{3}{4} m^2 - \frac{3}{8} m^2 e^2 + \frac{9}{8} m^2 e_{\prime}^2\right\} e \sin \xi$$

$$= \left\{2 + \frac{5}{2} m^2 - \frac{e^2}{4} + \frac{15}{16} m^2 e^2 + \frac{15}{4} m^2 e_{\prime}^2\right\} e \sin \xi$$

If $a\,\delta \frac{1}{r}$ and $\delta\left\{h - \int \frac{dR}{d\lambda'} dt\right\}$ are restricted to mean the terms in $\frac{a}{r}$ and in $h - \int \frac{dR}{d\lambda'} dt$ which are multiplied by γ^2, and if $\left[\frac{a}{r}\right]$ and $\left[\frac{d\lambda'}{n\,dt}\right]$ represent the terms in those quantities independent of γ^2;

$$\frac{d\lambda'}{dt} = \left[\frac{d\lambda'}{dt}\right] s^2 + \left[\frac{1}{r^2}\right] \delta \left\{h - \int \frac{dR}{d\lambda'} dt\right\} + \left[\frac{2h}{r}\right] \delta \frac{1}{r}$$

In order to have the terms of Arg. 2 multiplied by $e^3 \gamma^2$ and $m^2 e \gamma^2$ it is sufficient to take

$$\left[\frac{\mathrm{d}\lambda'}{\mathrm{n\,d}t}\right] = 1 + \left\{2 + m^2 - \frac{e^2}{4}\right\} e\cos\xi + \frac{15}{4} m\, e \cos(2\tau - \xi)$$

$$s^2 = \frac{\gamma^2}{2} + e^2\gamma^2 + \frac{91}{384} m^2\gamma^2 - \frac{3}{8} m\,\gamma^2 \cos 2\tau$$

$$+ \frac{109}{32} m^2 \gamma^2 e \cos\xi$$

$$\left[\frac{\mathrm{a}^2}{r^2}\right] = 1 + \left\{2 + \frac{5}{3} m^2 + \frac{3}{4} e^2\right\} e\cos\xi$$

$$\frac{\mathrm{a}}{\mathrm{n}\mu}\,\delta\left\{h - \int \frac{\mathrm{d}R}{\mathrm{d}\lambda'}\mathrm{d}t\right\} = -\frac{\gamma^2}{2} - \frac{3}{4} e^2\gamma^2 - \frac{9}{128} m^2\gamma^2$$

$$\left[\frac{h}{\mathrm{n\,a}\,r}\right] = 1 + \left\{1 - \frac{5}{8} e^2\right\} e\cos\xi \qquad \mathrm{a}\,\delta\frac{1}{r} = -\frac{3}{4} m^2\gamma^2 e\cos\xi$$

$$\frac{\mathrm{d}\lambda'}{\mathrm{n\,d}t} = \left\{2 + \left\{-\frac{1}{8} + 2 - \frac{3}{8} - \frac{3}{4}\right\} e^2\gamma^2\right.$$

$$\left. + \left\{\frac{109}{32} + \frac{1}{2} + \frac{91}{192} - \frac{45}{64} - \frac{9}{64} - \frac{5}{6} - \frac{3}{4}\right\} m^2\gamma^2\right\} e\cos\xi$$

$$= \left\{2 + \frac{3}{8} e^2\gamma^2 + \frac{375}{192} m^2\gamma^2\right\} e\cos\xi.$$

Uniting terms previously obtained

$$\lambda' = \left\{2 + \frac{5}{2} m^2 - \frac{e^2}{4} + \frac{15}{16} m^2 e^2 + \frac{15}{4} m^2 e_{,}^2\right.$$

$$\left. + \frac{3}{8} e^2\gamma^2 - \frac{67}{64} m^2\gamma^2\right\} e\sin\xi.$$

The coefficient of $e\sin\xi$ in the expression for the longitude, according to M. Plana, is

$$2 + \frac{3}{2} m^2 - \frac{e^2}{4} - 17\, m^2 e^2 - \frac{45}{4} m^2 e_{,}^2 - \frac{\gamma^2}{2} + \frac{23}{16} e^2\gamma^2 - \frac{63}{32} m^2\gamma^2$$

and of $e\cos\xi$ in that for $\frac{\mathrm{a}}{r}$ is

$$1 + \frac{m^2}{6} - \frac{e^2}{8} - \frac{1067}{96} m^2 e^2 - \frac{45}{8} m^2 e_{,}^2 - \frac{\gamma^2}{4} + \frac{13}{32} e^2\gamma^2 - \frac{265}{384} m^2\gamma^2.$$

I fear that my values are *inconsistent* with those of M. Plana.

In order to distinguish M. Plana's constants from mine, I inclose the latter between brackets, and as the coefficients of cos ξ in the value of $\frac{a}{r}$, of sin ξ in λ' and of sin η in the latitude must be the same,

$$e\, r_2 = (e)\,(r_2) \qquad e\, \lambda_2 = (e)\,(\lambda_2) \qquad \gamma\, s_{146} = (\gamma)\,(s_{146}).$$

The second equation gives

$$e\left\{2 + \frac{5}{2} m^2 + \frac{285}{32} m^3 - \frac{e^2}{4} + \frac{15}{16} m^2 e^2 + \frac{15}{4} m^2 e_{\prime}^2 \right.$$
$$\left. + \frac{3}{8} e^2 \gamma^2 - \frac{67}{64} m^2 \gamma^2 \right\}$$
$$= (e)\left\{2 + \frac{3}{2} m^2 - \frac{75}{64} m^3 - \frac{(e)^2}{4} - 17\, m^2 (e^2) - \frac{45}{4} m^2 e_{\prime}^2 \right.$$
$$\left. - \frac{(\gamma^2)}{2} + \frac{23}{16} (e^2 \gamma^2) - \frac{63}{32} m^2 (\gamma^2) \right\}$$
$$e = (e)\left\{1 - \frac{m^2}{2} - \frac{645}{128} m^3 - \frac{\gamma^2}{4}\right\} \text{ nearly.}$$

The third equation gives

$$\gamma\left\{1 - e^2 + \frac{m^2}{6} - \frac{3}{16} m^3\right\} = (\gamma)\left\{1 - (e^2) + \frac{33}{128} m^3\right\}$$
$$\gamma = (\gamma)\left\{1 - \frac{m^2}{6} + \frac{57}{128} m^3\right\} \text{ nearly.}$$

The following equation of condition must also be satisfied:

$$\lambda_2\, (r_2) = (\lambda_2)\, r_2$$
$$\left\{2 + \frac{5}{2} m^2 + \frac{285}{32} m^3 - \frac{e^2}{4} + \frac{15}{16} m^2 e^2 \right.$$
$$\left. + \frac{15}{4} m^2 e_{\prime}^2 + \frac{3}{8} e^2 \gamma^2 - \frac{57}{64} m^2 \gamma^2 \right\}$$
$$\left\{1 + \frac{m^2}{6} - \frac{645}{128} m^3 - \frac{(e^2)}{8} - \frac{1067}{96} m^2 (e^2) - \frac{45}{8} m^2 e_{\prime}^2 \right.$$
$$\left. - \frac{(\gamma^2)}{4} + \frac{13}{32} (e^2 \gamma^2) - \frac{265}{384} m^2 (\gamma^2) \right\}$$

$$= \left\{2 + \frac{3}{2} m^2 - \frac{75}{64} m^3 - \frac{e^2}{4} - 17\, m^2 (e^2) \right.$$

$$\left. - \frac{45}{4} m^2 e_{\prime}^2 - \frac{(\gamma^2)}{2} + \frac{23}{16} (e^2 \gamma^2) - \frac{63}{32} m^2 (\gamma^2) \right\}$$

$$\left\{1 + \frac{2}{3} m^2 - \frac{e^2}{8} + \frac{2}{3} m^2 e^2 + m^2 e_{\prime}^2 - \frac{3}{4} m^2 \gamma^2 \right\}$$

In order that the values of r_2, λ_2, and s_{146}, found in this treatise, may be consistent with (r_2), (λ_2), and (s_{146}), the present equation must obtain, which does not appear to be the case.

The numerical value of the coefficient of one inequality (that of sin ξ in the longitude is of course to be preferred) at least, must be determined empirically, in order to ascertain the numerical value of the eccentricity, which may then be substituted in the coefficients of the other arguments. Theory cannot dispense with the accurate empirical determination of the arbitrary constants. This of course can only be effected by a careful discussion of *observed* longitudes and latitudes in the manner explained, *Théorie Analytique du Système du Monde*, vol. iii. p. 523. But in order to employ this method it is necessary to possess in the first instance exact analytical expressions, embracing all quantities of the order in magnitude of the corrections desired in the elliptic elements. This condition is not yet accomplished in the Lunar Theory; perhaps it never will be, on account of the extreme difficulty of carrying the approximation to a sufficient extent in figures devoid of uncertainty. This difficulty at present can only be avoided by attempting to determine empirically more coefficients than would be required for the complete numerical solution of the problem, if the analytical formulæ could be safely trusted. I suggested the importance of this determination, in the present state of the Lunar Theory, at the last Meeting of the British Association for the Advancement of Science, held at Bristol.

Calculation of terms in the longitude of arguments 24 and 27.

In p. 219 I have given the following equation:

$$\frac{d\,\lambda'}{d\,t} = \frac{h\,(1+s^2)}{2\,r^2} + \frac{\mu}{h\,r} - \frac{\mu}{2\,h\,a} - \frac{d\,r^2}{2\,h\,d\,t^2} - \frac{r^2}{2\,h\,(1+s^2)^2} \frac{d\,s^2}{d\,t^2} \qquad (A)$$

$$- \frac{1}{h} \int d\,R - \frac{(1+s^2)}{2\,h\,r^2} \left\{ \int \frac{d\,R}{d\,\lambda'}\, d\,t \right\}^2.$$

Multiplying the equation

$$\frac{\mathrm{d}^2 . r^2}{2\,\mathrm{d}t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}R + \frac{r\,\mathrm{d}R}{\mathrm{d}r} = 0$$

by $\frac{2}{h}$ and adding the result to the former equation I obtain

$$\begin{aligned}\frac{\mathrm{d}\lambda'}{\mathrm{d}t} = {} & \frac{h(1+s^2)}{2r^2} - \frac{\mu}{hr} + \frac{3\mu}{2ha} + \frac{\mathrm{d}^2 . r^2}{h\,\mathrm{d}t^2} - \frac{\mathrm{d}r^2}{2h\,\mathrm{d}t^2} \\ & - \frac{r^2}{2h(1+s^2)^2}\frac{\mathrm{d}s^2}{\mathrm{d}t^2} + \frac{1}{h}\left\{3\int \mathrm{d}R + \frac{2r\,\mathrm{d}R}{\mathrm{d}r}\right\} \quad \ldots \text{(B)} \\ & - \frac{(1+s^2)}{2hr^2}\left\{\int \frac{\mathrm{d}R}{\mathrm{d}\lambda'}\mathrm{d}t\right\}^2 .\end{aligned}$$

By treating the equations in p. 218 after the manner explained in the *Théorie Analytique*, vol. i. p. 490, it is easy to arrive at the following equation.

$$\begin{aligned}\frac{\mathrm{d}.\delta\lambda'}{\mathrm{d}t} = {} & \frac{\mathrm{d}\{2r\,\mathrm{d}.\delta r + \delta r\,\mathrm{d}r\}}{h\,\mathrm{d}t^2} \\ & + \frac{1}{h}\left\{3\int \mathrm{d}R + \frac{2r\,\mathrm{d}R}{\mathrm{d}r} - \frac{\delta r}{r}\left[\frac{r\,\mathrm{d}R}{\mathrm{d}r}\right] + \left[\int \frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t\right]\frac{\mathrm{d}.\delta\lambda'}{\mathrm{d}t}\right\} \quad \ldots \text{(C)} \\ & + \left[\frac{h}{2r^2}\right]s^2 - \left[\frac{r^2}{2h}\right]\frac{\mathrm{d}s^2}{\mathrm{d}t^2} - \frac{s^2}{r^2}\left\{\int \frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t\right] \\ & + \frac{s^2}{2hr^2}\left\{\int \frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t\right\}^2\end{aligned}$$

M. de Pontécoulant has employed this equation in order to determine the coefficient of m in the equation 77 of the longitude. See the *Comptes rendus des Séances de l'Académie des Sciences*, 1837, No. 8.

When the terms multiplied by γ^2 are neglected

$$\begin{aligned}\frac{\mathrm{d}.\delta\lambda}{\mathrm{d}t} = {} & \frac{\mathrm{d}\{2r\,\mathrm{d}.\delta r + \delta r\,\mathrm{d}r\}}{h\,\mathrm{d}t^2} \\ & + \frac{1}{h}\left\{3\int \mathrm{d}R + \frac{2r\,\mathrm{d}R}{\mathrm{d}r} - \frac{\delta r}{r}\left[\frac{r\,\mathrm{d}R}{\mathrm{d}r}\right] + \left[\int \frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t\right]\frac{\mathrm{d}.\delta\lambda}{\mathrm{d}t}\right\}.\end{aligned}$$

This is the equation (T), *Méc. Céleste*, vol. i. p. 256.

No doubt equation (C) is to be considered as a transformation of equation (A) or (B); but the equations (A) and (B) are rigorous and generally applicable, while the equation (C) is only approximative, and depends upon the neglect of $(\delta r)^2$, $\delta r \, \delta \lambda$, &c. I shall now endeavour to calculate terms in the longitude belonging to arguments 24 and 27 as examples of the application of equation (C). In these instances δr and $\delta \lambda$ may be considered as including all terms multiplied by $e_{,}$.

Argument 24 results from the combinations

$$\{0,24\} \quad \{1,23\} \quad \{2,12\} \quad \{3,11\} \quad \{8,6\} \text{ and } \{9,5\}.$$

Argument 27 results from the combinations

$$\{0,27\} \quad \{1,26\} \quad \{2,15\} \quad \{3,14\} \quad \{8,7\} \text{ and } \{9,5\}.$$

Hence it is sufficient to take in δr, the terms multiplied by m^2 belonging to arguments 5, 6, 7, 11, 12, 14, 24 and 27, and those multiplied by m belonging to arguments 23 and 26.

I find

$$\begin{aligned}
\delta r = {} & \frac{3}{2} m^2 e_{,} \cos \xi_{,} - \frac{7}{2} m^2 e_{,} \cos (2\tau - \xi_{,}) + \frac{m^2}{2} e_{,} \cos (2\tau + \xi_{,}) \\
& + \left\{ \frac{21}{8} m + \frac{741}{64} m^2 \right\} e e_{,} \cos (\xi + \xi_{,}) + \left\{ - \frac{35}{8} m - \frac{1045}{64} m^2 \right\} e e_{,} \cos (2\tau - \xi - \xi_{,}) \\
& + \left\{ - \frac{21}{8} m - \frac{1209}{64} m^2 \right\} e e_{,} \cos (\xi - \xi_{,}) + \left\{ \frac{15}{8} m + \frac{65}{64} m^2 \right\} e e_{,} \cos (2\tau - \xi + \xi_{,}) \\
& + \frac{21}{8} m e^2 e_{,} \cos (2\xi + \xi_{,}) - \frac{21}{8} m e^2 e_{,} \cos (2\xi - \xi_{,}) \\
& + \frac{35}{8} m e^2 e_{,} \cos (2\tau - 2\xi - \xi_{,}) - \frac{15}{8} m e^2 e_{,} \cos (2\tau - 2\xi + \xi_{,})
\end{aligned}$$

$$\begin{aligned}
& \frac{2 r \, \mathrm{d} . \delta r}{\mathrm{d} t} + \frac{\mathrm{d} r}{\mathrm{d} t} \delta r \\
= {} & \left\{ \left\{ - \frac{35}{8} + \frac{35}{16} \right\} m + \left\{ \frac{105}{4} - \frac{205}{64} - \frac{315}{64} - \frac{7}{2} \right. \right. \\
& \qquad \left. \left. + \frac{1045}{128} + \frac{315}{128} + \frac{7}{4} \right\} m^2 \right\} e^2 e_{,} \sin (2\tau - 2\xi - \xi_{,}) \\
& + \left\{ \left\{ \frac{15}{8} - \frac{15}{16} \right\} m + \left\{ - \frac{15}{4} - \frac{55}{64} + \frac{315}{64} + \frac{1}{2} \right. \right. \\
& \qquad \left. \left. - \frac{65}{128} - \frac{315}{128} - \frac{1}{4} \right\} m^2 \right\} e^2 e_{,} \sin (2\tau - 2\xi + \xi_{,})
\end{aligned}$$

$$= \left\{ -\frac{35}{16} m + 27 m^2 \right\} e^2 e_{,} \sin(2\tau - 2\xi - \xi_{,})$$

[24]

$$+ \left\{ \frac{15}{16} m - \frac{77}{32} m^2 \right\} e^2 e_{,} \sin(2\tau - 2\xi + \xi_{,})$$

[27]

$$R = m^4 \left\{ \frac{1}{2} + \frac{3}{2} \cos 2\tau \right\} \left\{ \frac{315}{16} e^2 e_{,} \sin(2\tau - \xi_{,}) \right.$$

$$+ \frac{45}{16} e^2 e_{,} \sin(2\tau + \xi_{,}) + \frac{63}{10} e^2 e_{,} \sin(2\xi + \xi_{,})$$

$$- \frac{105}{16} e^2 e_{,} \sin(2\tau - 2\xi - \xi_{,}) - \frac{21}{16} e^2 e_{,} \sin(2\xi - \xi_{,})$$

$$\left. + \frac{15}{16} e^2 e_{,} \sin(2\tau - 2\xi + \xi_{,}) \right\}$$

$$+ m^4 \left\{ - e \cos\xi - \frac{9}{2} e \cos(2\tau - \xi) + \frac{3}{2} e \cos(2\tau + \xi) \right.$$

$$\left. + \frac{3}{2} e_{,} \cos\xi_{,} + \frac{21}{4} e_{,} \cos(2\tau - \xi_{,}) - \frac{3}{4} e_{,} \cos(2\tau + \xi_{,}) \right\}$$

$$\left\{ \frac{45}{8} e^2 \sin 2\tau - \frac{15}{8} e^2 \sin(2\tau - 2\xi) + \frac{21}{8} e e_{,} \sin(\xi + \xi_{,}) \right.$$

$$\left. + \frac{105}{8} e e_{,} \sin(2\tau - \xi - \xi_{,}) + \frac{21}{8} e e_{,} \sin(\xi - \xi_{,}) - \frac{15}{8} e e_{,} \sin(2\tau - \xi + \xi_{,}) \right\}$$

$$+ m^4 \left\{ - \frac{3}{2} e e_{,} \cos(\xi + \xi_{,}) - \frac{63}{4} e e_{,} \cos(2\tau - \xi - \xi_{,}) - \frac{3}{2} e e_{,} \cos(\xi - \xi_{,}) \right.$$

$$\left. + \frac{9}{4} e e_{,} \cos(2\tau - \xi + \xi_{,}) \right\} \frac{15}{4} e \sin(2\tau - \xi)$$

$$+ \frac{3}{2} m^4 \sin 2\tau \left\{ - \frac{525}{16} e^2 e_{,} \cos(2\tau - \xi_{,}) + \frac{75}{16} e^2 e_{,} \cos(2\tau + \xi_{,}) \right.$$

$$\left. - \frac{105}{16} e^2 e_{,} \cos(2\xi + \xi_{,}) - \frac{105}{16} e^2 e_{,} \cos(2\xi - \xi_{,}) \right\}$$

$$+ m^4 \left\{ - \frac{9}{2} e \sin(2\tau - \xi) + \frac{3}{2} e \sin(2\tau - \xi) + \frac{21}{4} e_{,} \sin(2\tau - \xi_{,}) \right.$$

$$\left. - \frac{3}{4} e_{,} \sin(2\tau + \xi_{,}) \right\} \left\{ - \frac{75}{8} e^2 \cos 2\tau - \frac{21}{4} e e_{,} \cos(\xi + \xi_{,}) \right.$$

$$\left. - \frac{105}{4} e e_{,} \cos(2\tau - \xi - \xi_{,}) - \frac{21}{4} e e_{,} \cos(\xi - \xi_{,}) + \frac{15}{4} e e_{,} \cos(2\tau - \xi + \xi_{,}) \right\}$$

$$+ m^4 \left\{ \frac{15}{4} e^2 \sin(2\tau - 2\xi) - \frac{63}{4} e e_{,} \sin(2\tau - \xi - \xi_{,}) + \frac{9}{4} e e \sin(2\tau - \xi + \xi_{,}) \right\}$$

$$\left\{ - \frac{15}{2} e \cos(2\tau - \xi) - 3 e_{,} \cos\xi_{,} \right\}$$

$$= \left\{ -\frac{105}{32} - \frac{189}{64} - \frac{105}{16} + \frac{189}{32} - \frac{45}{32} \quad \frac{45}{16} \right.$$

$$\left. -\frac{315}{64} + \frac{189}{16} - \frac{45}{8} \right\} m^4 e^2 e_, \sin(2\tau - 2\xi - \xi_,)$$

$$+ \left\{ \frac{15}{32} + \frac{63}{64} + \frac{15}{16} + \frac{189}{32} - \frac{45}{32} - \frac{45}{16} \right.$$

$$\left. -\frac{315}{64} + \frac{189}{16} - \frac{45}{8} \right\} m^4 e^2 e_, \sin(2\tau - 2\xi + \xi_,)$$

$$= -\frac{315}{32} m^4 e^2 e_, \sin(2\tau - 2\xi - \xi_,) + \frac{171}{32} m^4 e^2 e_, \sin(2\tau - 2\xi + \xi_,)$$

$$[24] \qquad\qquad\qquad\qquad [27]$$

$$\int \mathrm{d} R = -\frac{105}{32} m^3 e^2 e_, \cos(2\tau - 2\xi - \xi_,)$$

$$+ \frac{171}{32} m^3 e^2 e_, \cos(2\tau - 2\xi + \xi_,)$$

$$\frac{2 r \,\mathrm{d} R}{\mathrm{d} r} = \left\{ -\frac{105}{4} m^2 - \frac{75}{16} m^3 \right\} e^2 e_, \cos(2\tau - 2\xi - \xi_,)$$

$$+ \left\{ \frac{15}{4} m^2 - \frac{567}{16} m^3 \right\} e^2 e_, \cos(2\tau - 2\xi + \xi_,)$$

$$\frac{\delta r}{r} = \frac{35}{16} m e^2 e_, \cos(2\tau - \xi_,) - \frac{15}{16} m e^2 e_, \cos(2\tau + \xi_,)$$

$$+ \frac{21}{8} m e e_, \cos(\xi + \xi_,) - \frac{35}{8} m e e_, \cos(2\tau - \xi - \xi_,) - \frac{21}{8} m e e_, \cos(\xi - \xi_,)$$

$$+ \frac{15}{8} m e e_, \cos(2\tau - \xi + \xi_,) + \frac{63}{16} m e^2 e_, \cos(2\xi + \xi_,)$$

$$- \frac{63}{16} m e^2 e_, \cos(2\xi - \xi_,) + \frac{35}{16} m e^2 e_, \cos(2\tau - 2\xi - \xi_,)$$

$$- \frac{15}{16} m e^2 e_, \cos(2\tau - 2\xi + \xi_,)$$

$$\frac{\delta r}{r} \left[\frac{r \,\mathrm{d} R}{\mathrm{d} r} \right] = \left\{ -\frac{35}{32} - \frac{189}{64} - \frac{35}{16} + \frac{189}{32} \right\} m^3 e^2 e_, \cos(2\tau - 2\xi - \xi_,)$$

$$+ \left\{ \frac{15}{32} + \frac{189}{64} + \frac{15}{16} - \frac{189}{32} \right\} m^3 e^2 e_, \cos(2\tau - 2\xi + \xi_,)$$

$$= -\frac{21}{64} m^3 e^2 e_, \cos(2\tau - 2\xi - \xi_,)$$

$$- \frac{99}{64} m^3 e^2 e_, \cos(2\tau - 2\xi + \xi_,)$$

$$-\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t = \frac{3}{4}m^2\cos 2\tau - \frac{9}{2}m^2 e\cos(2\tau - \xi) + \frac{m^2}{2}e\cos(2\tau + \xi)$$

$$-\frac{15}{8}m e^2\cos(2\tau - 2\xi) + \frac{3}{8}m^2 e^2\cos(2\tau + 2\xi)$$

$$\frac{\mathrm{d}\,.\,\delta\lambda}{\mathrm{d}t} = -3m^2 e_{,}\cos\xi_{,} - \frac{21}{4}m e e_{,}\cos(\xi + \xi_{,}) + \frac{35}{4}m e e_{,}\cos(2\tau - \xi - \xi_{,})$$

$$+\frac{21}{4}m e e_{,}\cos(\xi - \xi_{,}) - \frac{15}{4}m e e_{,}\cos(2\tau - \xi + \xi_{,}) - \frac{105}{8}m e^2 e_{,}\cos(2\xi + \xi_{,})$$

$$+\frac{105}{8}m e^2 e_{,}\cos(2\xi - \xi_{,})$$

$$-\left[\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t\right]\frac{\mathrm{d}\,.\,\delta\lambda}{\mathrm{d}t} = \left\{-\frac{315}{64} + \frac{189}{16} + \frac{45}{16}\right\} m^3 e^2 e_{,}\cos(2\tau - 2\xi - \xi_{,})$$

$$+\left\{\frac{315}{64} - \frac{189}{16} + \frac{45}{16}\right\} m^3 e^2 e_{,}\cos(2\tau - 2\xi + \xi_{,})$$

$$= \frac{621}{64}m^3 e^2 e_{,}\cos(2\tau - 2\xi - \xi_{,}) - \frac{261}{64}m^3 e^2 e_{,}\cos(2\tau - 2\xi + \xi_{,})$$

Hence finally

$$\delta\lambda = \left\{\left\{-\frac{35}{16} + \frac{35}{4}\right\}m + \left\{27 - \frac{105}{32} + \frac{105}{32}\right.\right.$$

$$\left.\left. + \frac{91}{16} - \frac{7}{64} + \frac{207}{64}\right\}m^2\right\} e^2 e_{,}\sin(2\tau - 2\xi - \xi_{,})$$

[24]

$$+\left\{\left\{\frac{15}{16} - \frac{15}{4}\right\}m + \left\{-\frac{77}{32} + \frac{135}{32} - \frac{513}{32}\right.\right.$$

$$\left.\left. + \frac{477}{16} - \frac{99}{64} - \frac{261}{64}\right\}m^2\right\} e^2 e_{,}\sin(2\tau - 2\xi + \xi_{,})$$

[27]

$$= \left\{\frac{105}{16}m + \frac{577}{16}m^2\right\} e^2 e_{,}\sin(2\tau - 2\xi - \xi_{,})$$

[24]

$$+ \left\{-\frac{45}{15}m + \frac{319}{32}m^2\right\} e^2 e_{,}\sin(2\tau - 2\xi + \xi_{,})$$

[27]

The coefficient of argument 24 agrees with that of M. Plana, but M. Plana has $\frac{193}{32}$ instead of $\frac{319}{32}$ argument 27.

On Burckhardt's Expressions for the longitude and latitude of the Moon.

The places of the moon given in the Nautical Almanac are at present derived from Burckhardt's *Tables de la Lune* (Paris, 1812), which were constructed by the aid of more than four thousand observations. The form in which they are presented renders it difficult to deduce from them expressions for the longitude and latitude of the moon comparable with theory.

The following is the explanation given by Burckhardt of the construction of the tables:

" Epoques 1801, le 1^{er} janvier à minuit, méridien de Paris, et l'anomalie étant comptée du périhélie: de la longitude, $3^{s}\,21^{\circ}36'37''5$; de l'anomalie, $6^{s}\,25^{\circ}\,29'\,37''4$; du nœud, $0^{s}\,13^{\circ}\,54'\,58''7$.

" Mouvemens en cent années juliennes de la longitude, $10^{s}\,7^{\circ}52'53''5$; de l'anomalie, $6^{s}\,18^{\circ}\,49'\,5''3$; du nœud, $4^{s}\,14^{\circ}\,10'\,1''2$.

Equations de la longitude, où a = an. ⊙; A = an. ☽; D = long. ☽ − long. ⊙; δ = long. ☽ + supplém. nœud.

Equat.		Equat.	
1	− 659″3 sin *a* − 7″1 sin 2 *a*	16	+ 13″7 sin (D + *a*)
2	+ 147″3 sin (2 D − *a*)	17	− 6″6 sin (2 δ − 2 D + A)
3	− 57″7 sin (2 D + A)	18	+ 6″7 sin (2 D − A − 2 *a*)
4	+ 190″3 sin (2 D − A − *a*)	19	− 4″6 sin (2 A − 2 D − *a*)
5	+ 109″4 sin (A − *a*)	20	+ 7″4 sin 2 (δ − A)
6	− 83″8 sin (2 δ − A)	21	+ 2″8 sin (2 A − 2 D + *a*)
7	− 59″2 sin (2 δ − 2 D)	22	− 1″8 sin (2 δ − 2 D + *a*)
8	− 70″6 sin (A + *a*)	23	+ 2″1 sin (2 D + A − *a*)
	− 0″3 sin 2 (A + *a*)	24	+ 1″1 sin (4 D − 3 A)
9	+ 23″5 sin (A − D)	25	+ 0″9 sin (3 A − 2 D)
	+ 57″9 sin 2 (A − D)	26	− 0″9 sin (2 D − A + 2 *a*)
10	− 2″3 sin (A + D)	27	+ 0″8 sin (2 A − *a*)
	− 4″3 sin 2 (A + D)	28	− 0″7 sin (A + 2 *a*)
11	+ 2″3 sin (D − *a*)	29	+ 0″7 sin (A − 2 *a*)
	+ 7″3 sin 2 (D − *a*)	30	− 1″1 sin (♀ − ♁)
12	− 17″7 sin (2 D + *a*)		+ 0″4 sin 2 (♁ − ♁)
13	− 18″4 sin (2 D − A + *a*)	31	+ 0″8 sin (♀ − ♃)
14	− 12″2 sin (4 D − A)		− 0″2 sin 2 (♁ − ♃)
15	− 10″0 sin (2 D − 2 δ + A)	32	− 7″0 sin suppl. ☊

" Les argumens suivans se corrigent en y ajoutant la somme des équations précédentes :

vection : $+ 4825''5 \sin(2D - A) + 35''5 \sin(4D - 2A)$

nomalie : $\begin{cases} + 22692''4 \sin A + 777''1 \sin 2A + 37''2 \sin 3A \\ + 1''8 \sin 4A \end{cases}$

ariation : $\begin{cases} - 122''7 \sin D + 2138''6 \sin 2D + 2''9 \sin 3D \\ + 9''1 \sin 4D \end{cases}$

éduction : $- 412''2 \sin 2\delta.$

Equations de la latitude.

I	$18518''3 \sin\delta - 5''7 \sin 3\delta$	VII	$+ 22''4 \sin(2D - \delta - a)$
II	$+ 526''2 \sin(2D - \delta)$	VIII	$- 10''1 \sin(2D - \delta + a)$
III	$-$ 8″0 sin long. vraie ☽	IX	$+ 27''0 \sin(2A - D)$
IV	$+ 14''7 \sin(\delta - A)$	X	$+ 5''1 \sin(\delta + 2A - 2D)$
V	$- 25''9 \sin(\delta - a)$	XI	$+ 2''5 \sin(2D + A - \delta)$
VI	$+ 23''9 \sin(\delta + a)$	XII	$+ 16''3 \sin(A + \delta - 2D)$

“ Dans les argumens de la latitude, c’est la longitude vraie de la
ne dans son orbite qu’il faut employer : quant à la parallaxe,
i suivi la théorie de M. Laplace.”

Hence if S denote the sum of Burckhardt’s small equations; E,
e evection; A, the anomaly; V, the variation; and R, the reduc-
n, using the words, evection, anomaly, variation and reduction in
e same sense as M. Burckhardt, then, in the notation of this
eatise,

$$\begin{aligned}
= & - 57''7 \sin(2\tau + \xi) - 659''3 \sin\xi_{,} + 147''3 \sin(2\tau - \xi_{,}) \\
& - 17''7 \sin(2\tau + \xi_{,}) - 57''9 \sin(2\tau - 2\xi) - 4''3 \sin(2\tau + 2\xi) \\
& - 70''6 \sin(\xi + \xi_{,}) + 190''3 \sin(2\tau - \xi - \xi_{,}) \\
& + 109''4 \sin(\xi - \xi_{,}) - 18''4 \sin(2\tau - \xi - \xi_{,}) \\
& + 2''1 \sin(2\tau + \xi - \xi_{,}) - 7''1 \sin 2\xi_{,} + 7''3 \sin(2\tau - 2\xi_{,}) \\
& - 0''9 \sin(2\tau - 3\xi) - 2''8 \sin(2\tau - 2\xi - \xi_{,}) + 0''8 \sin(2\xi - \xi_{,}) \\
& + 4''6 \sin(2\tau - 2\xi + \xi_{,}) - 0''7 \sin(\xi + 2\xi_{,}) \\
& + 6''7 \sin(2\tau - \xi - 2\xi_{,}) + 0''7 \sin(\xi - 2\xi_{,}) - 0''9 \sin(2\tau - \xi + 2\xi_{,}) \\
& - 0''3 \sin(2\xi + 2\xi_{,}) + 59''2 \sin(2\tau - 2\eta) + 83''8 \sin(\xi - 2\eta) \\
& + 6''6 \sin(2\tau - \xi - 2\eta) - 10''0 \sin(2\tau + \xi - 2\eta) \\
& + 1''8 \sin(2\tau - \xi_{,} - 2\eta) - 7''4 \sin(2\xi - 2\eta) - 23''5 \sin(\tau - \xi) \\
& - 2''3 \sin(\tau + \xi) + 2''3 \sin(\tau - \xi_{,}) - 13''7 \sin(\tau + \xi_{,}) \\
& - 12''2 \sin(4\tau - \xi) + 1''1 \sin(4\tau - 3\xi)
\end{aligned}$$

$$\mathrm{E} = 4825''5 \sin(2\tau - \xi + \mathrm{S}) + 35''5 \sin(4\tau - 2\xi + 2\mathrm{S})$$

$$\mathrm{A} = 22692''4 \sin(\xi + \mathrm{S} + \mathrm{E}) + 777''1 \sin(2\xi + 2\mathrm{S} + 2\mathrm{E})$$
$$+ 37''2 \sin(3\xi + 3\mathrm{S} + 3\mathrm{E}) + 1''8 \sin(4\xi + 4\mathrm{S} + 4\mathrm{E})$$

$$\mathrm{V} = -122''7 \sin(\tau + \mathrm{S} + \mathrm{E} + \mathrm{A}) + 2138''6 \sin(2\tau + 2\mathrm{S} + 2\mathrm{E} + 2\mathrm{A})$$
$$+ 2''9 \sin(3\tau + 3\mathrm{S} + 3\mathrm{E} + 3\mathrm{A}) + 9''1 \sin(4\tau + 4\mathrm{S} + 4\mathrm{E} + 4\mathrm{A})$$

$$\mathrm{R} = -412''2 \sin\left\{2g\{\mathrm{n}t + \mathrm{S} + \mathrm{E} + \mathrm{A} + \mathrm{V}\}\right\}.$$

According to Burckhardt's expression for the longitude,

$$\lambda' = \mathrm{n}t + \mathrm{S} + \mathrm{E} + \mathrm{A} + \mathrm{V} + \mathrm{R}$$

rckhardt's xpression the longi- ude of the moon.

$$= \{-\cdot1 - 3\cdot2 + 265\cdot4 - \cdot2 + 2138\cdot6 - 1\cdot2 - 25\cdot8 - \cdot1\} \sin 2\tau$$
$$+ \{\cdot7 + 22692\cdot4 - \cdot1 - 3\cdot1 - \cdot6 + \cdot3 - \cdot2 - \cdot6 - 50\cdot0 + \cdot2 + \cdot9 - \cdot2\} \sin\xi$$
$$+ \{4825\cdot5 - 3\cdot2 - \cdot2 - \cdot9 - \cdot3 - \cdot1 - 235\cdot2 + 1\cdot4\} \sin(2\tau - \xi)$$
$$+ \{-57\cdot7 - \cdot2 + 18\cdot2 - \cdot9 - \cdot2 + 235\cdot2 - \cdot4 - 1\cdot4\} \sin(2\tau + \xi)$$
$$+ \{-659\cdot3 - 2\cdot2 - \cdot2 - 3\cdot9 - 6\cdot0 - \cdot1 + \cdot2 - 1\cdot5 - \cdot2 - \cdot1\} \sin\xi_{,}$$
$$+ \{147\cdot3 + 1\cdot2 + \cdot4 + 10\cdot5 + \cdot1 + \cdot4 + 6\cdot8 + \cdot1\} \sin(2\tau - \xi_{,})$$
$$+ \{-17\cdot7 - \cdot8 - \cdot4 - 1\cdot0 - \cdot4 - \cdot1 - 6\cdot8 - \cdot1\} \sin(2\tau + \xi_{,})$$
$$+ \{-\cdot7 + 777\cdot1 - \cdot4 - 5\cdot5 + \cdot6 - \cdot1 - 2\cdot7\} \sin 2\xi$$
$$+ \{-57\cdot9 + 265\cdot4 - \cdot6 + 12\cdot9 + \cdot4 - 8\cdot0\} \sin(2\tau - 2\xi)$$
$$+ \{-4\cdot3 - 3\cdot2 + 1\cdot3 + 12\cdot9 + 8\cdot0\} \sin(2\tau + 2\xi)$$
$$+ \{-70\cdot6 - \cdot2 - \cdot1 - 36\cdot3 - \cdot1 - \cdot4 + \cdot2 - \cdot2 - 2\cdot0 - \cdot1 - \cdot1\} \sin(\xi + \xi_{,})$$
$$+ \{190\cdot3 + 7\cdot7 + 8\cdot2 - \cdot2 - \cdot8 + \cdot7 + \cdot4\} \sin(2\tau - \xi - \xi_{,})$$
$$+ \{-1\cdot0 - \cdot8 - \cdot7 - \cdot4\} \sin(2\tau + \xi + \xi_{,})$$
$$+ \{109\cdot4 + 1\cdot7 - \cdot1 - \cdot1 + 36\cdot3 + \cdot1 + \cdot2 - \cdot2 - \cdot2 + \cdot2 + \cdot1 + \cdot1\} \sin(\xi - \xi_{,})$$
$$+ \{-18\cdot4 - 7\cdot7 - 1\cdot0 + \cdot2 + \cdot8 - 1\cdot1 - \cdot4\} \sin(2\tau - \xi + \xi_{,})$$
$$+ \{2\cdot1 + \cdot1 + 8\cdot2 + \cdot7 + \cdot1 + \cdot8 + 1\cdot1 + \cdot4\} \sin(2\tau + \xi - \xi_{,})$$
$$+ \{-7\cdot1 - \cdot1 - \cdot1\} \sin 2\xi_{,}$$
$$+ \{7\cdot3 + \cdot4 + \cdot1\} \sin(2\tau - 2\xi_{,}) - \cdot1 \sin(2\tau + 2\xi_{,})$$
$$+ \{-\cdot1 + 37\cdot2 - \cdot3 - \cdot2 - \cdot1 - \cdot3\} \sin 3\xi$$

Burckhard
expressio
for the lon
tude of th
moon.

$$+\{-\cdot9-3\cdot2+18\cdot2-\cdot1+\cdot9-\cdot4-\cdot5\}\sin(2\tau-3\xi)$$

$$+\{-\cdot2-\cdot2+\cdot9+\cdot4+\cdot5\}\sin(2\tau+3\xi)+\{-3\cdot9-2\cdot5-\cdot1\}\sin(2\xi+\xi_{,})$$

$$+\{-2\cdot8+\cdot8-\cdot4+10\cdot5+\cdot4+\cdot5+\cdot1\}\sin(2\tau-2\xi-\xi_{,})$$

$$+\{\cdot8+6\cdot0+\cdot1+2\cdot5-\cdot2-\cdot1\}\sin(2\xi-\xi_{,})$$

$$+\{4\cdot6-1\cdot2+\cdot4-\cdot4-1\cdot0+\cdot1-\cdot1\}\sin(2\tau-2\xi+\xi_{,})$$

$$+\{\cdot1+\cdot5+\cdot1+\cdot1\}\sin(2\tau+2\xi-\xi_{,})+\{-\cdot7-\cdot4-\cdot1\}\sin(\xi+2\xi_{,})$$

$$+\{6\cdot7+\cdot1+\cdot4\}\sin(2\tau-\xi-2\xi_{,})$$

$$+\{\cdot7+\cdot1+\cdot4\}\sin(\xi-2\xi_{,})+\{-\cdot9-\cdot1\}\sin(2\tau-\xi+2\xi_{,})$$

$$+4\sin(2\tau+\xi-2\xi_{,})+1\cdot8\sin4\xi+\{-\cdot2+1\cdot3\}\sin(2\tau-4\xi)$$

$$-\cdot2\sin(3\xi+\xi_{,})+\{-\cdot2+\cdot7\}\sin(2\tau-3\xi-\xi_{,})+\cdot4\sin(3\xi-\xi_{,})$$

$$+\cdot2\sin(2\tau-3\xi-\xi_{,})-\cdot3\sin(2\xi+2\xi_{,})+\cdot4\sin(2\tau-2\xi-2\xi_{,})$$

$$+\{-\cdot1-4\cdot6-\cdot6-412\cdot2+\cdot2+5\cdot0\}\sin2\eta$$

$$+\{59\cdot2+1\cdot0+\cdot4-\cdot5-\cdot1+1\cdot0-4\cdot7\}\sin(2\tau-2\eta)$$

$$+\{-\cdot1-1\cdot0-4\cdot7\}\sin(2\tau+2\eta)$$

$$+\{83\cdot8+\cdot7-\cdot4-\cdot1-\cdot1-\cdot3-45\cdot2+\cdot3\}\sin(\xi-2\eta)$$

$$+\{-\cdot3-\cdot1-\cdot1-45\cdot2+\cdot3+\cdot3\}\sin(\xi+2\eta)$$

$$+\{6\cdot6+3\cdot3-\cdot5-9\cdot2+\cdot1\}\sin(2\tau-\xi-2\eta)$$

$$+\{-\cdot9+\cdot5-9\cdot2+\cdot1\}\sin(2\tau-\xi+2\eta)$$

$$+\{-10\cdot0+\cdot1+3\cdot3+\cdot9+\cdot5-\cdot4\}\sin(2\tau+\xi-2\eta)$$

$$+\{-\cdot5-\cdot4\}\sin(2\tau+\xi+2\eta)+1\cdot3\sin(\xi_{,}-2\eta)$$

$$+1\cdot3\sin(\xi_{,}+2\eta)+\{1\cdot8-\cdot3\}\sin(2\tau-\xi_{,}-2\eta)$$

$$-\cdot3\sin(2\tau-\xi_{,}+2\eta)+\{-7\cdot4-\cdot1+4\cdot6+2\cdot5-1\cdot5\}\sin(2\xi-2\eta)$$

$$+\{-2\cdot5-1\cdot5\}\sin(2\xi+2\eta)+\{\cdot4+\cdot2-1\cdot0-\cdot4\}\sin(2\tau-2\xi-2\eta)$$

$$+\{-1\cdot0+\cdot1-\cdot1+1\cdot0-\cdot4\}\sin(2\tau-2\xi+2\eta)$$

$$+\{-\cdot5+\cdot2+\cdot1+\cdot1\}\sin(2\tau+2\xi-2\eta)$$

$$+\{\cdot2+\cdot1-\cdot1\}\sin(\xi+\xi_{,}-2\eta)+\{\cdot2+\cdot1-\cdot1\}\sin(\xi+\xi_{,}+2\eta)$$

$$+\{\cdot1-\cdot4\}\sin(2\tau-\xi-\xi_{,}-2\eta)-\cdot4\sin(2\tau-\xi-\xi_{,}+2\eta)$$

$$-\cdot3\sin(\xi-\xi_{,}-2\eta)-\cdot3\sin(\xi-\xi_{,}+2\eta)$$

khardt's
ression
e longi-
of the
oon.

$$+ \cdot 1 \sin(2\tau + \xi - \xi_{,} - 2\eta) - \cdot 1 \sin(2\tau - 3\xi + 2\eta)$$

$$+ \{\cdot 2 - 1\cdot 3 - \cdot 1 - 122\cdot 7 + \cdot 4 + \cdot 1 - \cdot 1\} \sin\tau$$

$$+ \{-23\cdot 5 - 1\cdot 4 + 6\cdot 7\}\sin(\tau-\xi) + \{-2\cdot 3 - 6\cdot 7 + \cdot 2\}\sin(\tau+\xi)$$

$$+ \{2\cdot 3 - \cdot 2 - \cdot 1\}\sin(\tau-\xi_{,}) + \{13\cdot 7 + \cdot 2\}\sin(\tau+\xi_{,})$$

$$+ \{-1\cdot 3 - \cdot 2 - \cdot 1 - \cdot 1\}\sin(\tau - 2\xi) + \{-\cdot 1 - \cdot 2\}\sin(\tau+2\xi)$$

$$+ \{-\cdot 2 + \cdot 1 - \cdot 1\}\sin(\tau-\xi-\xi_{,}) + \cdot 7\sin(\tau+\xi+\xi_{,})$$

$$+ \cdot 7\sin(\tau-\xi+\xi_{,}) + \cdot 1\sin(\tau+\xi-\xi_{,})$$

$$+ \{2\cdot 9 - \cdot 1 - \cdot 1\}\sin 3\tau + \{-1\cdot 4 - \cdot 2 - \cdot 5\}\sin(3\tau-\xi)$$

$$+ \{\cdot 1 + \cdot 5\}\sin(3\tau+\xi) + \cdot 1\sin(3\tau+\xi_{,})$$

$$+ \{-\cdot 2 + \cdot 1 - \cdot 1\}\sin(3\tau-2\xi) - \cdot 1\sin(3\tau-\xi-\xi_{,})$$

$$+ \cdot 2\sin(3\tau-\xi+\xi_{,}) + \{-\cdot 7 - \cdot 7 + \cdot 2 + 5\cdot 5 + 2\cdot 7 + 9\cdot 1\}\sin 4\tau$$

$$+ \{-12\cdot 2 + 1\cdot 5 + 1\cdot 9 - \cdot 3 + \cdot 3 + 50\cdot 0 - 2\cdot 0 - \cdot 6\}\sin(4\tau-\xi)$$

$$+ \{-\cdot 1 + \cdot 3 + \cdot 2 - \cdot 6 + \cdot 2 + 2\cdot 0 + \cdot 3\}\sin(4\tau+\xi)$$

$$+ \{\cdot 1 + \cdot 2 + 1\cdot 5 + \cdot 1 + \cdot 1\}\sin(4\tau-\xi)$$

$$- \cdot 2\sin(4\tau+\xi_{,}) + \{35\cdot 5 - \cdot 7 - 5\cdot 5 - \cdot 6 + 2\cdot 7\}\sin(4\tau-2\xi_{,})$$

$$- \cdot 1\sin(4\tau+2\xi) + \{1\cdot 7 + \cdot 1 + \cdot 1 + \cdot 2 - \cdot 2 + 2\cdot 0 + \cdot 1 + \cdot 1\}\sin(4\tau - \xi - \xi_{,})$$

$$+ \{-\cdot 2 - \cdot 2 - \cdot 2 - \cdot 1\}\sin(4\tau-\xi-\xi_{,}) + \cdot 1\sin(4\tau - 2\xi_{,})$$

$$+ \{\cdot 6 - \cdot 1\}\sin(4\tau-2\eta) + \{-\cdot 4 + \cdot 3\}\sin(3\xi-2\eta) - \cdot 1\sin(2\tau-4\eta)$$

$$+ \{1\cdot 1 - \cdot 7 - 1\cdot 5 + 1\cdot 9 - \cdot 3 - \cdot 2 + \cdot 1 + \cdot 3\}\sin(4\tau-3\xi)$$

$$+ \{\cdot 7 + \cdot 1 - \cdot 1 - \cdot 1\}\sin(4\tau-\xi-2\eta)$$

$$- \cdot 1\sin(4\tau+\xi-2\eta) + \{-\cdot 2 - \cdot 1 + \cdot 1\}\sin(4\tau-2\xi+\xi_{,})$$

$$+ \{2\cdot 2 + \cdot 1 + \cdot 1 - \cdot 2 + \cdot 1\}\sin(4\tau-2\xi-\xi_{,}) + \{\cdot 1 + \cdot 1\}\sin(4\tau-\xi-2\xi_{,})$$

$$- \cdot 1\sin(4\tau-3\xi-\xi_{,}) + \{\cdot 1 - \cdot 2\}\sin(4\tau-4\xi) + \{-\cdot 1 + \cdot 1\}\sin(4\tau-3\xi-\xi_{,})$$

$$- \cdot 1\sin(4\tau-3\xi-\xi_{,}) + \cdot 1\sin(4\tau-4\xi) - \cdot 1\sin(4\tau-3\xi-\xi_{,})$$

$$+ \cdot 1\sin(4\tau-3\xi+\xi_{,}) + \cdot 1\sin(4\tau-2\xi-2\xi_{,})$$

$$+ \{-\cdot 1 - \cdot 1\}\sin(4\tau-2\xi+2\eta) + \{\cdot 1 + \cdot 1 - \cdot 1\}\sin(4\tau-2\xi-2\eta)$$

$$- \cdot 1\sin(4\tau-\xi-\xi_{,}-2\eta) + \{-\cdot 1 - \cdot 1 + \cdot 4\}\sin(6\tau-\xi)$$

$$+ \{\cdot 6 + \cdot 4\}\sin(6\tau-2\xi) + \cdot 1\sin(6\tau-3\xi) - \cdot 2\sin(\xi-4\eta)$$

Retaining only the more considerable equations, (the coefficients of MM. Plana and Damoiseau are written underneath for the convenience of comparison,)

Comparison of Burckhardt's expression for the longitude of the moon with those of MM. Plana and Damoiseau

$= 2373{\cdot}4 \sin 2\tau + 22639{\cdot}7 \sin \xi + 4587{\cdot}0 \sin (2\tau - \xi)$
2370·3 [1] 22641·6 [2] 4585·6 [3]
2370·0 22639·7 4589·6

$+ 192{\cdot}6 \sin (2\tau + \xi) - 673{\cdot}3 \sin \xi_{\prime} + 166{\cdot}8 \sin (2\tau - \xi_{\prime})$
192·1 [4] 668·6 [5] 165·8 [6]
192·2 673·7 165 6

$- 27{\cdot}3 \sin (2\tau + \xi_{\prime}) + 768{\cdot}3 \sin 2\xi + 212{\cdot}2 \sin (2\tau - 2\xi)$
23·6 [7] 769·5 [8] 212·4 [9]
24·8 768·7 211·6

$+ 14{\cdot}7 \sin (2\tau + 2\xi) - 109{\cdot}9 \sin (\xi + \xi_{\prime}) + 206{\cdot}3 \sin (2\tau - \xi - \xi_{\prime})$
14·1 [10] 111·1 [11] 209·7 [12]
14·7 109·3 207·1

$- 2{\cdot}9 \sin (2\tau + \xi + \xi_{\prime}) + 147{\cdot}5 \sin (\xi - \xi_{\prime}) - 27{\cdot}6 \sin (2\tau - \xi + \xi_{\prime})$
2·9 [13] 148·1 [14] 28·8 [15]
3·0 147·7 28·7

$+ 13{\cdot}5 \sin (2\tau + \xi - \xi_{\prime}) - 7{\cdot}3 \sin 2\xi_{\prime} + 7{\cdot}8 \sin (2\tau - 2\xi_{\prime})$
14·0 [16] 7·9 [17] 7·8 [18]
14·7 7·3 7·9

$+ 36{\cdot}2 \sin 3\xi + 14{\cdot}0 \sin (2\tau - 3\xi)$
36·7 [20] 12·8 [21]
36·9 12·8

$+ 1{\cdot}4 \sin (2\tau + 3\xi) - 6{\cdot}5 \sin (2\xi + \xi_{\prime})$
3·3 [22] 7·3 [23]
1·3 7·7

$+ 9{\cdot}1 \sin (2\tau - 2\xi - \xi_{\prime}) + 9{\cdot}1 \sin (2\xi - \xi_{\prime}) + 2{\cdot}4 \sin (2\tau - 2\xi + \xi_{\prime})$
7·8 [24] 9·6 [26] 1·4 [27]
9·0 9·7 2·5

parison of
rckhardt's
pression
the longi-
le of the
oon with
se of MM.
ina and
moiseau.

$$-1{\cdot}2\sin(\xi+2\xi_{,})+7{\cdot}2\sin(2\tau-\xi-2\xi_{,})+1{\cdot}2\sin(\xi-2$$

1·2	[29]	7·5	[30]	2·1	[32
1·1		7·5		2·5	

$$-1{\cdot}0\sin(2\tau-\xi+2\xi_{,})+1{\cdot}8\sin4\xi+1{\cdot}1\sin(2\tau-4\xi)$$

2·0	[33]	2·0	[38]	·9	[39]
2·6		2·0		·9	

$$-412{\cdot}3\sin2\eta+56{\cdot}3\sin(2\tau-2\eta)-5{\cdot}8\sin(2\tau+2\eta)$$

411·0	[62]	54·9	[63]	3·4	[64]
411·7		54·8		5·7	

$$+38{\cdot}7\sin(\xi-2\eta)-45{\cdot}1\sin(\xi+2\eta)-9{\cdot}5\sin(2\tau-\xi+2\eta$$

37·2	[65]	45·2	[66]	9·4	[68]
39·5		45·1		9·6	

$$-5{\cdot}6\sin(2\tau+\xi-2\eta)+1{\cdot}3\sin(\xi_{,}-2\eta)$$

6·1	[69]	·1	[71]
6·6		·0	

$$+1{\cdot}3\sin(\xi_{,}+2\eta)+1{\cdot}5\sin(2\tau-\xi_{,}-2\eta)$$

·6	[72]	2·3	[73]
·4		2·9	

$$-1{\cdot}9\sin(2\xi-2\eta)-4{\cdot}0\sin(2\xi+2\eta)-123{\cdot}5\sin\tau$$

1·1	[77]	4·1	[78]	122·1	[101]
1·3		4·0		122·5	

$$-18{\cdot}2\sin(\tau-\xi)-8{\cdot}8\sin(\tau+\xi)+2{\cdot}0\sin(\tau-\xi_{,})$$

18.0	[102]	8·2	[103]	− ·4	[104]
17·2		8·4		·5	

$$+13{\cdot}9\sin(\tau+\xi_{,})-1{\cdot}7\sin(\tau-2\xi)+2{\cdot}7\sin3\tau$$

17·2	[105]	·8	[106]	·9	[116]
17·6		1·2		·0	

$$-2{\cdot}1\sin(3\tau-\xi)+16{\cdot}1\sin4\tau+38{\cdot}6\sin(4\tau-\xi)$$

2·9	[117]	14·4	[131]	38·0	[132]
3·0		14·8		38·6	

$$\begin{array}{llllll}
+ 2{\cdot}3 \sin(4\tau + \xi) & & + 2{\cdot}0 \sin(4\tau - \xi_{,}) & & + 31{\cdot}4 \sin(4\tau - 2\xi) & \\
\quad {\cdot}8 & [133] & \quad 1{\cdot}2 & [134] & \quad 34{\cdot}5 & [136] \\
- {\cdot}4 & & \quad {\cdot}8 & & \quad 31{\cdot}2 &
\end{array}$$

$$\begin{array}{lll}
+ 4{\cdot}1 \sin(4\tau - \xi - \xi_{,}) & & + {\cdot}7 \sin(4\tau - 3\xi) \\
\quad 3{\cdot}8 & [138] & \quad {\cdot}5 \\
\quad 3{\cdot}3 & & \quad 1{\cdot}4
\end{array}$$

$$\begin{array}{ll}
+ 2{\cdot}3 \sin(4\tau - 2\xi - \xi_{,}) & + 1{\cdot}0 \sin(6\tau - 2\xi) \\
\quad 1{\cdot}2 & \quad {\cdot}0 \\
\quad 3{\cdot}0 & \quad {\cdot}5
\end{array}$$

+ planetary terms.

M. Damoiseau's coefficient of the annual equation agrees nearly with that of Burckhardt, but not with that of M. Plana.

Burckhardt's expression seems to amount to making the moon's latitude

$$\begin{aligned}
&= 18518''3 \sin(g\lambda - \nu) + 526''2 \sin(2\tau - \eta + S + E + A + V) \\
&\quad - 14''7 \sin(\xi - \eta) - 16''3 \sin(2\tau - \xi - \eta) \\
&\quad + 2''5 \sin(2\tau + \xi - \eta + S + E + A + V) \\
&\quad + 25''9 \sin(\xi_{,} - \eta - S - E - A - V) \\
&\quad + 23''9 \sin(\xi_{,} + \eta + S + E + A + V) \\
&\quad + 22''4 \sin(2\tau - \xi_{,} - \eta + S + E + A + V) \\
&\quad - 10''1 \sin(2\tau + \xi_{,} + \eta + S + E + A + V) \\
&\quad + 27''0 \sin(2\xi - \eta + S + E + A + V) \\
&\quad - 5''1 \sin(2\tau - 2\xi - \eta) - 5''7 \sin 3\eta
\end{aligned}$$

$$\lambda = n t + S + E + A + V.$$

$$\begin{aligned}
\text{Lat.} &= \{18518{\cdot}3 - {\cdot}6 - 55{\cdot}8 - 2{\cdot}8 + 2{\cdot}7 - {\cdot}1\} \sin \eta \\
&\quad + \{-11{\cdot}3 + 105{\cdot}9 + 2{\cdot}7 - {\cdot}3 + 526{\cdot}2\} \sin(2\tau - \eta) \\
&\quad + \{11{\cdot}3 + 105{\cdot}9 - {\cdot}3\} \sin(2\tau + \eta) \\
&\quad + \{1{\cdot}2 + 1012{\cdot}2 + 3{\cdot}8 - 1{\cdot}5 - 6{\cdot}1 - 14{\cdot}7 - 1{\cdot}5 + {\cdot}1\} \sin(2\tau - \eta) \\
&\quad + \{-1{\cdot}2 + 1012{\cdot}2 - 1{\cdot}5\} \sin(\xi + \eta)
\end{aligned}$$

thardt's
ession
e lati-
of the
oon.

$$+\left\{5\cdot 8+205\cdot 1+\cdot 4-\cdot 6-28\cdot 9+\cdot 3-16\cdot 3\right\}\sin(2\tau-\xi-\nu)$$

$$+\left\{-5\cdot 8+205\cdot 1-\cdot 6-\cdot 1\right\}\sin(2\tau-\xi+\nu)$$

$$+\left\{-5\cdot 8+8\cdot 8-\cdot 2+\cdot 3+28\cdot 9+2\cdot 5+\cdot 3\right\}\sin(2\tau+\xi-\nu)$$

$$+\left\{5\cdot 8+8\cdot 8+\cdot 3\right\}\sin(2\tau+\xi+\nu)+\left\{-29\cdot 9+25\cdot 9-\cdot 1-\cdot 1\right\}\sin(\xi_{,}-\nu)$$

$$+\left\{-29\cdot 9+23\cdot 9+\cdot 1\right\}\sin(\xi_{,}+\nu)$$

$$+\left\{\cdot 2+7\cdot 4+\cdot 9+\cdot 1+22\cdot 4\right\}\sin(2\tau-\xi_{,}-\nu)$$

$$+\left\{\cdot 2+7\cdot 4-\cdot 1\right\}\sin(2\tau-\xi_{,}+\nu)$$

$$+\left\{\cdot 2-1\cdot 2-\cdot 9-\cdot 1-10\cdot 1\right\}\sin(2\tau+\xi_{,}-\nu)$$

$$+\left\{-\cdot 2-1\cdot 2+\cdot 1\right\}\sin(2\tau+\xi_{,}+\nu)$$

$$+\left\{-27\cdot 9+34\cdot 5+\cdot 1+27\cdot 0\right\}\sin(2\xi-\nu)+\left\{27\cdot 9+34\cdot 5\right\}\sin(2\xi+\nu)$$

$$+\left\{11\cdot 3+9\cdot 5+\cdot 1-1\cdot 0-5\cdot 1\right\}\sin(2\tau-2\xi-\nu)$$

$$+\left\{-11\cdot 3+9\cdot 5+\cdot 1+\cdot 1\right\}\sin(2\tau-2\xi+\nu)$$

$$+\left\{\cdot 7+\cdot 1+1\cdot 0+\cdot 1\right\}\sin(2\tau+2\xi-\nu)+\left\{\cdot 7+\cdot 1\right\}\sin(2\tau+2\xi+\nu)$$

$$+\left\{-1\cdot 6-4\cdot 9-\cdot 2-1\cdot 4\right\}\sin(\xi+\xi_{,}-\nu)$$

$$+\left\{-1\cdot 6-4\cdot 9+1\cdot 3\right\}\sin(\xi+\xi_{,}+\nu)$$

$$+\left\{\cdot 3+9\cdot 3+\cdot 1+\cdot 3-1\cdot 2\right\}\sin(2\tau-\xi-\xi_{,}-\nu)$$

$$+\left\{-\cdot 3+9\cdot 3-\cdot 3\right\}\sin(2\tau-\xi-\xi_{,}+\nu)+\left\{-\cdot 1-\cdot 5\right\}\sin(2\tau+\xi+\xi_{,}-\nu)$$

$$+\left\{-1\cdot 6+6\cdot 6+1\cdot 3-\cdot 2\right\}\sin(\xi-\xi_{,}-\nu)$$

$$+\left\{1\cdot 6+6\cdot 6-1\cdot 4+\cdot 1\right\}\sin(\xi-\xi_{,}+\nu)$$

$$+\left\{-\cdot 3-1\cdot 2-\cdot 1-\cdot 3+\cdot 5\right\}\sin(2\tau-\xi+\xi_{,}-\nu)$$

$$+\left\{\cdot 3-1\cdot 2+\cdot 3\right\}\sin(2\tau-\xi+\xi_{,}+\nu)$$

$$+\left\{\cdot 6+\cdot 1+1\cdot 2\right\}\sin(2\tau+\xi-\xi_{,}-\nu)$$

$$+\cdot 6\sin(2\tau+\xi-\xi_{,}+\nu)-\cdot 3\sin(2\xi_{,}-\nu)$$

$$-\cdot 3\sin(2\xi_{,}+\nu)+\cdot 3\sin(2\tau-2\xi_{,}-\nu)+\cdot 3\sin(2\tau-2\xi_{,}+\nu)$$

$$+\left\{-5\cdot 5+\cdot 1\right\}\sin(\tau-\nu)-5\cdot 5\sin(\tau+\nu)$$

$$-\cdot 8\sin(\tau-\xi-\nu)-\cdot 8\sin(\tau-\xi+\nu)-\cdot 4\sin(\tau+\xi-\nu)$$

$$-\cdot 4\sin(\tau+\xi+\nu)+\cdot 6\sin(\tau+\xi_{,}-\nu)$$

$$+\cdot 6\sin(\tau+\xi_{,}+\nu)+\left\{1\cdot 7+\cdot 5+1\cdot 5\right\}\sin(3\xi-\nu)+\left\{1\cdot 7+\cdot 5\right\}\sin(3\xi+\nu)$$

Burckhard expressio for the latitude of th moon.

$$+\{\cdot6-\cdot3-\cdot3\}\sin(2\tau-3\xi-\eta)+\{\cdot6-\cdot3+3\}\sin(2\tau-3\xi+\eta)$$

$$-\cdot3\sin(2\xi+\xi_{,}-\eta)-\cdot3\sin(2\xi+\xi_{,}+\eta)$$

$$+\cdot3\sin(2\xi-\xi_{,}-\eta)+\cdot3\sin(2\xi-\xi_{,}+\eta)$$

$$+\{-\cdot3+\cdot6+2\cdot7\}\sin(4\tau-\eta)+\{\cdot3+\cdot6\}\sin(4\tau+\eta)$$

$$+\{-1\cdot2+1\cdot5+6\cdot1\}\sin(4\tau-\xi-\eta)+\{1\cdot2+1\cdot5\}\sin(4\tau-\xi+\eta)$$

$$+\{\cdot1+\cdot1\}\sin(4\tau-\xi_{,}-\eta)+\cdot1\sin(4\tau+\xi_{,}-\eta)+\{1\cdot4+1\cdot2\}\sin(4\tau-2\xi+\eta)$$

$$+\{-1\cdot4+1\cdot2-\cdot1-\cdot1\}\sin(4\tau-2\xi-\eta)-5\cdot7\sin3\eta+2\cdot7\sin(2\tau-3\eta)$$

$$+3\cdot8\sin(\xi-3\eta)+\cdot4\sin(2\tau-\xi-3\eta)$$

$$+\{-\cdot2+\cdot1\}\sin(2\tau+\xi-3\eta)+\{\cdot2+\cdot2\}\sin(4\tau-\xi-\xi_{,}-\eta)$$

$$-\cdot1\sin(4\tau-\xi+\xi_{,}-\eta)+\cdot1\sin(4\tau-3\eta)+\cdot1\sin(4\tau-3\xi-\eta)$$

Retaining only the more considerable equations

$$\begin{array}{llllll}
\text{Lat.} = 18461\cdot7\sin\eta & +\ 623\cdot4\sin(2\tau-\eta) & & +\ 116\cdot9^{*}\sin(2\tau+\eta) & \\
\quad 18465\cdot5\,[146] & \quad 621\cdot5 & [147] & \quad 101\cdot7 & [148]
\end{array}$$

$$\begin{array}{llll}
+\ 993\cdot5\sin(\xi-\eta) & & +\ 1009\cdot5\sin(\xi+\eta) & \\
\quad 1000\cdot5 & [149] & \quad 1010\cdot9 & [150]
\end{array}$$

$$\begin{array}{llll}
+\ 165\cdot8\sin(2\tau-\xi-\eta) & & +\ 198\cdot6\sin(2\tau-\xi+\eta) & \\
\quad 166\cdot6 & [151] & \quad 196\cdot2 & [152]
\end{array}$$

$$\begin{array}{llll}
+\ 34\cdot5\sin(2\tau+\xi-\eta) & & +\ 14\cdot9\sin(2\tau+\xi+\eta) & \\
\quad 35\cdot0 & [153] & \quad 14\cdot7 & [154]
\end{array}$$

$$\begin{array}{llllll}
-\ 4\cdot2\sin(\xi_{,}-\eta) & & -\ 5\cdot9\sin(\xi_{,}+\eta) & & +\ 31\cdot0\sin(2\tau-\xi_{,}-\eta) & \\
\quad 4\cdot1 & [155] & \quad 5\cdot4 & [156] & \quad 29\cdot5 & [157]
\end{array}$$

$$\begin{array}{llll}
+\ 7\cdot5\sin(2\tau-\xi_{,}+\eta) & & -\ 12\cdot1\sin(2\tau+\xi_{,}-\eta) & \\
\quad 7\cdot6 & [158] & \quad 12\cdot3 & [159]
\end{array}$$

$$\begin{array}{llllll}
-\ 1\cdot3\sin(2\tau+\xi_{,}+\eta) & & +\ 33\cdot7\sin(2\xi-\eta) & & +\ 62\cdot4\sin(2\xi+\eta) & \\
\quad 1.2 & [160] & \quad 32\cdot3 & [161] & \quad 62\cdot5 & [162]
\end{array}$$

$$\begin{array}{llll}
+\ 14\cdot8\sin(2\tau-2\xi-\eta) & & -\ 1\cdot6\sin(2\tau-2\xi+\eta) & \\
\quad 16\cdot2 & [163] & \quad 1\cdot9 & [164]
\end{array}$$

---

* There appears to be an error with M. Plana in the numerical conversion. In vol. i. p. 719, line 1, I think + 20″,347 (4), should be + 36″,219 (4), and the coefficient of sin ($2\,E\,n\,t+g\,n\,t$) instead of 101″,708, 117″,580, agreeing with Burckhardt.

Burckhardt's expression for the latitude of the moon.

$$+ 1{\cdot}9 \sin (2\tau + 2\xi - \eta) + {\cdot}8 \sin (2\tau + 2\xi + \eta)$$

2·4 [165] ·9 [166]

$$- 8{\cdot}1 \sin (\xi + \xi_{\prime} - \eta) - 5{\cdot}2 \sin (\xi + \xi_{\prime} + \eta)$$

4·9 [167] 5·3 [168]

$$+ 8{\cdot}8 \sin (2\tau - \xi - \xi_{\prime} - \eta) + 8{\cdot}7 \sin (2\tau - \xi - \xi_{\prime} + \eta)$$

8·2 [169] 9·2 [170]

$$- {\cdot}6 \sin (2\tau + \xi + \xi_{\prime} - \eta) + 6{\cdot}1 \sin (\xi - \xi_{\prime} - \eta)$$

+ ·9 [171] 5·2 [173]

$$- 1{\cdot}4 \sin (2\tau - \xi + \xi_{\prime} - \eta) - {\cdot}7 \sin (2\tau - \xi + \xi_{\prime} + \eta)$$

1·9 [175] 2·7 [176]

$$+ 1{\cdot}9 \sin (2\tau + \xi - \xi_{\prime} - \eta) - 5{\cdot}4 \sin (\tau - \eta) - 5{\cdot}5 \sin (\tau + \eta)$$

1·6 [177] 4·6 [185] 5·0 [186]

$$+ 3{\cdot}7 \sin (3\xi - \eta) + 2{\cdot}2 \sin (3\xi + \eta) + 3{\cdot}0 \sin (4\tau - \eta)$$

1·7 [195] [196] 1·4

$$+ {\cdot}9 \sin (4\tau + \eta) + 6{\cdot}4 \sin (4\tau - \xi - \eta)$$

·4 6·5

$$+ 2{\cdot}7 \sin (4\tau - \xi + \eta) + 2{\cdot}6 \sin (4\tau - 2\xi + \eta)$$

1·4 1·2

$$- 5{\cdot}7 \sin 3\eta + 2{\cdot}7 \sin (2\tau - 3\eta) + 3{\cdot}8 \sin (\xi - 3\eta)$$

6·4 2·4 2·8

Mr. Airy has also executed the troublesome conversion of Burckhardt's expression for the longitude independently, and having favoured me with the communication of his results I am inclined to think that the expression for the longitude is accurate. These calculations are, however, so liable to error, that no reliance ought to be placed in the above expression for the latitude until it has been carefully verified by some other person. I attach but little value to Burckhardt's coefficients. I am not aware that the *reduced* places of the moon which either Burg or Burckhardt employed in the construction of their Lunar Tables have ever been published: they would be a valuable acquisition to astronomy, and no doubt they exist somewhere in MS., if not in print.

On the Calculation of λ_2, λ_3 *and* λ_4.

Page 204 for $+ 2 e \cos \xi \left\{ \frac{131}{18} m^4 \cos 2 \tau + \frac{7}{8} m^4 \cos 4 \tau \right\}$

$$\text{read } + 2 e \cos \xi \left\{ - \frac{179}{288} m^4 + \frac{131}{18} m^4 \cos 2 \tau + \frac{7}{8} m^4 \cos 4 \tau \right\}$$

Hence

$$\frac{d \lambda}{d t} = \left\{ 2 + m^2 - \frac{165}{32} m^3 - \frac{39691}{1152} m^4 \right\} e \cos \xi$$

$$\lambda_2 = \left\{ 2 + m^2 - \frac{165}{32} m^3 - \frac{39691}{1152} m^4 \right\} \left\{ 1 + \frac{3}{4} m^2 + \frac{225}{32} m^3 + \frac{4143}{128} m^4 \right\}$$

$$= 2 + \frac{5}{2} m^2 + \frac{285}{32} m^3 + \frac{35747}{1152} m^4.$$

In order that this value may be consistent with that of M. Plana, the following equation ought to obtain

$$\left\{ 2 + \frac{5}{2} m^2 + \frac{285}{32} m^3 + \frac{35747}{1152} m^4 \right\} \left\{ 1 + \frac{m^2}{6} - \frac{645}{128} m^3 - \frac{152129}{4608} m^4 \right\}$$

$$= \left\{ 2 + \frac{3}{2} m^2 - \frac{75}{64} m^3 - \frac{6559}{256} m^4 \right\} \left\{ 1 + \frac{2}{3} m^2 - \frac{1253}{288} m^4 \right\}$$

which, however, is not the case.

Page 198, line 2 from foot, read $+ \frac{765}{256}$ instead of $- \frac{765}{256}$, hence $\frac{a}{\mu} \delta R$ contains the term

$\frac{64427}{3072} m^5 e \cos(2\tau - \xi)$ instead of $\frac{43667}{3072} m^5 e \cos(2\tau - \xi)$

and $$\frac{\mathbf{a} R}{\mu} = \left\{ \frac{9}{4} m^2 + \frac{15}{16} m^3 + \frac{235}{64} m^4 + \frac{38147}{3072} m^5 \right\} e \cos(2\tau - \xi)$$

$$\mathbf{a}^2 \left(\delta \frac{1}{r}\right)^2 = \left\{ -\frac{2685}{1152} + \frac{33113}{4608} + \frac{19}{9} + \frac{495}{128} \right\} m^5 e \cos(2\tau - \xi)$$

$$= \frac{49921}{3072} m^5 e \cos(2\tau - \xi) \text{ instead of } \frac{22403}{1536} m^5 e \cos(2\tau - \xi).$$

(See p. 201.)

$$\mathbf{a}^3 \left(\delta \frac{1}{r}\right)^3 = \frac{95}{32} m^5 e \cos(2\tau - \xi)$$

$\mathfrak{r}_3$ contains the terms (See p. 202.)

$$\left\{ \frac{383}{18} + \frac{40193}{3072} - \frac{95}{16} \right\} m^5 e \cos(2\tau - \xi)$$

$$= \frac{269155}{9216} m^5 e \cos(2\tau - \xi).$$

Uniting to this terms found before, (See p. 122.)

$$\mathfrak{r}_3 = \frac{3}{2} m^2 + \frac{91}{16} m^3 + \frac{2657}{192} m^4 + \frac{269155}{9216} m^5.$$

Let $r_3 = \frac{15}{8} m + \frac{187}{32} m^2 + \frac{33113}{1536} m^3 + A m^4$, then

$$\left\{ \frac{15}{8} m + \frac{187}{32} m^2 + \frac{33113}{1536} m^3 + A m^4 \right\}$$

$$\left\{ -4m + \frac{11}{2} m^2 + \frac{177}{16} m^3 + \frac{2307}{64} m^4 \right\} \quad \text{(See p. 122.)}$$

$$= \left\{ 1 - 4m + \frac{11}{2} m^2 + \frac{177}{16} m^3 \right\}$$

$$\left\{ \frac{3}{2} m^2 + \frac{91}{16} m^3 + \frac{2657}{192} m^4 + \frac{269155}{9216} m^5 \right\}$$

$$-2 \left\{ \left\{ 2 + 2m + 4m^2 + \frac{13}{2} m^3 \right\} \left\{ \frac{9}{4} m^2 + \frac{15}{16} m^3 + \frac{235}{64} m^4 + \frac{38147}{3072} m^5 \right\} - \frac{15}{8} m^4 - \frac{259}{32} m^5 \right\}$$

Equating coefficients,

$$\frac{34605}{512}+\frac{33099}{512}+\frac{364243}{3072}-4A$$

$$=\frac{269155}{9216}-\frac{2657}{48}+\frac{1001}{32}+\frac{531}{32}-\frac{38147}{768}-\frac{235}{16}-\frac{15}{2}-\frac{117}{4}+\frac{259}{16}$$

$$A=\frac{1450501}{18432}$$

$$\frac{d\lambda}{n\,dt}=\left\{-\frac{141}{8}+\frac{1}{2}-\frac{3}{2}+\frac{5}{8}-\frac{203}{48}+\frac{1450501}{9216}+\frac{131}{18}\right.$$

$$\left.+\frac{251}{96}\right\}m^4 e\cos(2\tau-\xi).\quad \text{(See p. 204.)}$$

Uniting to these terms found before, (See p. 196.)

$$\frac{d\lambda}{n\,dt}=\left\{\frac{15}{4}m+\frac{143}{16}m^2+\frac{28729}{768}m^3+\frac{1336805}{9216}m^4\right\}e\cos(2\tau-\xi)$$

$$\lambda_3=\left\{\frac{15}{4}m+\frac{143}{16}m^2+\frac{28729}{768}m^3+\frac{1336805}{9216}m^4\right\}$$

$$\left\{1+2m+\frac{13}{4}m^2-\frac{65}{32}m^3\right\}$$

$$=\frac{15}{4}m+\frac{263}{16}m^2+\frac{51817}{768}m^3+\frac{2223797}{9216}m^4.$$

If M. Plana's *Eccentricity* be denoted by (e)

$$e=(e)\left\{1-\frac{m^2}{2}-\frac{645}{128}m^3\right\}\ \text{nearly.}\quad \text{(See p. 319.)}$$

Hence, with M. Plana's system of constants,

$$\lambda_3=\left\{\frac{15}{4}m+\frac{263}{16}m^2+\frac{51817}{768}m^3+\frac{2223797}{9216}m^4\right\}\left\{1-\frac{m^2}{2}-\frac{645}{128}m^3\right\}$$

$$=\frac{15}{4}m+\frac{263}{16}m^2+\left\{\frac{51817}{768}-\frac{15}{8}\right\}m^3+\left\{\frac{2223797}{9216}-\frac{263}{32}-\frac{9675}{512}\right\}m^4$$

$$=\frac{15}{4}m+\frac{263}{16}m^2+\frac{50377}{768}m^3+\frac{1973903}{9216}m^4$$

which agrees with the value of λ_3 given by M. Plana.

Page 205, line 1, read $\frac{24227}{576} m^4$ instead of $\frac{13989}{576} m^4$

$$\lambda_4 = \left\{\frac{51}{8} m^2 + \frac{135}{8} m^3 + \frac{24227}{575} m^4\right\} \left\{\frac{1}{3} + \frac{2}{9} m + \frac{25}{108} m^2\right\}$$

$$= \frac{17}{8} m^2 + \frac{169}{24} m^3 + \frac{33257}{1728} m^4.$$

With M. Plana's system of constants

$$\lambda_4 = \left\{\frac{17}{8} m^2 + \frac{169}{24} m^3 + \frac{33257}{1728} m^4\right\} \left\{1 - \frac{m^2}{2}\right\}$$

$$= \frac{17}{8} m^2 + \frac{169}{24} m^3 + \frac{21421}{1728} m^4.$$

M. Plana has

$$\frac{17}{8} m^2 + \frac{169}{24} m^3 + \frac{10495}{576} m^4$$

The difference in the numerical coefficient is $2''\cdot 06$ sex.

The above numerical corrections in pp. 204 and 205 are due to M. de Pontécoulant.

On the Calculation of λ_9.

There are several mistakes in the calculation of λ_9 (page 225), in consequence of which I obtained a value inconsistent with that of M. Plana. I am indebted to M. de Pontécoulant for notice of some of these mistakes. This inequality of longitude is one of the most troublesome, in consequence of the nature of the divisor introduced by integration, which is a multiple of m.

At the foot of page 222 read for the value of $\frac{a}{\mu} \delta \frac{dR}{d\lambda}$

$$= \left\{ -\frac{15}{8} + \frac{675}{128} + \frac{9}{4} - \frac{5}{4} - \frac{129}{32} + \frac{3375}{512} + \frac{45}{4} \right.$$

$$\left. - \frac{2025}{512} - \frac{675}{64} - \frac{675}{64} \right\} m^4 e^2 \sin(2\tau - 2\xi)$$

Hence it appears that $\frac{dR}{d\lambda}$ contains the term

$$\left\{\frac{15}{4} m^2 - \frac{1751}{256} m^4\right\} e^2 \sin(2\tau - 2\xi)$$

$$\frac{\mathrm{a}}{\mathrm{n}\,\mu}\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t=\left\{\frac{15}{4}m-\frac{1751}{256}m^3\right\}\frac{1}{2}\left\{1+\frac{3}{4}m+\frac{243}{32}m^2\right\}e^2\cos\left(2\,\tau-2\,\xi\right)$$

$$=\left\{\frac{15}{8}m+\frac{45}{32}m^2+\frac{5539}{512}m^3\right\}e^2\cos\left(2\,\tau-2\,\xi\right)$$

$$\frac{\mathrm{a}\,R}{\mu}=\left\{-\frac{15}{8}m^2-\frac{45}{32}m^3-\frac{8203}{512}m^4\right\}e^2\cos\left(2\,\tau-2\,\xi\right)$$

(See p. 166.)

Hence $\frac{\mathrm{a}}{\mu}\int\mathrm{d}\,R=-\frac{333}{64}m^4\,e^2\cos\left(2\,\tau-2\,\xi\right)$

$$\mathfrak{r}_9=\frac{45}{16}m+\frac{561}{64}m^2$$

Let $r_9=-\frac{15}{4}m^2-\frac{225}{16}m^3+A\,m^4$.

$$(4\,m^2-1)\left\{-\frac{15}{4}m^2-\frac{225}{16}m^3+A\,m^4\right\}=(4\,m^2-6\,m^3)\left\{\frac{45}{16}m+\frac{561}{32}m^2\right\}$$

$$+\frac{15}{4}m^2+\frac{45}{16}m^3+\frac{8203}{256}m^4+\frac{333}{32}m^4$$

$$-15-A=\frac{561}{16}-\frac{135}{8}+\frac{8203}{256}+\frac{333}{32}$$

$$A=-\frac{19363}{256}$$

$$r_9=-\frac{15}{4}m^2-\frac{225}{16}m^3-\frac{19363}{256}m^4$$

I now employ the equation

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,t}=\frac{h}{2\,r^2}+\frac{\mu}{h\,r}-\frac{\mathrm{d}\,r^2}{2\,h\,\mathrm{d}\,t^2}-\frac{\mu}{h\,a}-\frac{1}{h}\int\mathrm{d}\,R-\frac{1}{2\,h\,r^2}\left\{\int\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t\right\}^2.$$

When the fourth powers of the eccentricities are neglected, Arg. 9 results only from the combinations of Arg. 2 with Arg. 3, 8 with 1, and 136 with 1; and it will be found sufficient, in order to obtain the term of Arg. 9 multiplied by m^4, to take

the terms multiplied by m^2 in Arg. 1
. . . . m^3 . . 2
. . . . m^4 . . 3
. . . . m^2 . . 8
. . . . m^2 . . 136

$\frac{1}{2\,h\,r^2}\left\{\int\frac{d\,R}{d\,\lambda}\,d\,t\right\}^2$ gives no term multiplied by m^4 of Arg. 9.

$$r = \mathrm{r} - \mathrm{r}^2\,\delta\,\frac{1}{r} + \mathrm{r}^3\left(\delta\,\frac{1}{r}\right)^2 - \mathrm{r}^4\left(\delta\,\frac{1}{r}\right)^3 + \&c.$$

$$\frac{\mathrm{r}^3}{\mathrm{a}}\left(\delta\,\frac{1}{r}\right)^2 = \{1 - 3\,e\cos\xi\}\left\{\frac{m^4}{3}\cos 2\,\tau + \frac{15}{8}\,m^3\,e\cos\xi\right.$$

$$\left. + \left\{\frac{5}{8}\,m^3 + \frac{251}{96}\,m^4\right\}e\cos(2\,\tau - \xi) + \frac{225}{128}\,m^2\,e^2\cos(4\,\tau - 2\,\xi)\right\}$$

$$= \frac{15}{8}\,m^3\,e\cos\xi + \left\{\frac{5}{8}\,m^3 + \frac{203}{96}\,m^4\right\}e\cos(2\,\tau - \xi)$$

$$+ \frac{225}{128}\,m^2\,e^2\cos(4\,\tau - 2\,\xi)$$

$$\mathrm{r}^2 = -\,2\,e\cos\xi - \frac{e^2}{2}\cos 2\,\xi$$

It will be found sufficient, in order to procure $\frac{d\,r}{d\,t}$, to take

$$\frac{r}{\mathrm{a}} = -\left\{r_1 - \frac{m^4}{3}\right\}\cos 2\,\tau - \left\{1 + r_2 - 2\,r_0 - \frac{15}{8}\,m^3\right\}e\cos\xi$$

$$-\left\{r_3 - r_1 - \frac{5}{8}\,m^3 - \frac{203}{96}\right\}m^4\,e\cos(2\,\tau - \xi)$$

$$-\left\{\frac{1}{2} + r_8 - r_2 - \frac{1}{2}\,r_0\right\}e^2\cos 2\,\xi$$

$$-\left\{r_{136} - \frac{225}{128}\,m^2\right\}e^2\cos(4\,\tau - 2\,\xi)$$

$$\frac{h}{2\,r^2} = \frac{h}{2\,\mathrm{r}^2} + \frac{h}{\mathrm{r}}\,\delta\,\frac{1}{r} + \frac{h}{2}\left(\delta\,\frac{1}{\mathrm{r}}\right)^2$$

$$\frac{\mathrm{a}}{\mathrm{r}} = 1 + e\cos\xi + e^2\cos 2\,\xi \qquad h = \sqrt{\mathrm{a}\,\mu}\,\left(1 - \frac{m^2}{3}\right)$$

$$\frac{d\,r^2}{2\,h\,d\,t^2} = \left\{\left\{r_3 - r_1 - \frac{5}{8}\,m^3 - \frac{203}{96}\,m^4\right\}\left\{1 + r_2 - 2\,r_0\right.\right.$$

$$\left.\left. - \frac{15}{8}\,m^3\right\}\frac{\mathrm{c}}{2}\{2 - 2\,m - \mathrm{c}\}\left(1 + \frac{m^2}{3}\right)\right.$$

$$+\left\{r_1-\frac{m^2}{3}\right\}\left\{\frac{1}{2}+r_3-r_2-\frac{1}{2}r_1\right\}c\left\{2-2m\right\}\left\{1+\frac{m^2}{3}\right\}$$

$$\left.+2r_1\left\{r_{136}-\frac{225}{128}m^2\right\}\right\}n\,e^2\cos(2\tau-2\xi)$$

$$=\left\{\left\{r_3-r_1-\frac{5}{8}m^3-\frac{203}{96}m^4\right\}\left\{\frac{1}{2}-m+\frac{m^2}{3}-\frac{41}{48}m^3\right\}\right.$$

$$\left.+r_1\left\{1-m+\frac{17}{12}m^2\right\}+\frac{225}{64}m^2r_1-\frac{m^4}{3}\right\}m\,e^2\cos(2\tau-2\xi)$$

$$=\left\{r_3\left\{\frac{1}{2}-m+\frac{m^2}{3}-\frac{41}{48}m^3\right\}+r_1\left\{\frac{1}{2}+\frac{883}{192}m^2\right\}\right.$$

$$\left.-\frac{5}{16}m^3-\frac{147}{192}m^4\right\}n\,e^2\cos(2\tau-2\xi)$$

$$\delta\,\frac{1}{r}+\frac{\mu}{h}\delta\,\frac{1}{r}=\left\{2r_9+\frac{1}{2}\left(1-\frac{m^2}{3}\right)r_3\right.$$

$$\left.+\frac{1}{2}\left(1-\frac{m^2}{3}\right)r_1\right\}n\,e^2\cos(2\tau-2\xi)$$

$$\frac{\lambda}{d\,t}=\left\{2r_9+r_3\left\{m-\frac{m^2}{2}-\frac{41}{48}m^3\right\}-\frac{305}{64}m^4+\frac{5^*}{8}m^3+\frac{501}{128}m^4\right.$$

$$\left.+\frac{5}{16}m^3+\frac{147}{192}m^4+\frac{333}{64}m^4\right\}e^2\cos(2\tau-2\xi)$$

$$=-\left\{-\frac{15}{2}m-\frac{225}{8}m^2-\frac{19363}{128}m^3\right.$$

$$+\left\{\frac{15}{8}m+\frac{187}{32}m^2+\frac{33113}{1536}m^3\right\}\left\{1-\frac{m}{2}-\frac{41}{48}m^2\right\}$$

$$-\frac{305}{64}m^3+\frac{5}{8}m^2+\frac{501}{128}m^3+\frac{5}{16}m^2$$

$$\left.+\frac{147}{192}m^3+\frac{333}{64}m^3\right\}\frac{1}{2}\left\{1+\frac{3}{4}m+\frac{243}{32}m^2\right\}$$

$$=\left\{\frac{15}{4}-\frac{15}{16}\right\}m+\left\{\frac{225}{16}-\frac{187}{64}+\frac{15}{32}-\frac{5}{16}-\frac{5}{32}+\frac{135}{64}\right\}m^2$$

$$+\left\{\frac{19363}{256}-\frac{31085}{3072}+\frac{305}{128}-\frac{501}{256}-\frac{147}{384}-\frac{333}{128}+\frac{2139}{256}+\frac{10935}{512}\right\}m^3$$

---

* For this term and the following, see p. 120 and p. 171.

$$= \frac{45}{16} m + \frac{53}{4} m^2 + \frac{284689}{3072} m^3$$

The value with M. Plana's system of constants is

$$\left\{ \frac{45}{16} m + \frac{53}{4} m^2 + \frac{284689}{3072} m^3 \right\} \{1 - m^2\}$$

$$= \frac{45}{16} m + \frac{53}{4} m^2 + \frac{276049}{3072} m^3$$

agreeing with M. Plana.

In this investigation certain quantities have been left indeterminate, in order to show more clearly the nature of the reductions which ensue, and without which λ_9 would contain a term independent of m. Without these reductions the term in r_3 multiplied by m^4 would have been required.

I shall now endeavour to calculate the same quantity λ_9, by means of an equation often employed by Laplace, in order to afford an example of the application of that method.

Neglecting s^2 in the equation B of p. 321, I find the equation

$$\frac{d\lambda}{dt} = \frac{h}{2r^2} - \frac{\mu}{hr} + \frac{3\mu}{2ha} + \frac{d^2 . r^2}{h\,dt^2} - \frac{dr^2}{2h\,dt^2}$$
$$+ \frac{1}{h}\left\{ 3\int dR + 2r\frac{dR}{dr} \right\} - \frac{1}{2hr^2}\left\{ \int \frac{dR}{d\lambda} dt \right\}^2.$$

$$\frac{d\lambda}{dt} = \frac{h}{\mathrm{r}^2} - \frac{\mu}{hr} + \frac{3\mu}{2ha} + \frac{d^2\mathrm{r}}{h\,dt^2}\delta r$$
$$+ \frac{d\{2\mathrm{r}\,d\delta\mathrm{r} + dr\delta r\}}{h\,dt^2} + \frac{d^2(\delta r)^2}{h\,dt^2} - \frac{(d . \delta r)^2}{2h\,dt^2}$$
$$+ \frac{1}{h}\left\{ 3\int dR + 2r\frac{dR}{dr} \right\} - \frac{1}{2hr^2}\left\{ \int \frac{dR}{d\lambda} dt \right\}^2$$

$$\delta r = - \mathrm{r}^2 \delta \frac{1}{r} + \mathrm{r}^3 \left(\delta \frac{1}{r}\right)^2 - \mathrm{r}^4 \left(\delta \frac{1}{r}\right)^3 + \&c.$$

Neglecting $(\delta r)^3$, $\left(\delta \frac{1}{r}\right)^3$ the equation may be put in the form

$$\frac{d\delta\lambda}{dt} = \left\{ \frac{h^2}{\mathrm{r}^3} - \frac{\mu}{\mathrm{r}^2} - \frac{d^2 r}{dt^2} \right\} \frac{\mathrm{r}^2}{h} \delta \frac{1}{r} + \frac{d\{2r\,d\delta r + dr\delta r\}}{h\,dt^2}$$

$$+\frac{h}{2}\left(\delta\frac{1}{r}\right)^2+\frac{\mathrm{r}^3}{h}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}\left(\delta\frac{1}{r}\right)^2+\frac{\mathrm{d}^2(\delta r)^2}{h\,\mathrm{d}t^2}-\frac{(\mathrm{d}\,.\,\delta r)^2}{2h\,\mathrm{d}t^2}$$

$$+\frac{1}{h}\left\{3\int\mathrm{d}R+\frac{2r\,\mathrm{d}R}{\mathrm{d}r}\right\}-\frac{1}{2hr^2}\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t\right\}^2.$$

In the elliptic theory

$$\frac{h^2}{\mathrm{r}^3}-\frac{\mu}{\mathrm{r}^3}-\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}=0$$

$$\delta r=-\mathrm{r}^2\delta\frac{1}{r}+\mathrm{r}^3\left(\delta\frac{1}{r}\right)^2-\mathrm{r}^4\left(\delta\frac{1}{r}\right)^3$$

$(\delta r)^2=\mathrm{r}^4\left(\delta\frac{1}{r}\right)^2$ nearly.

Making a and n equal to unity

$$\frac{h^2}{\mathrm{r}^3}-\frac{\mu}{\mathrm{r}^2}-\frac{\mathrm{d}^2\mathrm{r}}{\mathrm{d}t^2}$$

$$=\left\{1-\frac{2}{3}m^2\right\}\left\{1+\frac{3}{2}e^2+3e\cos\xi+\frac{9}{2}e^2\cos 2\xi\right\}$$

$$-\left\{1+\frac{e^2}{2}+2e\cos\xi+\frac{5}{2}e^2\cos 2\xi\right\}$$

$$-\left(1-\frac{3}{2}m^2-\frac{225}{16}m^3\right)\left\{e\cos\xi+2e^2\cos 2\xi\right\}$$

This quantity multiplied by $\frac{\mathrm{r}^2}{h}$ gives

$$-\left\{\frac{2m^2}{3}+\frac{m^2}{2}e\cos\xi\right\}\left\{1+\frac{3}{2}e^2-2e\cos\xi-\frac{e^2}{2}\cos 2\xi\right\}$$

$$=-\frac{2}{3}m^2+\left\{-\frac{m^2}{2}+\frac{225}{16}m^3\right\}e\cos\xi$$

$$\delta\frac{1}{r}=\frac{m^2}{6}+\left\{m^2+\frac{19}{6}m^3\right\}\cos 2\tau+\frac{2}{3}m^2 e\cos\xi$$

$$+\left\{\frac{15}{8}m+\frac{187}{32}m^2+\frac{33113}{1536}m^3\right\}e\cos(2\tau-\xi)+\frac{5}{3}m^2e^2\cos 2\xi$$

$$+\left\{-\frac{15}{4}m^2-\frac{225}{16}m^3\right\}e^2\cos(2\tau-2\xi)+\frac{225}{64}m^2e^2\cos(4\tau-2\xi)$$

$$\mathrm{r}^2 = 1 + \frac{3}{2} e^2 - 2 e \cos \xi - \frac{e^2}{2} \cos 2 \xi$$

$$\mathrm{r}^2 \, \delta \, \frac{1}{r} = \left\{ m^2 + \frac{19}{6} m^3 \right\} \cos 2 \tau + \left\{ \frac{2}{3} - \frac{1}{3} \right\} m^2 e \cos \xi$$

$$+ \left\{ \frac{15}{8} m + \left\{ -1 + \frac{187}{32} \right\} m^2 + \left\{ -\frac{19}{6} + \frac{33113}{1536} \right\} m^3 \right\} e \cos (2 \tau - \xi)$$

$$+ \left\{ \frac{5}{3} - \frac{2}{3} - \frac{1}{12} \right\} m^2 e^2 \cos 2 \xi$$

$$+ \left\{ -\frac{15}{8} m + \left\{ -\frac{187}{32} - \frac{1}{4} - \frac{15}{4} \right\} m^2 \right\} e^2 \cos (2 \tau - 2 \xi)$$

$$= \left\{ m^2 + \frac{19}{6} m^3 \right\} \cos 2 \tau + \frac{m^2}{3} e \cos \xi$$

$$+ \left\{ \frac{15}{8} m + \frac{155}{32} m^2 + \frac{28249}{1536} m^3 \right\} e \cos (2 \tau - \xi)$$

$$+ \frac{11}{12} m^2 e^2 \cos 2 \xi + \left\{ -\frac{15}{8} m - \frac{315}{32} m^2 \right\} e^2 \cos (2 \tau - 2 \xi)$$

$$+ \frac{225}{64} m^2 e^2 \cos (4 \tau - 3 \xi)$$

$$\left\{ \frac{h^2}{\mathrm{r}^3} - \frac{\mu}{\mathrm{r}^2} - \frac{\mathrm{d}^2 \, \mathrm{r}}{\mathrm{d} \, t^2} \right\} \frac{\mathrm{r}^2}{h} \, \delta \, \frac{1}{r}$$

$$= \left\{ \left\{ \frac{5}{4} - \frac{15}{32} \right\} m^3 + \left\{ \frac{105}{16} - \frac{155}{128} + \frac{3375}{256} \right\} m^4 \right\} e^2 \cos (2 \tau - 2 \xi)$$

$$= \left\{ \frac{25}{32} m^3 + \frac{4745}{256} m^4 \right\} e^2 \cos (2 \tau - 2 \xi)$$

$$\left(\delta \frac{1}{r} \right)^2 = \frac{19}{36} m^4 + \frac{m^4}{3} \cos 2 \tau + \frac{15}{8} m^3 e \cos \xi$$

$$+ \left\{ \frac{5}{8} m^3 + \frac{251}{96} m^4 \right\} e \cos (2 \tau - \xi)$$

$$+ \left\{ \frac{5}{4} m^3 + \frac{501}{64} m^4 \right\} e^2 \cos (2 \tau - 2 \xi)$$

$$+ \frac{15}{8} m^3 e \cos (4 \tau - \xi) + \frac{225}{128} m^2 e^2 \cos (4 \tau - 2 \xi)$$

See p. 55, p. 120 and p. 171.

$$\mathrm{r}^3 = 1 - 3\,e\cos\xi \qquad \mathrm{r} = 1 - e\cos\xi - \frac{e^2}{2}\cos 2\xi$$

$$\delta r = -\mathrm{r}^2\,\delta\frac{1}{r} + \mathrm{r}^3\left(\delta\frac{1}{r}\right)^2$$

$$= -\left\{m^2 + \frac{19}{6}m^3\right\}\cos 2\tau + \left\{-\frac{m^3}{3} + \frac{15}{8}m^3\right\}e\cos\xi$$

$$-\left\{\frac{15}{8}m + \frac{155}{32}m^2 + \frac{27289}{1536}m^3\right\}e\cos(2\tau-\xi)$$

$$-\frac{11}{12}m^2e^2\cos 2\xi - \left\{-\frac{15}{8}m - \frac{315}{32}m^2\right\}e^2\cos(2\tau - 2\xi)$$

$$-\frac{225}{128}m^2e^2\cos(4\tau - 2\xi)$$

$$\frac{2\,\mathrm{r}\,\mathrm{d}\,.\,\delta r + \mathrm{d}\,r\,\delta r}{\mathrm{d}t}$$

$$= \left\{2 - 2\,e\cos\xi - e^2\cos 2\xi\right\}\left\{m^2 + \frac{19}{6}m^3\right\}\{2 - 2m\}\sin 2\tau$$

$$+\frac{15}{8}m + \frac{155}{32}m^2 + \frac{27289}{1536}m^3\Big\}\left\{1 - 2m + \frac{3}{4}m^2\right\}e\sin(2\tau-\xi)$$

$$-\left\{-\frac{15}{8}m - \frac{315}{32}m^2\right\}\left\{2m - \frac{3}{2}m^2\right\}e^2\sin(2\tau - 2\xi)$$

$$-\left\{1 - \frac{3}{4}m^2\right\}\{e\sin\xi + e^2\sin 2\xi\}\Bigg\{\left\{m^2 + \frac{19}{6}m^3\right\}\cos 2\tau$$

$$+\left\{\frac{15}{8}m + \frac{155}{32}m^2 + \frac{27289}{1536}m^3\right\}e\cos(2\tau - \xi)\Bigg\}$$

$$= \Bigg\{\left\{\frac{15}{8}m + \frac{315}{32}m^2\right\}\{4m - 3m^2\}$$

$$+\left\{\frac{15}{8}m + \frac{155}{32}m^2 + \frac{27289}{1536}m^3\right\}\Big\{-1 + 2m$$

$$-\frac{3}{4}m^2 + \frac{1}{2} - \frac{3}{8}m^2\Big\}$$

$$+ \left\{m^2 + \frac{19}{6} m^3\right\} \left\{-1 + m + \frac{1}{2}\right\} \Bigg\} e^2 \sin (2 \tau - 2 \xi)$$

$$\frac{2 \mathrm{r} \, \mathrm{d} \, . \, \delta r + \mathrm{d} r \, \delta r}{h \, \mathrm{d} t}$$

$$= \left\{ -\frac{15}{16} m + \left\{\frac{15}{2} + \frac{15}{4} - \frac{155}{64} - \frac{1}{2}\right\} m^2 \right.$$

$$\left. + \left\{\frac{315}{8} - \frac{45}{8} - \frac{27289}{3072} + \frac{155}{16} - \frac{135}{64} + 1 - \frac{19}{12} - \frac{5}{16}\right\} m^3 \right\} e^2 \sin (2 \tau - 2 \xi)$$

$$= \left\{ -\frac{15}{16} m + \frac{533}{64} m^2 + \frac{96919}{3072} m^2 \right\} e^2 \sin (2 \tau - 2 \xi)$$

$$\frac{h}{2} \left(\delta \frac{1}{r}\right)^2 = \left\{\frac{5}{8} m^3 + \frac{501}{128} m^4\right\} e^2 \cos (2 \tau - 2 \xi)$$

$$\frac{\mathrm{r}^3}{h} \frac{\mathrm{d}^2 r}{\mathrm{d} t^2} \left(\delta \frac{1}{r}\right)^2$$

$$= \{1 - 3 e \cos \xi\} \{e \cos \xi + 2 e^2 \cos 2 \xi\}$$

$$\left\{ \frac{m^4}{3} \cos 2 \tau + \left\{\frac{5}{8} m^3 + \frac{251}{96} m^4\right\} e \cos (2 \tau - \xi) \right\}$$

$$= \left\{\frac{5}{16} m^3 + \frac{89}{64} m^4\right\} e^2 \cos (2 \tau - 2 \xi)$$

$$\mathrm{d} \, . \, \delta r = 2 m^2 \sin 2 \tau + \left\{\frac{m^2}{3} - \frac{15}{8} m^3\right\} e \sin \xi$$

$$+ \left\{\frac{15}{8} m + \frac{155}{32} m^2\right\} \{1 - 2 m\} e \sin (2 \tau - \xi) + \frac{11}{6} m^2 e^2 \sin 2 \xi$$

$$+ \frac{225}{64} m^2 e^2 \sin (4 \tau - 2 \xi)$$

$$\frac{(\mathrm{d} \, . \, \delta r)^2}{h \, \mathrm{d} t^2} = \left\{ \frac{5}{8} m^3 + \left\{\frac{35}{96} + \frac{11}{3} + \frac{225}{32} - \frac{225}{64}\right\} m^4 \right\} e^2 \cos (2 \tau - 2 \xi)$$

$$= \left\{\frac{5}{8} m^3 + \frac{483}{64} m^4\right\} e^2 \cos (2 \tau - 2 \xi)$$

$$\frac{1}{h} \left\{ 3 \int \mathrm{d} R + 2 r \frac{\mathrm{d} R}{\mathrm{d} r} \right\}$$

$$= \left\{1 + \frac{m^2}{3}\right\} \left\{ -\frac{999}{64} m^4 - \frac{15}{2} m^2 - \frac{45}{8} m^3 - \frac{8203}{128} m^4 \right\} e^2 \cos (2 \tau - 2 \xi)$$

$$= \left\{ -\frac{15}{2} m^2 - \frac{45}{8} m^3 + \left\{ -\frac{999}{64} - \frac{5}{\;} - \frac{8203}{128} \right\} m^4 \right\} e^2 \cos (2\,\tau - 2\,\xi)$$

$$= \left\{ -\frac{15}{2} m^2 - \frac{45}{8} m^3 - \frac{10521}{128} m^4 \right\} e^2 \cos (2\,\tau - 2\,\xi).$$

$$\lambda_9 = - \left\{ -\frac{15}{2} m^2 + \left\{ \frac{25}{32} + \frac{5}{8} + \frac{5}{16} - \frac{5}{16} - \frac{45}{8} \right\} m^3 \right.$$

$$\left. + \left\{ \frac{4745}{256} + \frac{501}{128} + \frac{89}{64} - \frac{483}{128} - \frac{10521}{128} \right\} m^4 \right\} \frac{1}{m} \left\{ \frac{1}{2} + \frac{3}{8} m \right.$$

$$\left. + \frac{243}{64} m^2 \right\}$$

$$-\frac{15}{16} m + \frac{533}{64} m^2 + \frac{96919}{3072} m^3$$

$$= - \left\{ \frac{15}{2} m - \frac{135}{32} m^2 - \frac{15905}{256} m^3 \right\} \left\{ \frac{1}{2} + \frac{3}{8} m + \frac{243}{64} m^2 \right\}$$

$$-\frac{15}{16} m + \frac{533}{64} m^2 + \frac{96919}{3072} m^3$$

$$= \frac{45}{16} m + \frac{53}{4} m^2 + \frac{28469}{3072} m^3$$

agreeing with the value found in p. 344.

I have given this example of the calculation of λ_9 by the extension of Laplace's equation, not so much with a view of confirming the accuracy of the result previously obtained as with a view of affording an example of the application of this method.

On the quantity $\int$(d R).

If (d R) denote the differential of the disturbing function R, with respect to the coordinates x, y, z of the disturbed body only,

$$(\mathrm{d}\,R) = \frac{\mathrm{d}\,R}{\mathrm{d}\,x} \mathrm{d}\,x + \frac{\mathrm{d}\,R}{\mathrm{d}\,y} \mathrm{d}\,y + \frac{\mathrm{d}\,R}{\mathrm{d}\,z} \mathrm{d}\,z,$$

and by the equations of motion,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + \frac{\mu\,x}{r^3} + \frac{\mathrm{d}\,R}{\mathrm{d}\,x} = 0$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mu y}{r^3} + \frac{\mathrm{d}R}{\mathrm{d}y} = 0$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{\mu z}{r^3} + \frac{\mathrm{d}R}{\mathrm{d}z} = 0.$$

If $[\mathrm{d}R]$ denote the differential of R, considering only $\mathrm{n}\,t$ variable, then

$$(\mathrm{d}R) = [\mathrm{d}R] + \frac{\mathrm{d}R}{\mathrm{d}x}\,\mathrm{d}'x + \frac{\mathrm{d}R}{\mathrm{d}y}\,\mathrm{d}'y + \frac{\mathrm{d}R}{\mathrm{d}z}\,\mathrm{d}'z,$$

$\mathrm{d}'x$, $\mathrm{d}'y$, and $\mathrm{d}'z$ denoting the differentials of the coordinates x, y, z with respect to $n_{,}t$ only, or with respect to $m\,\mathrm{n}\,t$ only in the Theory of the Moon. I have often employed this theorem in the course of this work. I shall now endeavour to effectuate the integration indicated by the symbol $\int(\mathrm{d}R)$ in certain cases where it may be useful.

The quantities $\mathrm{d}'x$, $\mathrm{d}'y$, and $\mathrm{d}'z$ contain of course no term of which the argument is independent of $n_{,}t$ or $m\,\mathrm{n}\,t$; and it is easy to show that if X is any function of x, y, z only, explicitly the development of the quantity

$$\mathrm{X}\,\mathrm{d}'\mathrm{X}$$

contains no term of which the argument is independent of $n_{,}t$. For let

$$\mathrm{X} = \mathrm{A}\cos(i + jm)\,t + \mathrm{A}'\cos(i' + j'm)\,t$$

$$\mathrm{d}'\mathrm{X} = -jm\,\mathrm{A}\sin(i + jm)\,t + -j'm\,\mathrm{A}'\sin(i' + j'm)\,t$$

$$\begin{aligned}\mathrm{X}\,\mathrm{d}'\mathrm{X} = {} & -jm\,\frac{\mathrm{A}^2}{2}\sin(2i + 2jm)\,t \\ & -\frac{\mathrm{A}\,\mathrm{A}'}{2}\,jm\,\{\sin(i + i' + jm + j'm)\,t \\ & \qquad + \sin(i - i' + jm - j'm)\,t\} \\ & -\frac{\mathrm{A}\,\mathrm{A}'}{2}\,j'm\,\{\sin(i + i' + jm + j'm)\,t \\ & \qquad - \sin(i - i' + jm - j'm)\,t\}.\end{aligned}$$

Suppose, for instance, the angle

$$(i + i' + jm + j'm)\,t$$

is independent of $m\,t$,

$$j + j' = 0;$$

and it will be seen that precisely in this case the coefficient of the sine becomes equal to zero, and so in the other cases which may be imagined. This nearly self-evident theorem will facilitate the integration required.

I now revert to the theorem

$$(\mathrm{d}\,\boldsymbol{R}) = [\mathrm{d}\,R] + \frac{\mathrm{d}\,\boldsymbol{R}}{\mathrm{d}\,x}\,\mathrm{d}'x + \frac{\mathrm{d}\,\boldsymbol{R}}{\mathrm{d}\,y}\,\mathrm{d}'y + \frac{\mathrm{d}\,\boldsymbol{R}}{\mathrm{d}\,z}\,\mathrm{d}'z$$

$$= [\mathrm{d}\,R] - \left\{\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + \frac{\mu\,x}{r^3}\right\}\mathrm{d}'x$$

$$- \left\{\frac{\mathrm{d}^2 y}{\mathrm{d}\,t^2} + \frac{\mu\,y}{r^3}\right\}\mathrm{d}'y$$

$$- \left\{\frac{\mathrm{d}^2 z}{\mathrm{d}\,t^2} + \frac{\mu\,z}{r^3}\right\}\mathrm{d}'z$$

$$= [\mathrm{d}\,\boldsymbol{R}] - \frac{\mathrm{d}^2 z}{\mathrm{d}\,t^2}\,\mathrm{d}'z - \frac{\mathrm{d}^2 y}{\mathrm{d}\,t^2}\,\mathrm{d}'y + \frac{\mathrm{d}^2 z}{\mathrm{d}\,t^2}\,\mathrm{d}'z + \mathrm{d}'\,\frac{\mu}{r}.$$

If we consider any argument independent of $n_{\prime}\,t$, of course

$$\mathrm{d}'\,\frac{\mu}{r} = 0;$$

also, by what has preceded,

$$\frac{\mathrm{d}\,x\,\mathrm{d}'\,\mathrm{d}\,x}{\mathrm{d}\,t^2} = \frac{\mathrm{d}\,x\,\mathrm{d}\,\mathrm{d}'\,x}{\mathrm{d}\,t^2} = 0$$

$$\mathrm{d}\,\boldsymbol{R} = [\mathrm{d}\,R] - \frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2}\,\mathrm{d}'x - \frac{\mathrm{d}\,x\,\mathrm{d}\,\mathrm{d}'x}{\mathrm{d}\,t^2}$$

$$- \frac{\mathrm{d}^2 y}{\mathrm{d}\,t^2}\,\mathrm{d}'y - \frac{\mathrm{d}\,y\,\mathrm{d}\,\mathrm{d}'y}{\mathrm{d}\,t^2}$$

$$- \frac{\mathrm{d}^2 z}{\mathrm{d}\,t^2}\,\mathrm{d}'z - \frac{\mathrm{d}\,z\,\mathrm{d}\,\mathrm{d}'z}{\mathrm{d}\,t^2}$$

$$= [\mathrm{d}\,R] - \frac{\mathrm{d}\,\{\mathrm{d}\,x\,\mathrm{d}'x\}}{\mathrm{d}\,t^2} - \frac{\mathrm{d}\,\{\mathrm{d}\,y\,\mathrm{d}'y\}}{\mathrm{d}\,t^2} - \frac{\mathrm{d}\,\{\mathrm{d}\,z\,\mathrm{d}'z\}}{\mathrm{d}\,t^2}.$$

If $r^{\backprime}$ is the projected distance, $\lambda^{\backprime}$ the longitude, as throughout this work,

$$\mathrm{d}\,x = \mathrm{d}\,r^{\backprime}\cos\lambda^{\backprime} - r^{\backprime}\sin\lambda^{\backprime}\,\mathrm{d}\,\lambda^{\backprime} \qquad \mathrm{d}\,y = \mathrm{d}\,r^{\backprime}\sin\lambda^{\backprime} + r^{\backprime}\cos\lambda^{\backprime}\,\mathrm{d}\,\lambda^{\backprime}$$

$$\mathrm{d}'x = \mathrm{d}'r^{\backprime}\cos\lambda^{\backprime} - r^{\backprime}\sin\lambda^{\backprime}\,\mathrm{d}'\lambda^{\backprime} \qquad \mathrm{d}'y = \mathrm{d}'r^{\backprime}\sin\lambda^{\backprime} + r^{\backprime}\cos\lambda^{\backprime}\,\mathrm{d}'\lambda^{\backprime}$$

$$\mathrm{d}\,x\,\mathrm{d}'x + \mathrm{d}\,y\,\mathrm{d}'y = \mathrm{d}\,r^{\backprime}\,\mathrm{d}'r^{\backprime} + r^{\backprime 2}\,\mathrm{d}\,\lambda^{\backprime}\,\mathrm{d}'\lambda^{\backprime}.$$

Also, when the terms only are considered independent of $n_{\prime} t$,

$$[\mathrm{d} R] = \mathrm{d} R \qquad \int [\mathrm{d} R] = R$$

$$\int (\mathrm{d} R) = R - \frac{\mathrm{d} x \, \mathrm{d}' x}{\mathrm{d} t^2} - \frac{\mathrm{d} y \, \mathrm{d}' y}{\mathrm{d} t^2} - \frac{\mathrm{d} z \, \mathrm{d}' z}{\mathrm{d} t^2}$$

$$= R - \frac{\mathrm{d} r^{\backprime} \, \mathrm{d}' r^{\backprime}}{\mathrm{d} t^2} - \frac{r^{\backprime 2} \, \mathrm{d} \lambda^{\backprime} \, \mathrm{d}' \lambda^{\backprime}}{\mathrm{d} t^2} - \frac{\mathrm{d} z \, \mathrm{d}' z}{\mathrm{d} t^2}.$$

In the elliptic theory $\mathrm{d}' r^{\backprime}$, $\mathrm{d}' \lambda^{\backprime}$, $\mathrm{d}' z$ are equal to zero, and it is obvious that only terms will occur in $\mathrm{d}' r^{\backprime}$ of the order of the disturbing mass; nor can the term $n_{\prime} t$ be got rid of in the product $\mathrm{d} r^{\backprime} \, \mathrm{d}' r^{\backprime}$ except by terms in $\mathrm{d} r^{\backprime} \, \mathrm{d}' r^{\backprime}$, neither of which occur in the elliptic theory; moreover, $\mathrm{d}' r^{\backprime}$, $\mathrm{d}' \lambda^{\backprime}$, $\mathrm{d}' z$ are multiplied by m. This theorem shows that when any argument is considered in the Theory of the Moon independent of $m \, n \, t$, which includes, of course, all those *of long period*, the integration denoted by the symbol $\int (\mathrm{d} R)$ may be effected, and it shows the nature of this quantity.

Suppose, for example, it were required to find the term in $\int (\mathrm{d} R)$ multiplied by m^4 in Argument 2 (ξ), it is sufficient to take

$$\mathrm{d} r = 2 m^2 \sin 2 \tau + \frac{15}{8} m e \sin (2 \tau - \xi)$$

$$\mathrm{d} r = - 2 m^3 \sin 2 \tau - \frac{15}{4} m^2 e \sin (2 \tau - \xi)$$

$$h - \int \frac{\mathrm{d} R}{\mathrm{d} \lambda} \mathrm{d} t = \frac{3}{4} m^2 \cos 2 \tau \qquad \mathrm{d}' \lambda = - \frac{15}{2} m^2 e \cos \; 2 \tau - \xi)$$

$$\mathrm{d} r \, \mathrm{d}' r + \left\{ h - \int \frac{\mathrm{d} R}{\mathrm{d} \lambda} \mathrm{d} t \right\} \mathrm{d}' \lambda = \left\{ - \frac{15}{8} - \frac{15}{4} - \frac{45}{16} \right\} m^4 e \cos \xi$$

$$= - \frac{135}{16} m^4 e \cos \xi$$

$$\int (\mathrm{d} R) = \left\{ \frac{5999}{384} + \frac{135}{16} \right\} m^4 e \cos \xi, \qquad \text{as before, p. 54.}$$

In such an instance there is no utility in having recourse to this method. Now, however, suppose it were required to calculate the term in $\int (\mathrm{d} R)$, Argument 77 ($2 \xi - 2 \eta$), where the divisor introduced by integration is a multiple of m^2.

It will easily be seen that the only Arguments necessary to be taken into account in calculating the quantity $\mathrm{d}\, r^{\backprime}\, \mathrm{d}'\, r^{\backprime}$ are 3 and 69, corresponding to the angles $2\,\tau - \xi$ and $2\,\tau + \xi - 2\,\eta$. If we consider, for example, the Argument 80 corresponding to the angle $2\,\tau - 2\,\xi + 2\,\eta$, it is true that this angle by combination with the angle $2\,\tau$ would produce the argument in question, but then the coefficient of $\cos 2\,\tau$ in the expression for $r^{\backprime}$ is multiplied by the square of m. I find without difficulty

$$r^{\backprime} = -\frac{15}{8}\, m\, e \cos(2\,\tau - \xi) + \frac{39}{64}\, m\, e\, \gamma^2 \cos(2\,\tau + \xi - 2\,\eta)$$

$$[3] \qquad\qquad\qquad\qquad [69]$$

$$\mathrm{d}\, r^{\backprime} = \frac{15}{8}\, m\, e \sin(2\,\tau - \xi) - \frac{39}{64}\, m\, e\, \gamma^2 \sin(2\,\tau + \xi - 2\,\eta)$$

$$\mathrm{d}'\, r^{\backprime} = \frac{15}{4}\, m^2\, e \sin(2\,\tau - \xi) + \frac{39}{32}\, m^2\, e\, \gamma^2 \cos(2\,\tau + \xi - 2\,\eta$$

$$\mathrm{d}\, r^{\backprime}\, \mathrm{d}'\, r^{\backprime} = \frac{585}{256}\, m^3\, e^2\, \gamma^2 \cos(2\,\xi - 2\,\eta)$$

$$[77]$$

$$h - \int \frac{\mathrm{d}\, R}{\mathrm{d}\, \lambda^{\backprime}}\, \mathrm{d}\, t = -\frac{15}{8}\, m\, e^2 \cos(2\,\tau - 2\,\xi) - \frac{3}{8}\, m\, \gamma^2 \cos(2\,\tau - 2\,\eta).$$

See p. 184 and p. 263.

$$\mathrm{d}'\, \lambda^{\backprime} = -\frac{45}{8}\, m\, e^2 \cos(2\,\tau - 2\,\xi) - \frac{9}{8}\, m\, \gamma^2 \cos(2\,\tau - 2\,\eta)$$

$$\left\{h - \int \frac{\mathrm{d}\, R}{\mathrm{d}\, \lambda^{\backprime}}\, \mathrm{d}\, t\right\} \mathrm{d}'\, \lambda^{\backprime} = \frac{135}{64}\, m^3\, e^2\, \gamma^2 \cos(2\,\xi - 2\,\eta).$$

The following expression for z will be found sufficient:

$$z = \frac{3}{8}\, m\, \gamma \sin(2\,\tau - \eta) + \frac{15}{16}\, m\, e\, \gamma \sin(2\,\tau - \xi + \eta)$$

$$+ \frac{3}{16}\, m \sin(2\,\tau + \xi - \eta) + \frac{45}{64}\, m\, e^2\, \gamma \sin(2\,\tau - 2\,\xi + \eta).$$

See p. 241.

$$\mathrm{d}\, z = \frac{3}{8}\, m\, \gamma \cos(2\,\tau - \eta) + \frac{15}{8}\, m\, e\, \gamma \cos(2\,\tau - \xi + \eta)$$

$$+ \frac{3}{8}\, m\, e\, \gamma \cos(2\,\tau + \xi - \eta) + \frac{45}{64}\, m\, e^2\, \gamma \cos(2\,\tau - 2\,\xi + \eta)$$

$$\mathrm{d}'z = -\frac{3}{4}m^2\gamma\cos(2\tau-\eta) - \frac{15}{8}m^2e\gamma\cos(2\tau-\xi+\eta)$$

$$-\frac{3}{8}m^2e\gamma\cos(2\tau+\xi-\eta) - \frac{45}{32}m^2e^2\gamma\cos(2\tau-2\xi+\eta)$$

$$\mathrm{d}z\,\mathrm{d}'z = -\frac{315}{256}m^3e^2\gamma^2\cos(2\xi-2\eta).$$

$$\int \mathrm{d}R = R + \left\{-\frac{585}{256}-\frac{135}{64}+\frac{315}{256}\right\}m^3e^2\gamma^2\cos(2\xi-2\eta)$$

$$= \left\{R_{77}-\frac{405}{128}m^3\right\}e^2\gamma^2\cos(2\xi-2\eta). \qquad [77]$$

In order to calculate R_{77} I make

$$R = -2[R][r]\,\delta\frac{1}{r} + \left[\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right]\delta\lambda + \left[-R+\frac{m_{\prime}r^2}{2r_{\prime}^3}\right]s^2,$$

and after the proper substitutions in this equation I find

$$R = \frac{405}{128}m^3e^2\gamma^2\cos(2\xi-2\eta).$$

The details of the calculation of this coefficient have already been published by M. de Pontécoulant, and therefore it is unnecessary to repeat them here. This value of R_{77} gives zero for the corresponding coefficient in $\int(\mathrm{d}'R)$.

Conversely, if we adopt the analysis of M. Poisson in order to conclude that the coefficient of the argument in $\int(\mathrm{d}R)$ is equal to zero, the coefficient of the corresponding argument in the disturbing function R may be arrived at with great facility by means of the equation

$$R = \frac{\mathrm{d}r'\,\mathrm{d}'r'}{\mathrm{d}t^2} + \left\{h - \int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\mathrm{d}t\right\}\mathrm{d}'\lambda' + \frac{\mathrm{d}z\,\mathrm{d}'z}{\mathrm{d}t^2}.$$

THE END.

PRINTED BY RICHARD AND JOHN E. TAYLOR,
RED LION COURT, FLEET STREET.